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Approximate Consensus in Multi-agent Stochastic Systems with Switched Topology and Noise

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Abstract— **In this paper the approximate consensus problem in multi-agent stochastic systems with noisy information about the current state of the nodes and randomly switched topology for agents with nonlinear dynamics is considered. The control is formed by the local voting protocol with step size not tending to zero. To analyze closed loop system we propose to use method of continuous models (ODE approach or Derevitskii-Fradkov-Ljung (DFL)-scheme). The usage of this method allows one to reduce the computation load. The bounds of the mean proximity of trajectories of the discrete stochastic system to its continuous deterministic model are obtained. Based on those bounds the conditions for achieving mean square ε-consensus are established.**

The method is applied to the load balancing problem in decentralized stochastic dynamic network with incomplete information about the current state of nodes and changing set of communication links is considered. The load balancing problem is reformulated as consensus problem in noisy model with switched topology. The conditions to achieve the optimal level of nodes load are obtained.

The performance of the system is evaluated both analytically and by simulation.

Obtained results are important for control of production networks, multiprocessor or multicomputer networks, etc.

I. INTRODUCTION

Distributed coordination in networks of dynamic agents has attracted an interest numerous researchers in recent years. It is mostly due to broad applications of multi-agent systems in many areas including formation control [1], flocking, distributed sensor networks, congestion control in communication networks, cooperative control of unmanned air vehicles (UAVs)[2], attitude alignment of clusters of satellites, and others [3]. Many of these problems can be reformulated in terms of achieving consensus in multi-agent systems [4], [5], [6].

In [7] the stochastic approximation algorithm for solving consensus problem was proposed and justified for the group of cooperating agents that communicate with imperfect information in discrete time, under condition of switching topology and delay. Stochastic gradient algorithms were used for such problems before [8], [9], [10]. Stochastic approximation with decreasing step-sizes allows each agent both to extract state information from its neighbors and to reduce the noise influence.

Under dynamic changes of the external conditions (getting new task, etc.), stochastic approximation algorithms with decreasing step-size are not efficient. In [11], [12] the efficiency of stochastic approximation algorithms with constant step-size was studied. Their applicability to the problem of load balancing in centralized network system where noisy information about load and productivity of nodes was analyzed in [13], [14].

Analyzing of discrete stochastic systems may be complicated in practical applications. On the one hand, it is because of imperfect information exchange, which is, moreover, usually measured with noise. On the other hand, it is due to the effects of quantization effect common to all digital systems. Additional complication may be due to switching topology of networks.

To analyze the dynamics of the stochastic discrete systems the method of continuous models (ODE approach or Derevitskii-Fradkov-Ljung (DFL)-scheme) was developed and used in the control theory, dynamical systems theory and nonlinear mechanics. This method was described in [15], [16], [17], [18]. The employment of continuous-time models for analysis and synthesis of discrete-time stochastic systems has started in the 1970s. In [19], [20], [21], [22] the method of continuous models was used. In the present paper the results of [20] are expanded and improved, the new conditions for approximate mean square consensus are obtained.

These problems show the relevance of study in the properties of stochastic approximation type algorithms with small constant or not decreasing to zero step size in the nonlinear formulation of the problem with switched topology and noise.

In this paper we consider the approximate consensus problem in multi-agent stochastic system with nonlinear dynamics and measurement noise and uncertainties in topology and control protocol. As an example of such system we consider load balancing system in network with noisy information about the load and switched topology. Such problem is important for control of production networks, multiprocessor or multicomputer networks, etc.

The paper is organized as follows. Section II introduces the basic concepts and problem statement. In Section III we introduce the main assumptions. In Section IV the main results are given. In Section V the problem of load balancing in network is posed. The simulation results are presented in Section VI. In Section VII the analytical results are given.

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Conclusions are outlined in Section VIII.

II. Preliminaries. Consensus Problem in Networks

We explain some notation used in this article. The node index is used as a superscript but not an exponent in different variables. For column vectors $x_1,\dots,x_l$, $[x_1;\dots;x_l]$ denotes the column vector obtained by vertical concatenation of the l vectors.

Under the dynamic network we mean a set of dynamic systems (agents) which interact according to the graph of information connections. Consider a dynamic network as a set of agents $N=\{1,2,\dots,n\}$.

A *directed graph (digraph)* (N,E) consists of N and a set of directed edges E. Denote the *neighbor set* of node i as $N^i=\{j:(j,i)\in E\}$.

Associate with each edge $(j,i)\in E$ a weight $a^{i,j}>0$. Denote an *adjacency or connectivity matrix* $A=[a^{i,j}]$ of the graph, denoted hereinafter $\mathscr{G}_A$. Define the weighted in-degree of node i as the i-th row sum of A: $d^i(A)=\sum_{j=1}^n a^{i,j}$. Denote $\mathscr{L}(A)$ as *Laplacian* of graph $\mathscr{G}_A$.

A *directed tree* is a digraph where each node i, except the root, has exactly one parent node j so that $(j,i)\in E$. We call $\overline{\mathscr{G}_A}=(\overline{N},\overline{E})$ a *subgraph* of $\mathscr{G}_A$ if $\overline{N}\subset N$ and $\overline{E}\subset E\cap\overline{N}\times\overline{N}$. The digraph $\mathscr{G}_A$ is said to contain a *spanning tree* if there exists a directed tree $\mathscr{G}_{tr}=(N,E_{tr})$ as a subgraph of $\mathscr{G}_A$.

Endow each node $i\in N$ of a graph at time $t=0,1,2\dots,T$ with a time-varying state vector $x_t^i\in\mathbb{R}$ with dynamics

$$x_{t+1}^i=x_t^i+f^i(x_t^i,u_t^i), \tag{1}$$

with some functions $f^i(\cdot,\cdot):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ depending on the states in the previous time x_t^i and on control actions $u_t^i\in\mathbb{R}$.

We consider the network (multi-agent) system, consisting of dynamic agents with inputs u_i, outputs $y_t^{i,i}$ and states x_t^i.

Nodes i and j *agree* in a network at time t if and only if $x_t^i=x_t^j$.

The n nodes of a network have *reached a consensus* at time t if and only if $x_t^i=x_t^j \quad \forall i,j\in N, i\neq j$.

The n nodes to achieve *mean square ε-consensus* if $\mathrm{E}||x_t^i||^2<\infty$, $i\in N$, and there exists a random variable $x^\star$ such that $\mathrm{E}||x_t^i-x^\star||^2\leq\varepsilon$ for all $i\in N$.

The dynamic network topology is modeled by a sequence of digraphs $\{(N,E_t)\}_{t\geq 0}$, where $E_t\subset E$ changes with time. And denote A_t as a corresponding adjacency matrix. Denote the maximal set of communication links $E_{\max}=\{(j,i):\sup_{t\geq 0}a_t^{i,j}>0\}$.

If $(j,i)\in E_t$, node i receives information from node j for the purposes of feedback control.

To form its control strategy each node uses its own state

$$y_t^{i,i}=x_t^i+w_t^{i,i}, \tag{2}$$

(possibly noisy) and if $N_t^i\neq\emptyset$, noisy measurements of its neighbors states

$$y_t^{i,j}=x_t^j+w_t^{i,j},\ j\in N_t^i, \tag{3}$$

where $w_t^{i,i},w_t^{i,j}$ is the noise.

A *protocol* (*control algorithm*) with topology (N,E_t) is a feedback on observations

$$u_t^i=K_t^i(y_t^{i,j_1},\dots,y_t^{i,j_{m_i}}), \tag{4}$$

where $\{j_1,\dots,j_{m_i}\}\in\{i\}\bigcup\overline{N}_t^i$, $\overline{N}_t^i\subseteq N_t^i$.

Consider the protocol, called the *local voting protocol*

$$u_t^i=\alpha_t\sum_{j\in\bar{N}_t^i}b_t^{i,j}(y_t^{i,j}-y_t^{i,i}), \tag{5}$$

where $\alpha_t>0$ are step-sizes of control protocol, $b_t^{i,j}>0\ \forall j\in\bar{N}_t^i$, K_t^i is a smooth function of its arguments for each time $t=0,1,2,\dots$. We set $b_t^{i,j}=0$ for other pairs i,j and denote $B_t=[b_t^{i,j}]$ as the matrix of control protocol.

Matrix $A_{\max}$ of size $n\times n$ is denoted as $a_{\max}^{i,j}=p_a^{i,j}b^{i,j}$, $i\in N$, $j\in N$.

For the vector or matrix M denote the Frobenius norm: $||M||=[Tr(M^{\mathrm{T}}M)]^{1/2}$, where $Tr(\cdot)$ is a trace (sum of the diagonal elements) of matrix. In some cases for matrix A the vector norm will be used, i. e. square root of the sum of the squares of all its elements, which we denote $||A||_2$.

Under some general assumption in [7] author proved a necessary and sufficient condition for the asymptotic mean square consensus when step-size α_t tends to zero and simple second part in (1): $f^i(x_t^i,u_t^i)=u_t^i$. We will analyze more general case of $f^i(x_t^i,u_t^i)$ and α_t nondecreasing to zero.

III. Main assumptions

Let $(\Omega,\mathscr{F},P)$ be the underlying probability space. Let E be symbol of mathematical expectation and E_x be conditional expectation under the condition x.

In the formulation of further results we assume that the following conditions are satisfied.

A1. $\forall i\in N$ functions $f^i(x,u)$ are Lipschitz in x and u: $|f^i(x,u)-f^i(x',u')|\leq L_1(L_x|x-x'|+|u-u'|)$, and for any fixed x the function $f^i(x,\cdot)$ is such that $\mathrm{E}_xf^i(x,u)=f^i(x,\mathrm{E}_x u)$;

It follows from Lipschitz condition that the growth rate is bounded: $|f^i(x,u)|^2\leq L_2(L_c+L_x|x|^2+|u|^2)$;

A2. **a)** $\forall i\in N, j\in N_{\max}^i$ the noises $w_t^{i,j}$ are centered, independent and have bounded variance: $E(w_t^{i,j})^2\leq\sigma_w^2$.

b) $\forall i\in N, j\in N_{\max}^i$ the appearances of variable edges (j,i) in the graph $\mathscr{G}_{A_t}$ are independent random events with probability $p_a^{i,j}$ (i. e. matrices A_t are independent, identically distributed random matrices).

c) $\forall i\in N, j\in N_{\max}^i$ weights $b_t^{i,j}$ in the control protocol are bounded random variables: $\underline{b}\leq b_t^{i,j}\leq\bar{b}$ with probability 1, and there exist limits $b^{i,j}=\lim_{t\to\infty}\mathrm{E}b_t^{i,j}$.

Moreover, all of these random variables and matrices are independent of each other and their components have a limited variance.

A3. Graph $(N,E_{\max})$ has a spanning tree.

IV. Analysis of the Closed Loop System Dynamics

Denote $\bar{x}_t=[x_t^1;\dots;x_t^n]$. Rewrite the dynamics of the nodes in vector-matrix form:

$$\bar{x}_{t+1}=\bar{x}_t+F(\alpha_t,\bar{x}_t,\bar{w}_t), \tag{6}$$

where $F(\alpha_t,\bar{x}_t,\bar{w}_t)$ is the vector of dimension n:

$$F(\alpha_t,\bar{x}_t,\bar{w}_t)=\begin{pmatrix} \cdots \\ f^i(x_t^i,\alpha_t \sum_{j\in\bar{N}_t^i} b_t^{i,j}((x_t^j-x_t^i)+(w_t^{i,j}-w_t^{i,i}))) \\ \cdots \end{pmatrix}. \quad (7)$$

The *method of continuous models* [15], [18], (also called ODE approach [16], or Derevitskii-Fradkov-Ljung (DFL)-scheme [23]) consists on the approximate replacement of initial stochastic difference equation (6) by ordinary differential equation

$$\frac{d\bar{x}}{d\tau}=R(\alpha,\bar{x}), \quad (8)$$

where

$$R(\alpha,\bar{x})=R\begin{pmatrix} & x^1 \\ \alpha, & \vdots \\ & x^n \end{pmatrix}=\begin{pmatrix} \cdots \\ \frac{1}{\alpha}f^i(x^i,\alpha s^i(\bar{x})) \\ \cdots \end{pmatrix}, \quad (9)$$

$$s^i(\bar{x})=\sum_{j\in N_{\max}^i} a_{\max}^{i,j}(x^j-x^i)=-d^i(A_{\max})x^i+\sum_{j=1}^n a_{\max}^{i,j}x^j,\ i\in N.$$

Note, that if the last part of the condition A1 is not satisfied, then instead of (10) one can use the following definition

$$R(\alpha,\bar{x})=\frac{1}{\alpha}\mathrm{E}_x F(\alpha_t,\bar{x}_t,\bar{w}_t). \quad (10)$$

Conditions of closeness the trajectories $\{\bar{x}_t\}$ from (6)-(7) and $\{\bar{x}(\tau_t)\}$, $\tau_t=\alpha_0+\alpha_1+\ldots+\alpha_{t-1}$, (in particular, if $\forall t\ \alpha_t=\alpha=const$, then $\tau_t=T\alpha$), from (8)-(10) in a finite time interval follows from [18].

Theorem *1:* If conditions **A1**, **A2a–c** are satisfied, $\forall i\in N$ function $f^i(x,u)$ is smooth in u, $f^i(x,0)=0$ for any x and $0<\alpha_t\le\bar{\alpha}$, **then** there exists $\tilde{\alpha}$ such that for $\bar{\alpha}<\tilde{\alpha}$ the following inequality holds:

$$\mathrm{E}\max_{0\le\tau_t\le\tau_{\max}}||\bar{x}_t-\bar{x}(\tau_t)||^2\le C_1e^{C_2\tau_{\max}}\bar{\alpha}, \quad (11)$$

where $C_1>0$, $C_2>0$ are some constants.

Proof:

In further proofs the following facts will be useful.

Proposition *1:* For $\bar{z}\in\mathbb{R}^n$ and matrix $A_{\max}$ the following inequality holds

$$\sum_{i=1}^n(\sum_{j\in N_{\max}^i} a_{\max}^{i,j}z^j)^2\le||A_{\max}||_2^2||\bar{z}||^2.$$

Proof: Using the Cauchy-Schwarz inequality we obtain

$$\sum_{i=1}^n(\sum_{j\in N_{\max}^i} a_{\max}^{i,j}z^j)^2\le\sum_{i=1}^n(\sum_{j\in N_{\max}^i} {a_{\max}^{i,j}}^2)(\sum_{j\in N_{\max}^i} {z^j}^2)\le$$

$$\le(\sum_{i=1}^n\sum_{j=1}^n {a_{\max}^{i,j}}^2)(\sum_{j=1}^n {z^j}^2)\le||A_{\max}||_2^2||\bar{z}||^2.$$

■

Proposition *2:*

$$||\bar{s}(\bar{z})||^2\le2||\mathscr{L}(A_{\max})||_2^2||\bar{z}||^2.$$

Proof: Using the result of Proposition 1 yields

$$||\bar{s}(\bar{z})||^2=\sum_{i=1}^n(\sum_{j=1}^n a_{\max}^{i,j}(z^j-z^i))^2\le\sum_{i=1}^n(d^i(A_{\max})|z^i|+$$

$$+|\sum_{j\in N_{\max}^i} a_{\max}^{i,j}z^j|)^2\le2(\sum_{i=1}^n d^i(A_{\max})^2+||A_{\max}||_2^2)||\bar{z}||^2=$$

$$2||\mathscr{L}(A_{\max})||_2^2||\bar{z}||^2.$$

■

Proposition *3:* If **A2** is satisfied then $s^i(\bar{x})=\frac{1}{\alpha_t}\mathrm{E}_{\mathscr{F}_{t-1}}u_t^i$ and the following inequality holds

$$\frac{1}{\alpha_t^2}\mathrm{E}_{\mathscr{F}_{t-1}}{u_t^i}^2\le(n-1)\bar{b}^2||\bar{x}_t-x_t^i\underline{1}||^2+n\bar{b}^2\sigma_w^2,\ i\in N.$$

Proof: By the definition of the protocol (5)

$$\frac{1}{\alpha_t}u_t^i=\sum_{j\in\bar{N}_t^i} b_t^{i,j}((x^j-x^i)+(w_t^{i,j}-w_t^{i,i})).$$

It follows from conditions **A2** that $s^i(\bar{x})=\frac{1}{\alpha_t}\mathrm{E}_{\mathscr{F}_{t-1}}u_t^i$.

By the centrality of observation noise (on condition **A2a**) we consecutively derive

$$\frac{1}{\alpha_t^2}\mathrm{E}_{\mathscr{F}_{t-1}}{u_t^i}^2=\mathrm{E}_{\mathscr{F}_{t-1}}(\sum_{j\in\bar{N}_t^i} b_t^{i,j}((x_t^j-x_t^i)+(w_t^{i,j}-w_t^{i,i})))^2=$$

$$=\mathrm{E}_{\mathscr{F}_{t-1}}(\sum_{j\in\bar{N}_t^i} b_t^{i,j}(x_t^j-x_t^i))^2+\mathrm{E}_{\mathscr{F}_{t-1}}(\sum_{j\in\bar{N}_t^i} b_t^{i,j}(w_t^{i,j}-w_t^{i,i}))^2\le$$

$$\le||\bar{x}_t-x_t^i\underline{1}||^2\mathrm{E}_{\mathscr{F}_{t-1}}\sum_{j\in\bar{N}_t^i}(b_t^{i,j})^2+\mathrm{E}_{\mathscr{F}_{t-1}}\sum_{j\in\bar{N}_t^i}{b_t^{i,j}}^2({w_t^{i,j}}^2+$$

$$+{w_t^{i,i}}^2)\le(n-1)\bar{b}^2||\bar{x}_t-x_t^i\underline{1}||^2+n\bar{b}^2\sigma_w^2.$$

■

Proposition *4:* By assumptions **A2** the following inequality holds

$$\mathrm{E}_{\mathscr{F}_{t-1}}|\frac{1}{\alpha_t}u_t^i-s^i(\bar{x})|^2\le(n-1)\bar{b}^2||\bar{x}_t-x_t^i\underline{1}||^2+n\bar{b}^2\sigma_w^2,\ i\in N.$$

Proof: Using assumptions **A2** and taking the conditional expectation with respect to σ-algebra $\tilde{\mathscr{F}}_t$ yields that $s^i(\bar{x})=\mathrm{E}u_t^i$.

Denote

$$\tilde{s}_t^i=\sum_{j\in\bar{N}_t^i} b_t^{i,j}(x_t^j-x_t^i),\ \tilde{w}_t^i=\sum_{j\in\bar{N}_t^i} b_t^{i,j}(w_t^{i,j}-w_t^{i,i}).$$

By the centrality of observation noise (condition **A2a**) derive

$$\mathrm{E}_{\tilde{\mathscr{F}}_t}|\frac{1}{\alpha_t}u_t^i-s^i(\bar{x})|^2=\mathrm{E}_{\tilde{\mathscr{F}}_t}|\tilde{s}_t^i+\tilde{w}_t^i-s^i(\bar{x})|^2=\mathrm{E}_{\tilde{\mathscr{F}}_t}(\tilde{s}_t^i)^2+$$

$$\mathrm{E}_{\tilde{\mathscr{F}}_t}(\tilde{w}_t^i)^2+(s^i(\bar{x}))^2-2s^i(\bar{x})\mathrm{E}_{\tilde{\mathscr{F}}_t}\tilde{w}_t^i-2s^i(\bar{x})\mathrm{E}_{\tilde{\mathscr{F}}_t}\tilde{s}_t^i+$$

$$+2\mathrm{E}_{\tilde{\mathscr{F}}_t}\tilde{s}_t^i\tilde{w}_t^i=\mathrm{E}_{\tilde{\mathscr{F}}_t}(\tilde{s}_t^i)^2+\mathrm{E}_{\tilde{\mathscr{F}}_t}(\tilde{w}_t^i)^2-(s^i(\bar{x}))^2=$$

$$=\mathrm{E}_{\tilde{\mathscr{F}}_t}(\tilde{s}_t^i)^2-(s^i(\bar{x}))^2+n\bar{b}^2\sigma_w^2.$$

Taking the conditional expectation with respect to σ-algebra $\mathscr{F}_{t-1}$ we obtain

$$\mathrm{E}_{\mathscr{F}_{t-1}}(\tilde{s}_t^i)^2-(s^i(\bar{x}))^2+n\bar{b}^2\sigma_w^2\le(n-1)\bar{b}^2||\bar{x}_t-x_t^i\underline{1}||^2+n\bar{b}^2\sigma_w^2.$$

■

To proof the Theorem 1 we need to show that the Lipschitz and growth conditions from [24] hold. The first is a direct consequence of the Lipschitz continuous function $f^i(x,u)$ and the form of vector function $R(\alpha,\bar{z})$. Let $\bar{z},\bar{z}'\in\mathbb{R}^n$. By Proposition 1 we have

$$||R(\alpha,\bar{z})-R(\alpha,\bar{z}')||=\left(\frac{L_1^2}{\alpha^2}\sum_{i=1}^{n}(L_x|z^i-z'^i|+|\alpha s^i(\bar{z}-\bar{z}')|)^2\right)^{\frac{1}{2}}\le$$

$$\le L_1\sqrt{2}\sqrt{\frac{L_x}{\alpha^2}+2||\mathscr{L}(A_{\max})||_2^2}||(\bar{z}-\bar{z}')||=\bar{L}_1||\bar{z}-\bar{z}'||.$$

Similarly

$$||R(\alpha,\bar{z})-R(\alpha',\bar{z})||=$$

$$\left(\sum_{i\in N}(\frac{1}{\alpha}f^i(z^i,\alpha s^i(\bar{z}))-\frac{1}{\alpha'}f^i(z^i,\alpha' s^i(\bar{z})))^2\right)^{\frac{1}{2}}=$$

$$=(\sum_{i\in N}(\frac{1}{\alpha}(f^i(z^i,\alpha s^i(\bar{z}))-f^i(z^i,\alpha' s^i(\bar{z})))-$$

$$-(\frac{1}{\alpha'}-\frac{1}{\alpha})f^i(z^i,\alpha' s^i(\bar{z})))^2)^{\frac{1}{2}}\le(2\sum_{i\in N}\frac{L_1^2(\alpha-\alpha')^2}{\alpha^2}|s^i(\bar{z})|^2+$$

$$L_2(\frac{1}{\alpha'}-\frac{1}{\alpha})^2(L_c+L_x|z^i|^2+\alpha'^2|s^i(\bar{z})|^2))^{\frac{1}{2}}\le\bar{L}_\alpha(1+||\bar{z}||)|\alpha-\alpha'|.$$

Next let us prove that the growth condition. Let $\bar{z}\in\mathbb{R}^n$. Due to the limited growth rate $f^i(x,u)$ and Lipschitz property in u (by assumption **A1**) we have

$$\mathrm{E}||\frac{1}{\alpha_t}F(\alpha_t,\bar{z},\bar{w}_t)-R(\alpha,\bar{z})||^2=\sum_{i\in N}\mathrm{E}|\frac{1}{\alpha_t}f^i(z^i,\alpha_t\tilde{s}_t^i)-$$

$$-\frac{1}{\alpha}f^i(z^i,\alpha s^i(\bar{z}))|^2=\sum_{i\in N}\mathrm{E}|\frac{1}{\alpha_t}(f^i(z^i,\alpha_t\tilde{s}_t^i)-f^i(z^i,\alpha s^i(\bar{z})))-$$

$$-(\frac{1}{\alpha}-\frac{1}{\alpha_t})f^i(z^i,\alpha s^i(\bar{z}))|^2\le 2\sum_{i\in N}\mathrm{E}\frac{1}{\alpha_t^2}|f^i(z^i,\alpha_t\tilde{s}_t^i)-$$

$$-f^i(z^i,\alpha s^i(\bar{z}))|^2+(\frac{1}{\alpha}-\frac{1}{\alpha_t})^2|f^i(z^i,\alpha s^i(\bar{z}))|^2\le$$

$$\le 2\sum_{i\in N}\mathrm{E}\frac{L_1}{\alpha_t^2}|\alpha_t\tilde{s}_t^i-\alpha s^i(\bar{z})|^2+L_2(\frac{1}{\alpha}-\frac{1}{\alpha_t})^2(L_c+L_x|z^i|^2+$$

$$+|s^i(\bar{z})|^2)\le\gamma_t(nL_c+L_x||\bar{z}||^2)+\sum_{i\in N}\mathrm{E}4L_1|\tilde{s}_t^i-s^i(\bar{z})|^2+$$

$$+(\frac{4L_1(\alpha-\alpha_t)^2}{\alpha_t^2}+\gamma_t)s^i(\bar{z})^2,$$

where $\gamma_t=2L_2(1/\alpha-1/\alpha_t)^2$.

By Propositions 2, 3 and 4, denote $\beta_t=(4L_1((\alpha-\alpha_t)^2-1)/\alpha_t^2+\gamma_t$ consistently derive

$$\mathrm{E}||\frac{1}{\alpha_t}F(\alpha_t,\bar{z},\bar{w}_t)-R(\alpha,\bar{z})||^2\le\gamma_t(nL_c+L_x||\bar{z}||^2)+$$

$$+\sum_{i\in N}(4\frac{L_1}{\alpha_t^2}(\mathrm{E}(\tilde{s}_t^i)^2-s^i(\bar{z})^2)+(\frac{4L_1(\alpha-\alpha_t)^2}{\alpha_t^2}+\gamma_t)s^i(\bar{z})^2=$$

$$=\gamma_t(nL_c+L_x||\bar{z}||^2)+4\frac{L_1}{\alpha_t^2}\mathrm{E}||\tilde{s}_t||^2+\beta_t||\bar{s}(\bar{z})||^2\le$$

$$\le n\gamma_tL_c+4\frac{L_1}{\alpha_t^2}n^2\bar{b}^2\sigma_w^2+(\gamma_tL_x+8n(n-1)\bar{b}^2\frac{L_1}{\alpha_t^2}+$$

$$+2\beta_t||\mathscr{L}(A_{\max})||_2^2)||\bar{z}||^2=\bar{L}_2(1+||\bar{z}||^2).$$

■

We return to the problem of achieving consensus. We assume that in the continuous model (8)-(10) the $\frac{\varepsilon}{4}$-consensus is reached over time, i. e. all components of the vector $\bar{x}(\tau)$ become close to some common value $x^\star$ for all $i\in N$.

Theorem *2:* Let the conditions **A1**, **A2a–c** be satisfied, $\forall i\in N$ functions $f^i(x,u)$ are smooth by u, $f^i(x,0)=0$ for any x, $0<\alpha_t\le\bar{\alpha}$, for continuous model (8)-(10) $\frac{\varepsilon}{4}$-consensus is achieved for time $\mathscr{T}(\frac{\varepsilon}{4})$, consensus protocol parameters $\{\alpha_t\}$ are chosen so that $\tau_{\max}=\sum_{t=0}^{T}\alpha_t>\mathscr{T}(\frac{\varepsilon}{4})$ and for constants C_1,C_2 the following inequality holds

$$C_1e^{C_2\tau_{\max}}\max_{\alpha_t:\tau_t\le\tau_{\max}}\alpha_t\le\frac{\varepsilon}{4},$$

then mean square ε-consensus is achieved in stochastic discrete system (6)-(7) at time $t:\ \mathscr{T}(\frac{\varepsilon}{4})\le t\le\tau_{\max}$.

Proof: Denote $x^\star$ is the consensus value of the continuous model. From the first group of conditions of Theorem 2 the conditions of Theorem 1 hold, i. e. the result of the theorem is true. From other conditions of Theorem 2 and the result of Theorem 1 we obtain

$$\mathrm{E}||\bar{x}_t-x^\star\underline{1}||^2\le 2\mathrm{E}||\bar{x}_t-\bar{x}(\tau)||^2+2||\bar{x}(\tau)-x^\star\underline{1}||^2\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\le\varepsilon.$$

■

Consider an important particular case $\forall i\in N\ f^i(x,u)=u$.

Definition *1:* $T(\varepsilon)$ is called *time to ε-consensus*, if n nodes achieve ε-consensus for all $t\ge T(\varepsilon)$.

For time to $\frac{\varepsilon}{4}$-consensus in continuous model (8)-(10) the upper bound was obtained:

$$\mathscr{T}\left(\frac{\varepsilon}{4}\right)=\frac{1}{2Re(\lambda_2)}\ln\left(\frac{4(n-1)||\bar{x}_0-x^\star\underline{1}||^2}{\varepsilon}\right).\qquad(12)$$

Lemma *1:* If graph $\mathscr{G}_A$ has a spanning tree then control protocol (5) with $\alpha_t=\alpha$ and $B_t=A$ provides ε-consensus for continuous system (8) without noise for any $t\ge T(\varepsilon)$, where $T(\varepsilon)$ defined as

$$T(\varepsilon)=\frac{1}{2Re(\lambda_2)}\ln\left(\frac{(n-1)||x_0-x^\star\underline{1}||^2}{\varepsilon}\right)\qquad(13)$$

and its value $x^\star$ is given by $x^\star=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\bar{z}_1^ix_0^i$.

From Theorem 2 we can get the important consequence.

Theorem *3:* If $f^i(x,u)=u$ for any $i\in N$, conditions **A2a–c**, **A3** are satisfied, $\forall i\in N$ functions $f^i(x,u)$ are smooth in u, $f^i(x,0)=0$ for any x

then for any arbitrarily small positive number $\varepsilon>0$ for any $\tau_{\max}>\mathscr{T}(\frac{\varepsilon}{4})$ denoted in (12), when selecting sufficiently small α_t

$$\max_{\alpha_t:\tau_t\le\tau_{\max}}\alpha_t\le\frac{\varepsilon}{4C_1e^{C_2\tau_{\max}}}$$

at time $t:\ \mathscr{T}(\frac{\varepsilon}{4})\le t\le\tau_{\max}$ in stochastic discrete system (6)-(7) n nodes achieve mean square ε-consensus. $C_1,C_2,\bar{\alpha}$ are some constants and λ_2 is the closest to the imaginary axis eigenvalue of matrix $\mathscr{L}$ with nonzero real part.

Proof: In the conditions of Theorem 3 $R(\alpha,\bar{x})$ is a linear function. Therefore the dynamical system (8) takes the form:

$$\dot{\bar{x}} = -\mathscr{L}(A_{\max})\bar{x},$$

where $\mathscr{L}(A_{\max})$ is the Laplacian of $A_{\max}$. All amounts in rows of elements of the matrix $\mathscr{L}(A_{\max})$ are equal to zero and, moreover, all the diagonal elements are positive and equal to the absolute value of the sum of all the other elements in the row. The vector of 1's $\underline{1}$ is the right eigenvector corresponding to zero eigenvalue. The resulting continuous system is partially stable with respect to $h = \underline{1}^T\bar{x}$.

By condition **A3** it was obtained that in this continuous system the asymptotic consensus is achieved, and in Lemma 1 the ε-consensus is achieved, and the time to consensus is given by (13).

■

V. Load Balancing Problem

In recent years, distributed parallel computing systems were increasingly used in calculations[25]. For such systems the problem of separation package of jobs among several computing devices is important. Similar problems arise in transport networks [26], [27] or in production networks [28].

We consider the system of separation the same type of jobs between different nodes for parallel computing or production with feedback. Denote $N = \{1,\dots,n\}$ as a set of intelligent agents (nodes), each of which serves the incoming requests a first-in-first-out queue. Jobs are received at different times and on different nodes.

At any time t state of agent i, $i = 1,\dots,n$ is described by two characteristics:

- q_t^i is a load or queue length of the atomic elementary jobs of the node i at time t;
- r_t^i is a productivity of the node i at time t.

The dynamics of each agent are described by

$$q_{t+1}^i = q_t^i - r_t^i + z_t^i + u_t^i;\;\; i \in N,\; t = 0,1,\dots,T,\dots \qquad (14)$$

where z_t^i is the new job received by node i at time t.

At each time t node i can receive from its neighbors $j \in N_t^i$ the following information:

- the observations about the its loading — $y_t^{i,j}$;
- the productivity of the node — r_t^i.

The goal is to keep even load of all nodes in network.

To minimize the implementation time of all jobs redistribution of jobs among agents should be done. It will increase the capacity of the system.

In stationary case the best strategy is to reload jobs by such a way as

$$q_t^i/r_t^i = q_t^j/s_t^j,\; \forall i,j\; \in N.$$

Hence, if we consider $x_t^i = q_t^i/r_t^i$ as state of each node i then our goal is to achieve consensus.

Theorem 1 from previous section let us reformulate the problem of study the dynamics of load balancing as investigation of the continuous model which can be performed either analytically or numerically.

VI. Simulation Results

To illustrate the theoretical results we give an example of simulation for the load balancing system of computer network.

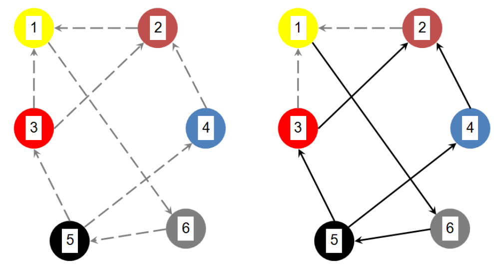

Fig. 1. 1) - Maximal set of communication links $E_{\max}$; 2) - Network topology.

Compare the dynamics of algorithm (5) and of continuous model described above.

Fig. 2 shows that trajectories of stochastic discrete system (dotted lines) are the same with the limiting trajectories of the differential equation (dashed lines).

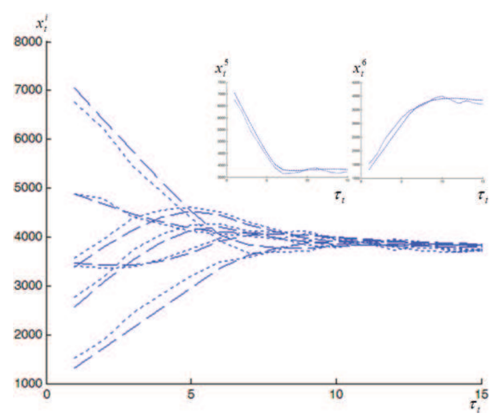

Fig. 2. Trajectories of stochastic discrete system and its continuous model.

Theorem 1 and its illustration shows that we can analyze the continuous model for the investigation of the work of the algorithm.

VII. Analytical results

Assume that $r_t^i \neq 0 \forall\, i$. Consider the control protocol (5), where $\forall\, i \in N$, $\forall\, t$ denote $\bar{N}_t^i = N_t^i$ and $b_t^{i,j} = r_t^j/r_t^i$, , $j \in N_t^i$.

For the considered case the dynamics of closed loop system (14) for protocol (5) is as follows:

$$x_{t+1}^i = x_t^i - 1 + z_t^i/r_t^i + \alpha_t \sum_{j \in N_t^i} b_t^{i,j}(y_t^{i,j}/r_t^j - y_t^{i,i}/r_t^i). \qquad (15)$$

where α_t is a sequence of positive step-sizes, $y_t^{i,j}$ are noisy measurements of the state x_t^j, z_t^i is the new job received by node i at time t.

It is assumed that the length of sent jobs at each step is small compared to the current queue length and, hence, the length of jobs and the queue length can be considered as continuous variables.

As an example we consider the system consisting of computing blocks. Fig. 1-a shows a computing network of the six agents, indicating the possible communication links, some of which may be "closed" and "open up" over time. Network topology is random at any time t. We have link 1-3 or 1-2 with probability $1/2$, i. e. $p_0^{13} = p_1^{12} = 1/2$ (Fig. 1-b). To simplify we assume that p_t^i do not depend on time, i. e. $p_t^i \equiv p^i$.

For the case without delay the equation (8) is as follows:

$$\frac{dX}{d\tau} = R(\alpha, \bar{x}), \tag{16}$$

where

$$R(\alpha,\bar{x}) = \begin{pmatrix} -1 & \frac{1}{2}b^{1,2} & \frac{1}{2}b^{1,3} & 0 & 0 & 0 \\ 0 & -1 & 0 & b^{2,4} & 0 & 0 \\ 0 & 0 & -1 & 0 & b^{3,5} & 0 \\ 0 & 0 & 0 & -1 & b^{4,5} & 0 \\ 0 & 0 & 0 & 0 & -1 & b^{5,6} \\ b^{6,1} & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{17}$$

VIII. Conclusion

In this paper we have studied the approximate consensus problem in multi-agent stochastic system with nonlinear dynamics, noise and switched topology.

As an example of such system the decentralized load balancing problem with incomplete information for networks with switching topology is analyzed.

Local voting protocol is used to achieve approximate consensus. Analytical bounds for the system behavior are derived and the simulation results of algorithm for decentralized load balancing system are given. The simulation results for decentralized load balancing computing system demonstrate good performance of the algorithm.

Method of continuous models was used to analyze system dynamics. The usage of this method can significantly reduce the computation load.

In future work it would be of interest to analyze the algorithm under the influence of different types of noise. Also attempts will be made to improve the algorithm for use in the case of biased measurement errors and delays.

One of future research direction could be extension of our approximation results to discontinuous systems [29].

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