



## Brief paper

Exciting multi-DOF systems by feedback resonance<sup>☆</sup>Denis Efimov<sup>a,1</sup>, Alexander Fradkov<sup>b,c,d</sup>, Tetsuya Iwasaki<sup>e</sup><sup>a</sup> Non-A project at INRIA - LNE, Parc Scientifique de la Haute Borne 40, avenue Halley Bât.A, Park Plaza 59650 Villeneuve d'Ascq, France<sup>b</sup> Institute for Problems of Mechanical Engineering, 61, Bolshoy av., V.O., St-Petersburg 199178, Russia<sup>c</sup> St. Petersburg State University, Russia<sup>d</sup> National Research University of Information Technology, Mechanics and Optics, Russia<sup>e</sup> Department of Mechanical and Aerospace Engineering, University of California, Los Angeles, CA 99095, United States

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## ABSTRACT

The mechanism of entrainment to natural oscillations in a class of (bio)mechanical systems described by linear models is investigated. Two new nonlinear control strategies are proposed to achieve global convergence to a prescribed resonance mode of oscillation within a finite time. The effectiveness of the proposed methods for resonance entrainment is demonstrated by examples of computer simulation for linear and nonlinear systems.

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## 1. Introduction

Robotic or animal locomotion is a periodical movement synchronized with the influence of the environment. To minimize energy consumption during movement, body oscillations have to be performed in a resonance mode. For example, the period of walking can be related to the natural frequency of a leg as a pendulum (Holt, Hamill, & Anders, 1991). This fact is of great importance for analysis of the locomotion phenomenon in animals and for robotic system design as well. In this way a fundamental question arises: how does one design a feedback controller ensuring reliable oscillations in (or near) a resonance mode?

An answer to this question is based on the fact that rhythmic animal motions are controlled by neural circuits called the Central Pattern Generator (CPG) (Delcomyn, 1980; Grillner, Buchanan, Walker, & Brodin, 1988; Orlovsky, Deliagina, & Grillner, 1999).

CPGs are composed by neural networks generating sustained oscillations and defining the locomotion rhythm. Such a control mechanism of oscillations can be related to the entrainment phenomenon, when one oscillator (CPG) entrains to the frequency and phase of rhythmic movement in another oscillator (in a series or feedback connection). This feedback mechanism is now used in robotics and mechanical systems to regulate locomotion (Futakata & Iwasaki, 2008, 2011). The conditions of resonance regime appearance are analyzed using (multivariable) Harmonic Balance (HB), and hence are approximate in nature.

The goal of this work is to make the next step and develop control methods that achieve resonance entrainment *exactly* and in a finite time (the system has to oscillate in a resonance mode after a finite time for any initial conditions). Our underlying idea is that not only is the resonance motion energy optimal, but its control mechanism may also be a result of a natural optimization. We use the speed gradient principle (Fradkov, 2007) to arrange the type of behavior that was called feedback resonance in Andrievsky and Fradkov (1999); Fradkov (1999). It is shown that the neural network implementation of the proposed algorithm can have a structure similar to that of Futakata and Iwasaki (2008, 2011), though its functioning can be completely different.

The main results present two dynamical output controls for a class of multiple degree-of-freedom (DOF) mechanical systems: a finite-time control (with an energy-optimal excitation mechanism) and an energy/phase regulator. The former achieves a prescribed mode of natural oscillations in a finite time, while the latter allows for regulation of the energy and phase for every mode. The

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properties of these control laws are precisely characterized with rigorous proofs. It is assumed that only the position of the mechanical system is available for measurements.

Preliminary results are presented in Section 2, where the resonance controls based on the speed gradient approach (Andrievsky & Fradkov, 1999; Efimov & Fradkov, 2007b, 2012; Fradkov, 1999) are introduced and applied to a single-DOF linear pendulum. Optimality of a form of the speed gradient algorithm for energy excitation in a linear pendulum has been shown in Andrievsky (2005). The problem statement of natural oscillation entrainment in linear mechanical multi-DOF systems is presented in Section 3: the relation between the entrainment of a single-DOF system and a multi-DOF one is shown. Two control algorithms based on the results from Efimov and Fradkov (2007b, 2012) are developed. The efficiency of the obtained control algorithms is demonstrated by examples of numerical simulation in Section 4. By computer simulation, it is shown that the proposed control strategy can also be applied to resonance control of nonlinear systems.

## 2. Preliminary results

In this section, we will develop basic control strategies by applying the results from Andrievsky (2005), Andrievsky and Fradkov (1999), Efimov and Fradkov (2007b, 2012), and Fradkov (1999) to the linear model of a pendulum:

$$\ddot{x} + \kappa \dot{x} + \omega^2 x = bu, \quad (1)$$

where  $x \in \mathbb{R}$  is the angle of pendulum rotation,  $\dot{x} \in \mathbb{R}$  is the angular velocity,  $u \in [-U, U]$  is the control,  $U \in \mathbb{R}_+$  is the maximal control amplitude,  $\kappa \in \mathbb{R}_+$  is the friction coefficient,  $\omega \in \mathbb{R}_+$  is the frequency of oscillations without friction, and  $b \in \mathbb{R}$ ,  $b \neq 0$ , is the control gain. It is assumed that  $\omega > 0.5\kappa$ ; in this case, the transfer function of system (1) has poles  $\lambda_{1,2} = -0.5\kappa \pm \omega_0 i$  with nonzero imaginary part  $\omega_0 = \sqrt{\omega^2 - \kappa^2/4}$ , where  $\omega_0$  is the natural frequency of oscillations for (1). Typically, oscillations with frequency  $\omega_0$  are called resonant for (1). It is worth noting that the Bode magnitude plot of system (1) reaches its maximum at the frequency  $\omega_1 = \sqrt{\omega^2 - \kappa^2/2}$  if  $\omega \geq \kappa/\sqrt{2}$ , and at zero frequency otherwise (for the variable  $\dot{x}$ , the resonant frequency is  $\omega$ ). The difference between the frequencies  $\omega$ ,  $\omega_0$ ,  $\omega_1$  is minor if the value of  $\kappa$  is sufficiently small, as it is usually in practice, and that is the case under consideration, i.e.,  $\omega \gg \kappa/\sqrt{2}$ .

Due to constraint on the control amplitude, the problem of excitation of the resonance oscillations becomes meaningful from a practical point of view. If it is required to ensure oscillations with the maximal amplitude for a given amplitude of the control, then the resonance mode is a solution. The question is how to design such a control  $u = u(x, \dot{x})$ ,  $|u| \leq U$  providing system oscillations with frequency  $\omega_0$  or  $\omega_1$ . An additional goal consists in the oscillation phase resetting.

### 2.1. Controls for stabilization/excitation of the linear pendulum

Let us first recall a simple control law that achieves convergence to the resonant oscillation in finite time. In the paper Andrievsky (2005), it is shown that the control

$$u = U \operatorname{sign}(\dot{x}) \quad (2)$$

is the optimal solution of the terminal energy maximization problem. The solutions of system (1) with control (2) are understood in the Filippov sense (Filippov, 1988) (see also Yakubovich, Leonov, & Gelig, 2004, for a more general definition); roughly speaking, it is an absolutely continuous function of time passed through the

given initial conditions and satisfying the differential inclusion (1), (2) with

$$\operatorname{sign}(s) = \begin{cases} s/|s| & \text{if } s \neq 0; \\ [-1, 1] & \text{if } s = 0; \end{cases}$$

the value  $\operatorname{sign}(0) = 0$  is used for simulations/experiments. Since control (2) is piecewise constant, solutions of system (1), (2) can be presented as follows:

$$x(t) = e^{-\kappa t/2} [\omega_0^{-1} (\dot{x}(0) + \kappa \{x(0) - s_0\}/2) \sin(\omega_0 t) + \{x(0) - s_0\} \cos(\omega_0 t)] + s_0,$$

$$\dot{x}(t) = e^{-\kappa t/2} [\dot{x}(0) \cos(\omega_0 t) - \omega_0^{-1} (\omega^2 \{x(0) - s_0\} + \kappa \dot{x}(0)/2) \sin(\omega_0 t)],$$

$$s_0 = b\omega^{-2} U \operatorname{sign}(\dot{x}(0))$$

for  $t \in [0, t_1)$ , where

$$t_1 = \frac{1}{\omega_0} \operatorname{atan} \left( \frac{\omega_0 \dot{x}(0)}{\omega^2 \{x(0) - s_0\} + \kappa \dot{x}(0)/2} \right)$$

is the instant of time when  $\dot{x}(t_1) = 0$  (for the first time). Introduce  $t_i$ ,  $i \geq 1$ , as the sequence of instants satisfying constraints  $\dot{x}(t_i) = 0$  and  $\dot{x}(t) \neq 0$  for  $t \neq t_i$ ,  $i \geq 1$ . For  $t \in [t_i, t_{i+1})$ ,  $i \geq 1$ , we have

$$x(t) = e^{-\kappa(t-t_i)/2} \{x(t_i) - s_i\} (\omega_0^{-1} \kappa/2 \sin[\omega_0(t-t_i)] + \cos[\omega_0(t-t_i)]) + s_i,$$

$$\dot{x}(t) = -e^{-\kappa(t-t_i)/2} \{x(t_i) - s_i\} \omega_0^{-1} \omega \sin[\omega_0(t-t_i)],$$

$$s_i = -s_{i-1}.$$

From the expression above we have that  $t_{i+1} - t_i = \pi/\omega_0$ ,  $i \geq 1$ . Therefore, control (2) ensures that the trajectory of system (1) converges to oscillations with frequency  $\omega_0$  (Efimov & Fradkov, 2007b) (the resonance oscillations) in finite time  $t_1$ . The analysis reviewed above previously led to the following result.

**Lemma 1** (Andrievsky, 2005; Efimov & Fradkov, 2007b). *For system (1) with control (2), the following statements are true:*

- (1) *there is convergence in finite time  $t_1$  to oscillations with frequency  $\omega_0$ ;*
- (2) *the asymptotic amplitude of oscillations for the variable  $x$  is  $\gamma_0 = -bU/\omega^2 (1 + e^{-\kappa\pi/\omega_0} + 2e^{-\kappa\pi/(2\omega_0)})/(1 - e^{-\kappa\pi/\omega_0})$ .*

Note that control (2) is independent of values  $\omega$  and  $\kappa$  of the system parameters.

In the paper (Efimov & Fradkov, 2007b), an adaptive algorithm for amplitude adjustment in control (2) is proposed ensuring a required amplitude of oscillations on the limit cycle  $\gamma_0$  for the case of unknown values of the parameters of pendulum (1). In the paper (Efimov & Fradkov, 2007a), the same problem (excitation to the resonance mode of pendulum (1) for unknown values of parameters) is solved using the adaptive observer technique. Control (2) is related to the mechanism of positive rate feedback with saturation discussed in Futakata and Iwasaki (2008).

Control law (2) does not regulate the phase of oscillations. In addition, the resulting oscillations in the resonance mode have non-constant energy  $E$ :

$$E(x, \dot{x}) = 0.5[\dot{x}^2 + \omega^2 x^2], \quad \dot{E} = -\kappa \dot{x}^2 + b \dot{x} u.$$

For  $\kappa = 0$ , each fixed value of the energy  $E$  corresponds to an oscillation with the angle amplitude  $\sqrt{2E}/\omega$ ; the presence of friction leads to energy dissipation. The control presented in Efimov and Fradkov (2012) makes the regulation of both the energy  $E$  and the phase  $\varphi \in [-\pi, \pi)$ , for 1D and 2D lattices of linear oscillators without friction. Below, we will generalize the previous control law to deal with damped systems. The idea is based on the “action–angle” transformation of coordinates  $(E, \varphi)$ :

$$E = 0.5[\dot{x}^2 + \omega^2 x^2], \quad \varphi = \operatorname{atan} \left( \frac{\dot{x}}{\omega x} \right),$$

which has a well-defined inverse:

$$x = \omega^{-1} \sqrt{2E} \cos(\varphi), \quad \dot{x} = \sqrt{2E} \sin(\varphi).$$

In the new coordinates, the pendulum equations (1) can be rewritten as follows:

$$\dot{E} = \sqrt{2E} \sin(\varphi) \left[ bu - \kappa \sqrt{2E} \sin(\varphi) \right],$$

$$\dot{\varphi} = \cos(\varphi) bu / \sqrt{2E} - \kappa \cos(\varphi) \sin(\varphi) - \omega,$$

which are well defined for any  $E > 0$  (if  $E = 0$ , then the system is at the origin). Let  $E_d > 0$  and  $\varphi_d(t) = \phi_d - \omega t$ ,  $\phi_d \in [-\pi, \pi)$ , be the desired energy and phase values, respectively; then the control  $u$  can be chosen in the form

$$u = \begin{cases} b^{-1} \kappa \sqrt{2E} \sin(\varphi) - \rho_1(\psi_1) & \text{if } V \leq 0.5E_d^2, \\ b^{-1} \kappa \sqrt{2E} \sin(\varphi) - \rho_2(\psi_2) & \text{if } (V, E) \in \Omega_{V,E}, \\ -\rho_2(\psi_2) & \text{otherwise,} \end{cases} \quad (3)$$

$$\Omega_{V,E} = \{(V, E) \in \mathbb{R}_+^2 : V > 0.5E_d^2, E \leq E_d\},$$

$$V = 0.5[(E - E_d)^2 + E_d^2 \pi^{-2} (\varphi - \varphi_d(t))^2],$$

$$\psi_1 = \psi_2 + bE_d^2 \pi^{-2} (\varphi - \varphi_d(t)) \cos(\varphi) / \sqrt{2E},$$

$$\psi_2 = b(E - E_d) \sqrt{2E} \sin(\varphi),$$

where  $\psi_i \rho_i(\psi_i) > 0$ ,  $i = 1, 2$  for any  $\psi_i \neq 0$  and  $|\rho_1(\psi_1)| \leq U - 2b^{-1} \kappa \sqrt{E_d}$ ,  $|\rho_2(\psi_2)| \leq U - b^{-1} \kappa \sqrt{2E_d}$  (it is assumed that  $U > 2b^{-1} \kappa \sqrt{E_d}$ ).

**Lemma 2.** For system (1), control (3) ensures the boundedness of solutions with the limit relations

$$\lim_{t \rightarrow +\infty} E(t) = E_d, \quad \lim_{t \rightarrow +\infty} [\varphi(t) - \varphi_d(t)] = 0 \quad (4)$$

for any  $E(0) \in \mathbb{R}_+$  and  $\varphi(0) \in [-\pi, \pi)$ ; in addition,  $|u| \leq U$ .

**Proof.** First, assume that  $V(t') \leq 0.5E_d^2$  for some  $t' \geq 0$ , and consider the Lyapunov function  $V$  behavior for system (1) and control (3) for  $t \geq t'$ ; we obtain

$$\dot{V} = -\psi_1 \rho_1(\psi_1) \leq 0.$$

Therefore,  $V(t) \leq V(t') \leq 0.5E_d^2$  for all  $t \geq t'$ . Since  $|\varphi - \varphi_d(t)| \leq \pi$  by the phase definition, the relation  $V(t) \leq 0.5E_d^2$  implies that  $E(t) \leq 2E_d$  for all  $t \geq t'$ . From the definition of control (3), for  $V \leq 0.5E_d^2$ , we get that  $|u| \leq U$  in this case. Next, the errors  $E - E_d$  and  $\varphi - \varphi_d(t)$  stay bounded for all  $t \geq 0$ . This fact, for any finite value  $E_d$ , implies boundedness of the state variables  $x(t)$  and  $\dot{x}(t)$ . According to the properties of function  $\rho_1$ ,  $\dot{V} = 0$  if and only if  $\psi_1 = 0$ . Since all trajectories of the system are bounded, asymptotically they converge to an invariant set where  $\psi_1 = 0$ . The desired asymptotic convergences  $E(t) \rightarrow E_d$  and  $\varphi(t) \rightarrow \varphi_d(t)$  follow the observability property:  $\psi_1(t) = 0$  for  $\forall t \geq 0$  implies that  $V(t) = 0$ . Indeed, assume that  $\psi_1(t) = 0$ ,  $V(t) \leq 0.5E_d^2$  for  $\forall t \geq 0$ ; then  $u = b^{-1} \kappa \sqrt{2E} \sin(\varphi)$  and system (1) with this control can be simplified as follows:

$$\dot{E} = 0, \quad \dot{\varphi} = -\omega.$$

Therefore,  $E(t) = E_0$  and  $\varphi(t) = \phi_0 - \omega t$  for all  $t \geq 0$  for some  $E_0 \geq 0$  and  $\phi_0 \in [-\pi, \pi)$ ; then

$$\psi_1(t) = b \left[ (E_0 - E_d) \sqrt{2E_0} \sin(\phi_0 - \omega t) + E_d^2 \pi^{-2} (\phi_0 - \phi_d) \cos(\phi_0 - \omega t) / \sqrt{2E_0} \right] = 0$$

for all  $t \geq 0$  by the assumption. However, the last quantity can be true only for  $E_0 = E_d$  and  $\phi_0 = \phi_d$ , that is necessary to prove (the relations (4) are satisfied).

Second, consider the general case when  $V(t) > 0.5E_d^2$  for  $t \geq 0$ . Consider an auxiliary Lyapunov function  $W = 0.5(E - E_d)^2$ , whose derivative has the form

$$\dot{W} = (E - E_d) \sqrt{2E} \sin(\varphi) \left\{ b[v - \rho_2(\psi_2)] - \kappa \sqrt{2E} \sin(\varphi) \right\},$$

where  $v = 0$  for  $E > E_d$  and  $v = b^{-1} \kappa \sqrt{2E} \sin(\varphi)$  for  $E \leq E_d$ . If  $E \leq E_d$ , then  $\dot{W} = -\psi_2 \rho_2(\psi_2)$ ; if  $E > E_d$ , then

$$\dot{W} = -\psi_2 \rho_2(\psi_2) - 2\kappa(E - E_d)E \sin^2(\varphi) \leq -\psi_2 \rho_2(\psi_2).$$

Therefore,  $\dot{W} = -\psi_2 \rho_2(\psi_2) \leq 0$  whenever  $V > 0.5E_d^2$ . Due to the form of  $W$  and  $\psi_2$ , the energy  $E(t)$  stays bounded, the error  $|E(t) - E_d|$  is not increasing, and  $|u| \leq U$  in this case also. Using partial detectability arguments (Shiriaev, 2000; Shiriaev & Fradkov, 2001), we can prove that in this case  $\lim_{t \rightarrow +\infty} E(t) = E_d$  (if  $V(t) > 0.5E_d^2$  for all  $t \geq 0$ ). However, if  $E(t') = E_d$ , then  $V(t') \leq 0.5E_d^2$ . Consequently, either  $\lim_{t \rightarrow +\infty} [\varphi(t) - \varphi_d(t)] = 0$  under the condition  $V(t) > 0.5E_d^2$ , which is satisfied for all  $t \geq 0$  (and the stabilization goals (4) are achieved simultaneously), or  $\varphi(t) \neq \varphi_d(t)$ , and there is a finite time instant  $t' \geq 0$  such that  $V(t') \leq 0.5E_d^2$ . Next, according to the consideration above,  $V(t) \leq 0.5E_d^2$  for all  $t \geq t'$ , and the required limit relations (4) hold. In both cases, the variables  $x$  and  $\dot{x}$  stay bounded under  $|u| \leq U$ .  $\square$

It is worth stressing that, if  $E(t') = 0$  for some  $t' \geq 0$ , then it becomes possible to have  $|\psi_1(t')| = \infty$ , but, due to the stated boundedness of  $\rho_1$ , the control input stays finite. In this case, control (3) generates an exciting pulse, which pushes the trajectory away from the origin ( $E(t') = 0$ ).

Later in this paper, controls (2) and (3) will be used to excite a resonance mode of oscillations in a finite time for multi-DOF systems. When achieving the resonance entrainment to a prescribed mode, it is important to attenuate the other oscillation modes at the same time. For this purpose, we are also interested in the finite-time stabilization of system (1). In applications, an asymptotic achievement of a control goal may not be considered satisfactory when one has to pass from one task to another within a finite time frame. During robotic locomotion for instance, it may be required that switching from walking to running be completed after a finite-time transient. At a conceptual level, such requirement may be formalized as a finite-time transition from one mode of resonance oscillation to another. To this end, we propose the following control to achieve finite-time convergence of every trajectory of (1) to the origin:

$$u = -\beta \operatorname{sign}(\dot{x}) - (\beta + \varepsilon) \operatorname{sign}(\kappa \dot{x} / \omega^2 + x), \quad (5)$$

where  $2\beta + \varepsilon = U$  and  $\beta > 0$ ,  $\varepsilon > 0$  are the design parameters.

**Lemma 3.** Control (5) ensures global stabilization of the origin for system (1) in a finite time  $t_0 \geq 0$ :

$$t_0 \leq T_0 = 2\mu^{-1} \sqrt{V[\kappa \dot{x}(0) / \omega^2 + x(0), \dot{x}(0)]},$$

where

$$V(\xi_1, \xi_2) = 0.5[0.5\omega^2 \xi_1^2 + \xi_2^2 + 0.5\omega^2 (\xi_1 - \kappa \xi_2 / \omega^2)^2] + b(\beta + \varepsilon) |\xi_1|$$

and

$$\mu = \min \left\{ b(\beta + \varepsilon) \kappa / \omega, b\beta \omega / \left( \sqrt{0.5} \omega + 0.5\kappa \right), 0.5\kappa \omega^2 / \sqrt{b(\beta + \varepsilon)}, 8\kappa b \sqrt{b(\beta + \varepsilon)} \varepsilon / \left[ 3\sqrt[3]{2\omega^2} \right] \right\}.$$

**Proof.** Consider the following coordinate transformation for system (1) and control (5):

$$\begin{aligned}\xi_1 &= \kappa \dot{x} / \omega^2 + x, & \xi_2 &= \dot{x}; \\ \dot{\xi}_1 &= \kappa [-\omega^2 \xi_1 + bu] / \omega^2 + \xi_2, \\ \dot{\xi}_2 &= -\omega^2 \xi_1 + bu; \\ u &= -\beta \operatorname{sign}(\xi_2) - (\beta + \varepsilon) \operatorname{sign}(\xi_1).\end{aligned}$$

Consider the Lyapunov function  $V(\xi_1, \xi_2)$  for this system; then

$$\begin{aligned}\dot{V} &= 0.5\omega^2 \xi_1 (\kappa [-\omega^2 \xi_1 + bu] / \omega^2 + \xi_2) + \xi_2 (-\omega^2 \xi_1 + bu) \\ &\quad + 0.5\omega^2 (\xi_1 - \kappa \xi_2 / \omega^2) (\kappa [-\omega^2 \xi_1 + bu] / \omega^2 + \xi_2) \\ &\quad + b(\beta + \varepsilon) \operatorname{sign}(\xi_1) (\kappa [-\omega^2 \xi_1 + bu] / \omega^2 + \xi_2) \\ &= -0.5\kappa \omega^2 \xi_1^2 - 0.5\kappa \xi_2^2 - b(\beta + \varepsilon) \kappa |\xi_1| + 0.5\kappa b \xi_1 u + b \xi_2 u \\ &\quad + \kappa b^2 (\beta + \varepsilon) \operatorname{sign}(\xi_1) u / \omega^2 + b(\beta + \varepsilon) \operatorname{sign}(\xi_1) \xi_2 \\ &= -0.5\kappa \omega^2 \xi_1^2 - 0.5\kappa \xi_2^2 - 1.5b(\beta + \varepsilon) \kappa |\xi_1| \\ &\quad - 0.5\beta \kappa b \xi_1 \operatorname{sign}(\xi_2) - b\beta |\xi_2| \\ &\quad + \kappa b^2 (\beta + \varepsilon) [-\beta \operatorname{sign}(\xi_1) \operatorname{sign}(\xi_2) \\ &\quad - (\beta + \varepsilon) \operatorname{sign}(\xi_1)^2] / \omega^2 \\ &\leq -0.5\kappa \omega^2 \xi_1^2 - 0.5\kappa \xi_2^2 - 1.5b(\beta + \varepsilon) \kappa |\xi_1| \\ &\quad + 0.5\beta \kappa b |\xi_1| - b\beta |\xi_2| + \kappa b^2 (\beta + \varepsilon) [\beta \operatorname{sign}(\xi_1)^2 \\ &\quad - (\beta + \varepsilon) \operatorname{sign}(\xi_1)^2] / \omega^2 \\ &\leq -0.5\kappa \omega^2 \xi_1^2 - 0.5\kappa \xi_2^2 - b(\beta + \varepsilon) \kappa |\xi_1| \\ &\quad - b\beta |\xi_2| - \kappa b^2 (\beta + \varepsilon) \varepsilon \operatorname{sign}(\xi_1)^2 / \omega^2.\end{aligned}$$

This inequality implies global asymptotic stability of the system. To prove the finite-time stability note that  $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$  for any  $a, b \in \mathbb{R}_+$ ; then

$$\begin{aligned}\sqrt{V} &\leq 0.5\omega |\xi_1| + \sqrt{0.5} |\xi_2| \\ &\quad + 0.5\omega |\xi_1 - \kappa \xi_2 / \omega^2| + \sqrt{b(\beta + \varepsilon)} \sqrt{|\xi_1|} \\ &\leq \omega |\xi_1| + (\sqrt{0.5} + 0.5\kappa / \omega) |\xi_2| + \sqrt{b(\beta + \varepsilon)} \sqrt{|\xi_1|}.\end{aligned}$$

Using the relation  $\sqrt{|a|} \leq 3\sqrt[3]{2}/8 \operatorname{sign}(a)^2 + a^2$ , which is valid for any  $a \in \mathbb{R}$ , we finally obtain

$$\begin{aligned}\sqrt{V} &\leq \omega |\xi_1| + (\sqrt{0.5} + 0.5\kappa / \omega) |\xi_2| \\ &\quad + \sqrt{b(\beta + \varepsilon)} \left[ 3\sqrt[3]{2}/8 \operatorname{sign}(\xi_1)^2 + \xi_1^2 \right].\end{aligned}$$

Then, for the defined  $\mu$ , we have

$$\dot{V} \leq -\mu \sqrt{V}.$$

The corresponding solution in the time domain satisfies the following inequality for all  $t \geq 0$ :

$$V(t) \leq (\sqrt{V(0)} - 0.5\mu t)^2.$$

Since  $V(t) \geq 0$ , the function  $V(t)$  has to reach zero in a time  $t_0$  smaller than time  $T_0$ .  $\square$

## 2.2. Differentiation

Frequently, in applications, only the angle position  $x$  is available for measurements, while controls (2), (3) and (5) are based on the velocity  $\dot{x}$ . We will use the following nonlinear filter, which has been proposed in Efimov and Fridman (2011) to evaluate  $\dot{x}$  in a finite time.

**Lemma 4** (Efimov & Fridman, 2011). Let  $|\dot{x}(t)| \leq L_1 \in \mathbb{R}_+$ ,  $|\ddot{x}(t)| \leq L_2 \in \mathbb{R}_+$ , for all  $t \geq 0$ . Then the filter

$$\begin{aligned}\dot{\zeta}_1 &= -\alpha \sqrt{|\zeta_1 - x(t)|} \operatorname{sign}[\zeta_1 - x(t)] + \zeta_2, \\ \dot{\zeta}_2 &= -\varpi \operatorname{sign}[\zeta_1 - x(t)] - \chi \operatorname{sign}(\zeta_2) - \zeta_2, \\ \zeta_1(0) &= x(0), & \zeta_2(0) &= 0,\end{aligned}\quad (6)$$

where

$$\begin{aligned}\chi &= 0.25\sqrt[4]{2}L_1 + \varrho, & \varpi &> L_1 + L_2 + 3\chi, \\ \alpha &= 4 \left[ \sqrt{2(\varpi + L_1 + L_2 + 2\chi)} \chi \right. \\ &\quad \left. + \sqrt{\varpi + L_1 + L_2 + 3\chi} (L_1 + L_2 + 2\chi) \right] / \\ &\quad [2\varpi - L_1 - L_2 - 2\chi]\end{aligned}\quad (7)$$

for some  $\varrho \in \mathbb{R}_+$ , has bounded solutions, and  $\zeta_2(t) = \dot{x}(t)$  for all  $t \geq T_d$  with  $T_d \leq L_1 / (0.25\sqrt[4]{2}L_1 + \varrho)$ .

## 3. Main results

This section extends the feedback resonance mechanism of controls (2) and (3) for the single-DOF system to the multi-DOF system:

$$J\ddot{x} + D\dot{x} + Sx = Bu, \quad y = Cx, \quad (8)$$

where  $x \in \mathbb{R}^n$  is the vector of generalized coordinates, and  $u \in \mathbb{R}^m$  and  $y \in \mathbb{R}^m$  are the input and the output, respectively.

**Assumption 5.** The inertia, damping, and stiffness matrices satisfy

$$\begin{aligned}J &= J^T > 0, & S &= S^T \geq 0, & D &= \varrho S, & \varrho &> 0, \\ \operatorname{rank}(B) &= \operatorname{rank}(C) = m, & m &= n.\end{aligned}$$

Eq. (8) with Assumption 5 captures a fairly general class of standard mechanical systems with mass, stiffness, and damping elements, linearized around an equilibrium point. Two restrictions are imposed by Assumption 5. One is that  $B$  and  $C$  are square invertible, implying that all the degrees of freedom are fully actuated and measured. The other is the Rayleigh damping  $D = \varrho S$ , which is commonly used in structural dynamics to capture (uncertain) dissipation effects. These assumptions make the analysis simple enough to provide insights into the natural entrainment mechanism.

In addition, the conditions in Assumption 5 are frequently met in biological systems (Futakata & Iwasaki, 2008, 2011). A musculo-skeletal body can be viewed as a set of rigid links connected by rotational joints. At each joint, the muscle and tendon serve as an origin of visco-elastic property, being also a torque actuator. The muscle length is fed back to the motor control system through stretch receptors, and collocation arises naturally in biomechanics. In this case, the stiffness  $S$  and the damping  $D$  matrices share the same structure.

A natural mode of (8) is defined by a pair of the natural frequency  $\omega_i \in \mathbb{R}_+$  and the mode shape  $\xi_i \in \mathbb{R}^n$  satisfying  $(S - \omega_i^2 J)\xi_i = 0$ ,  $i = \overline{1, n}$  (where the symbol  $\overline{1, n}$  is used to denote the sequence of integers  $1, 2, \dots, n$ ). For brevity of presentation we will assume that all  $\omega_i$ ,  $i = \overline{1, n}$  are different (the approach can be easily extended to the multiple case). We will also assume that the controller is allowed to have centralized information processing.



### 3.1. Natural oscillation entrainment problem

The problem of natural oscillation entrainment can be formulated now as follows. Design a control  $u$  ensuring that the system (8) oscillates at a selected natural frequency  $\omega_k$ ,  $k \in \{1, \dots, n\}$ .

Our approach is based on the transformation  $x = Pz$  of system (8) to the normal mode canonical form:

$$\ddot{z} + K\dot{z} + \Omega z = Ru, \quad (9)$$

where

$$\Omega = P^{-1}J^{-1}SP = \text{diag}[\omega_1^2, \dots, \omega_n^2],$$

$$K = P^{-1}J^{-1}DP = \text{diag}[\kappa_1, \dots, \kappa_n],$$

$$R = P^{-1}J^{-1}B, \quad \kappa_i = \varrho\omega_i^2, \quad i = \overline{1, n},$$

and the columns of the matrix  $P$  are composed of the vectors  $\xi_i$ . Under Assumption 5, we may assume that the vector  $x = C^{-1}y$  is available for measurements (matrix  $C$  has rank  $n$ ). By the same arguments, since  $\text{rank}(B) = n$ , matrix  $R$  is non-singular, and, by introducing an auxiliary control  $v = Ru$ , we may rewrite system (9) as a sequence of independent systems (1):

$$\ddot{z}_i + \kappa_i \dot{z}_i + \omega_i^2 z_i = v_i, \quad i = \overline{1, n}. \quad (10)$$

Thus, to choose  $v_i$ ,  $i = \overline{1, n}$ , controls (2), (3) and (5) can be used. The  $i$ th system in (10) is responsible for the oscillation of system (8) on the frequency  $\omega_i$  or  $\omega_{i,0} = \sqrt{\omega_i^2 - \kappa_i^2/4}$ . Stabilization of the  $i$ th normal mode at the origin (zero energy level) implies elimination of the frequency  $\omega_i$  from the frequency spectrum of the oscillation of system (8). Thus to solve the problem of natural oscillation entrainment to frequency  $\omega_k$ , it is necessary to stabilize at the origin all normal modes with  $\omega_i \neq \omega_k$  providing a sufficient level of excitation for the  $k$ th mode. For future reference, define the set of canceling normal modes  $\mathcal{J} = \{1, \dots, n\} \setminus \{k\}$ .

### 3.2. Finite-time control

We choose control (2) to excite the  $k$ th mode, and control (5) to attenuate the other modes. Differentiator (6) is used to calculate velocities.

**Theorem 6.** *Let Assumption 5 be satisfied and a constant  $U > 0$  be given. Then the control*

$$u = R^{-1}v, \quad v = [v_1, \dots, v_n]^T, \\ v_k = U \text{sign}(\hat{z}_k), \quad (11)$$

$$v_j = -\beta \text{sign}(\hat{z}_j) - (\beta + \varepsilon) \text{sign}(\kappa_j \hat{z}_j / \omega_j^2 + z_j), \quad j \in \mathcal{J},$$

where  $2\beta + \varepsilon = U$  and

$$\hat{z}_i = \zeta_{2,i}; \\ \dot{\zeta}_{1,i} = -\alpha_i \sqrt{|\zeta_{1,i} - z_i(t)|} \text{sign}[\zeta_{1,i} - z_i(t)] + \zeta_{2,i}, \quad (12)$$

$$\dot{\zeta}_{2,i} = -\varpi_i \text{sign}[\zeta_{1,i} - z_i(t)] - \chi_i \text{sign}(\zeta_{2,i}) - \zeta_{2,i},$$

$$\zeta_{1,i}(0) = z_i(0), \quad \zeta_{2,i}(0) = 0, \quad i = \overline{1, n},$$

with the parameters  $\alpha_i$ ,  $\varpi_i$ , and  $\chi_i$  calculated in accordance with (7) for

$$L_{1,i} = \sqrt{2\varepsilon_i / (\eta_i v_i)} U, \quad L_{2,i} = (\kappa_i + \omega_i^2) L_{1,i} + U, \quad (13)$$

$$\eta_i = \min\{0.5, 0.25\omega_i^2\},$$

$$v_i = \kappa_i \min \left\{ \frac{1}{3}, \frac{0.5}{1 + \omega_i^{-2} \kappa_i^2} \right\}, \quad \varepsilon_i = 0.25\kappa_i \omega_i^{-2} + \kappa_i^{-1},$$

ensures for system (8) boundedness of solutions and natural oscillation entrainment for all initial conditions in a finite time.

**Proof.** The control  $u$  and the auxiliary inputs  $v_i$ ,  $i = \overline{1, n}$  are always bounded by construction. Then the solutions of the systems (10) are bounded for all  $i = \overline{1, n}$  (since  $\kappa_i > 0$  and  $\omega_i^2 > 0$ ). Moreover, for any initial conditions, the trajectories of (10) have to converge in finite time to the set where  $|\dot{z}_i| \leq L_{1,i}$  and  $|z_i| \leq L_{2,i}$ ,  $i = \overline{1, n}$ . Indeed, for the Lyapunov function  $S(z_i, \dot{z}_i) = 0.5[0.5\omega_i^2(z_i + \kappa_i \dot{z}_i / \omega_i^2)^2 + \dot{z}_i^2 + 0.5\omega_i^2 z_i^2]$ , we have

$$\dot{S} \leq -0.25\omega_i^2 \kappa_i (z_i + \kappa_i \dot{z}_i / \omega_i^2)^2 - 0.25\kappa_i \dot{z}_i^2 + \varepsilon_i v_i^2 \\ \leq -v_i S + \varepsilon_i v_i^2,$$

which for  $\mathbf{z}_i = [z_i \ \dot{z}_i]^T$  implies that  $\eta_i \|\mathbf{z}_i(t)\|^2 \leq 0.5\dot{z}_i^2(t) + 0.25\omega_i^2 z_i^2(t) \leq S(t) \leq 2v_i^{-1} \varepsilon_i v_i^2$  for all  $t \geq t_c$ ,  $t_c \in \mathbb{R}_+$ . Therefore,  $\|\mathbf{z}_i(t)\| \leq \sqrt{2\varepsilon_i / (\eta_i v_i)} U$  and

$$|\dot{z}_i(t)| \leq \sqrt{2\varepsilon_i / (\eta_i v_i)} U = L_{1,i},$$

$$|z_i(t)| \leq (\kappa_i + \omega_i^2) L_{1,i} + U = L_{2,i}$$

for all  $t \geq t_c$ . In this case, according to Lemma 4, there is a finite time  $T_d \geq t_c$  such that  $\hat{z}_i = \zeta_{2,i}$ ,  $i = \overline{1, n}$ , the control  $v_k = U \text{sign}(\hat{z}_k)$  is identical to (1), and the controls  $v_j = -\beta \text{sign}(\hat{z}_j) - (\beta + \varepsilon) \text{sign}(\kappa_j \hat{z}_j / \omega_j^2 + z_j)$  for all  $j \in \mathcal{J}$  coincide with (5). Thus from Lemmas 1 and 3 there are time instants  $t_{1,k} \geq T_d$  and  $t_{0,j} \geq T_d$  such that the  $k$ th mode oscillates on the frequency  $\omega_{k,0}$  for  $t \geq t_{1,k}$ , while all other modes are canceled for  $t \geq t_{0,j}$ .  $\square$

In this theorem, the upper limit of the control  $U$  is introduced for  $v$  for brevity of formulation (it can be recalculated from an upper bound of the control  $u$ ). Unlike the algorithms of Futakata and Iwasaki (2008, 2011), the control methods proposed in this paper achieve resonance entrainment exactly and in finite time with an optimal performance. These advantages are based on a special structure of controls (2), (5) and a nonlinear differentiation algorithm used for estimation of the derivatives  $\hat{z}_j$ .

This control has a neural network interpretation given in Fig. 1. As in Futakata and Iwasaki (2011), the network has linear input and output layers and a nonlinear CPG hidden layer.

### 3.3. Energy/phase regulation

Let  $E_d > 0$  be the desired value of energy  $E_k = 0.5[\dot{z}_k^2 + \omega_k^2 z_k^2]$  for the  $k$ th mode excitation and  $\varphi_d(t) = \phi_d - \omega t$ ,  $\phi_d \in [-\pi, \pi)$  be the desired trajectory of the phase for this mode. In this case, control (3) can be applied to the  $k$ th mode for its excitation and phase resetting. For the  $j$ th modes with  $j \in \mathcal{J}$ , to attenuate the oscillations, control (5) can be used as before. Differentiator (6) calculates the estimates of velocities.

**Theorem 7.** *Let Assumption 5 be satisfied and a constant  $U > 2\kappa_k \sqrt{E_d}$  be given. Consider the control*

$$u = R^{-1}v, \quad v = [v_1, \dots, v_n]^T, \\ v_k = \begin{cases} \kappa_k \hat{z}_k - \rho_1(\hat{\psi}_1) & \text{if } \hat{V} \leq 0.5E_d^2, \\ \hat{z}_k - \rho_2(\hat{\psi}_2) & \text{if } \hat{V} > 0.5E_d^2 \text{ and } \hat{E}_k \leq E_d, \\ -\rho_2(\hat{\psi}_2) & \text{if } \hat{V} > 0.5E_d^2 \text{ and } \hat{E}_k > E_d, \end{cases} \\ \hat{V} = 0.5[(\hat{E}_k - E_d)^2 + E_d^2 \pi^{-2}(\hat{\varphi}_k - \varphi_d(t))^2], \\ \hat{\psi}_1 = \hat{\psi}_2 + 0.5\omega_k E_d^2 \pi^{-2}(\hat{\varphi}_k - \varphi_d(t)) z_k / \hat{E}_k, \\ \hat{\psi}_2 = (\hat{E}_k - E_d) \hat{z}_k, \\ \hat{E}_k = 0.5[\hat{z}_k^2 + \omega_k^2 z_k^2], \quad \hat{\varphi}_k = \text{atan} \left( \frac{\hat{z}_k}{\omega_k z_k} \right), \\ v_j = -\beta \text{sign}(\hat{z}_j) - (\beta + \varepsilon) \text{sign}(\kappa_j \hat{z}_j / \omega_j^2 + z_j), \quad j \in \mathcal{J}, \quad (14)$$

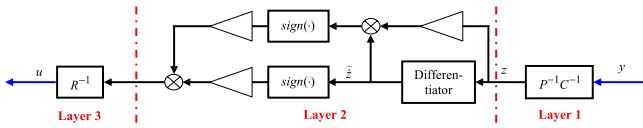


Fig. 1. The finite time control scheme.

where the functions  $\rho_i, i = 1, 2$  are chosen to satisfy  $\hat{\psi}_i \rho_i(\hat{\psi}_i) > 0$  for any  $\hat{\psi}_i \neq 0, |\rho_1(\hat{\psi}_1)| \leq U - 2\kappa_k \sqrt{E_d}, |\rho_2(\hat{\psi}_2)| \leq U - \kappa_k \sqrt{2E_d}$ , and the estimates  $\hat{z}_i$  are generated by differentiators (12) with parameters  $\alpha_i, \varpi_i$  and  $\chi_i$  calculated in accordance with (7), (13). Control (14) ensures boundedness of solutions of system (8) and natural oscillation entrainment with phase regulation for all initial conditions.

**Proof.** Again, the control  $u$  and the auxiliary inputs  $v_i, i = \overline{1, n}$  are always bounded by construction ( $|v_i| \leq U$ ). Then the solutions of systems (10) are bounded for all  $i = \overline{1, n}$  and for any initial conditions the trajectories of (10) have to converge in finite time to the set where  $|\dot{z}_i| \leq L_{1,i}$  and  $|\ddot{z}_i| \leq L_{2,i}, i = \overline{1, n}$  (the constants  $L_{1,i}, L_{2,i}$  are specified in (13); this can be proven using the Lyapunov function  $S$  as before). In this case, according to Lemma 4, there is a finite time  $T_d \geq t_c$  such that  $\hat{z}_i = \zeta_{2,i}, i = \overline{1, n}$ , the control  $v_k$  is identical to (3), and the controls  $v_j = -\beta \text{sign}(\hat{z}_j) - (\beta + \varepsilon) \text{sign}(\kappa_j \hat{z}_j / \omega_j^2 + z_j)$  for all  $j \in \mathcal{J}$  coincide with (5). Thus from Lemmas 2 and 3 there are time instants  $t_{0,j} \geq T_d$  such that the modes in  $\mathcal{J}$  are canceled for all  $t \geq t_{0,j}$ ; in addition,

$$\lim_{t \rightarrow +\infty} E_k(t) = E_d, \quad \lim_{t \rightarrow +\infty} [\varphi_k(t) - \varphi_d(t)] = 0. \quad \square$$

Control (14) also admits a neural network interpretation skipped for brevity.

#### 4. Application

In this section, we consider a mechanical arm on a horizontal plane from Futakata and Iwasaki (2011), which is formed as a chain of three rigid links connected to each other by two rotational joints. The  $i$ th link has mass  $m_i$  and length  $2\ell_i$  ( $i = 1, 2, 3$ ), and the first link is connected to the inertial frame through a rotational joint. A spring of stiffness  $k_i$  is mounted at the  $i$ th joint, a dashpot of damping coefficient is  $\rho k_i$ , and each joint is connected to an actuator that generates the controlling torque  $u_i$ . As in Futakata and Iwasaki (2011), we selected

$$m_i = 1, \quad \ell_i = 0.5, \quad k_i = 1, \quad \rho = 0.1.$$

First, let us take the linear part of the example from Futakata and Iwasaki (2011):

$$J = \frac{1}{12} I_3 + Q, \\ B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = B^T, \quad S = BC, \\ Q = L^T L, \quad L = \begin{bmatrix} 0.5 & 0 & 0 \\ 1 & 0.5 & 0 \\ 1 & 1 & 0.5 \end{bmatrix}, \quad \varrho = 0.1,$$

where  $I_3$  is the identity matrix with dimension  $3 \times 3$ . This system has natural frequencies

$$\omega = [5.146 \ 1.915 \ 0.292]^T.$$

In Fig. 2, the results of application of control (11) are shown, where

$$\mathcal{J} = \begin{cases} \{1, 3\} & \text{if } t \leq 25; \\ \{2, 3\} & \text{if } t > 25, \end{cases} \quad U = \begin{cases} 1 & \text{if } t \leq 25; \\ 5 & \text{if } t > 25, \end{cases}$$

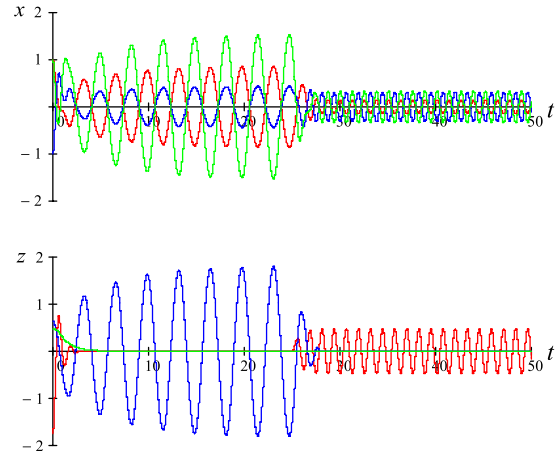


Fig. 2. Finite-time control.

$\varepsilon = U/3$  and  $\eta = 10$ . Thus, for  $t \leq 25$ , the system oscillates with frequency  $\omega_2 = 1.915$ ; next, for  $t > 25$ , control (11) activates natural oscillations with  $\omega_1 = 5.146$  in finite time. Such a change of oscillation frequency (or a normal mode) corresponds to a transition from the “walking” mode to the “running” one in animals. As we can see from this figure, the canonical form variables  $z$  are canceled or excited in a finite time corresponding to the chosen mode (one component of  $z$  is oscillating; the rest equal zero). For the state variable  $x$  this results in a common frequency of oscillation change. Since the modes in  $\mathcal{J}$  are canceled sufficiently quickly (comparing with the system time scales), the frequencies in  $\mathcal{J}$  almost do not influence the  $x$  behavior.

The results of the application of control (14) for the same time-varying set  $\mathcal{J}$  and  $U = 1, E_d = 1, \phi_d = \frac{2\pi}{3}, \rho_1(\psi_1) = (U - 2\kappa_k \sqrt{E_d}) \tanh(5\psi), \rho_2(\psi_2) = (U - \kappa_k \sqrt{2E_d}) \tanh(5\psi)$  are shown in Fig. 3 (the trajectories  $x(t)$  and  $z(t)$  are given in Fig. 3(a) and (b), respectively, and the energy variable  $E$  is plotted in Fig. 3(c) in a logarithmic scale; the phase variable  $\varphi$  is presented in Fig. 3(d)). As we can conclude, this control demonstrates a similar entrainment performance with additional phase resetting.

The results of the application of controls (11) and (14) confirm their entrainment abilities to the resonance oscillations at natural frequencies of (8). Switching between different frequencies (mode of oscillations) can be carried out by the incoming parameter setup ( $U, \mathcal{J}, E_d$  and  $\phi_d$ ).

Now let us verify applicability of the proposed approach on the nonlinear version of the example from Futakata and Iwasaki (2011):

$$J_x(x)\ddot{x} + G_x(x)\dot{x}^2 + D\dot{x} + Sx = Bu, \quad (15)$$

where  $J_x(x) = \frac{1}{12} I_3 + S_x Q S_x + C_x Q C_x$  and  $G_x(x) = S_x Q C_x - C_x Q S_x$  for  $S_x = \text{diag}\{\sin(x_i)\}, C_x = \text{diag}\{\cos(x_i)\}$ ; the remaining matrices are the same as before. The results of the application of control (11) with exactly the same parameters as in linear case are shown in Fig. 4. Comparing the curves presented in Figs. 2 and 4, it is possible to see that the canonical form variables  $z$  have the same qualitative behavior (one component is oscillating; the others are canceled by the control). The cancellation is not exact for system (15) due to appearance of nonlinear terms not covered by the control algorithm developed for linear system (8). The oscillations are also less frequent due to additional dependence of frequencies of a nonlinear system on amplitude of oscillations. Computing the average frequency of oscillations during the first 25 s numerically, we obtain 1.271 (1.915 in the linear case), and for  $t \geq 25$  we

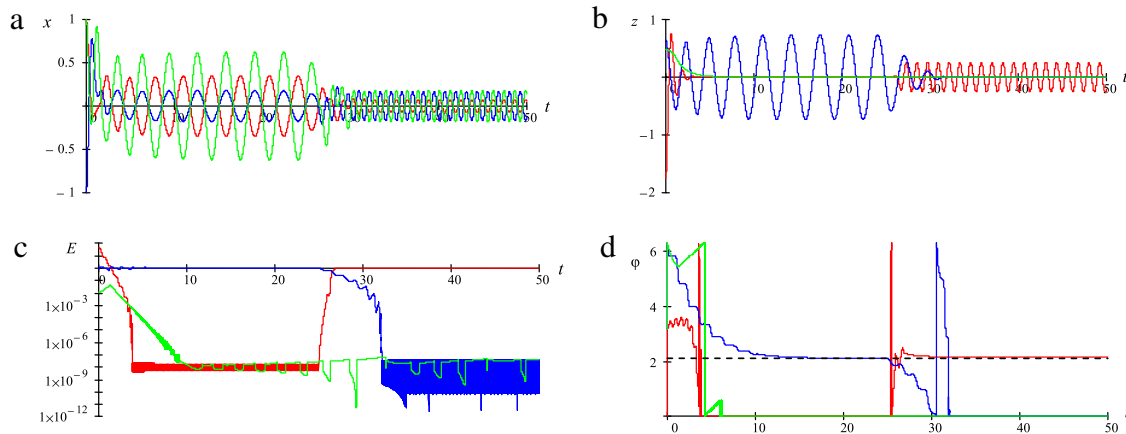


Fig. 3. Energy/phase regulation.

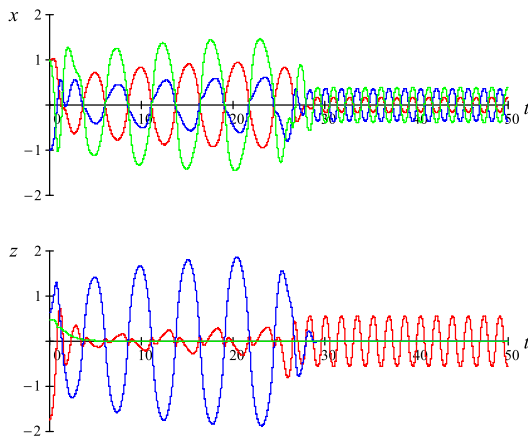


Fig. 4. Finite-time control for the nonlinear system.

get 3.607 (5.146 in the linear case). Nevertheless, control (11) ensures resonance mode activation and reliable switching among different frequencies even for sufficiently large deviations from the equilibrium point. However, the results of these simulations show that in a nonlinear case some quantitative deflections from the entrainment behavior guaranteed in the linear case can be expected.

## 5. Conclusions

The entrainment mechanism to a resonance mode based on CPG from Futakata and Iwasaki (2008, 2011) is extended. Two control algorithms for feedback resonance, finite-time control and energy/phase regulation, are used to design entrainment neural networks. The obtained conditions of oscillation at a natural frequency are based on Lyapunov analysis and optimal control theory (the harmonic balance method is avoided). One of the proposed controls provides the resonance oscillations in finite time. The proposed approach can be applied to multi-DOF mechanical systems representable in the normal mode canonical form. The results of computer simulation demonstrate good entrainment ability of the proposed approach, even for nonlinear systems.

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