CONTROL OF OSCILLATORY BEHAVIOR OF MULTISPECIES POPULATIONS

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Abstract. Control problem for the model of the multispecies Lotka-Volterra ecosystem is considered. The algorithm for control of oscillatory behavior of the ecosystem based on the speed-gradient method is proposed. The conditions of convergence of the algorithm are obtained. The conditions of convergence of the algorithm are obtained. The results of numerical experiments are presented demonstrating that the speed-gradient method can achieve control goal with a small control signal, which is important in controlling real ecosystems where control action should be sufficiently small. In turn, it allows to increase stability and robustness of the oscillation behavior by means of increase of the distance from boundaries of population survival area.
Keywords: Lotka-Volterra ecosystem, ecological stability, speed-gradient algorithm

1. Introduction

Mathematical models of populations are important for studying of ecosystems stability. Only stable ecosystems are able to exist for a long time and their stability limits define those maximum loadings which excess can lead to ecocatastrophes. The stability problem is connected with questions of an operation of natural populations, estimations of pollution limits of an environment, the forecast of farming [1].

An important tool for study and management of ecological systems is control theory that provides a variety of methods for improving stability of populations. Many applications of the control theory to ecological models are based on traditional methods of linear and nonlinear control aimed at stabilization of an equilibrium or a given reference trajectory of the system [2,3]. However, in many other cases natural system behavior is oscillatory, like in classical predator-prey models. In such problems some special methods of oscillation control [4] can be more efficient.

An advantageous feature of oscillatory behaviour is existence of one or several invariants: functionals on the system states space that remain constant along trajectories of the undisturbed (uncontrolled) system. Such invariants may be interpreted as energy-like or entropy-like functions [5-7]. A smart control should respect invariants. Moreover, it seems reasonable to reformulate the control goal properly in terms of invariants.

A general approach to control of invariants was proposed in [4,8,9] and consequent works, see [11], based on the Speed-Gradient (SG) method [4,12]. It was successfully applied to the classical Lotka-Volterra model for two species [4]. It was shown that SG-control allows one to change the amplitude of the oscillation cycles in an arbitrary manner keeping the amplitude of the control action as small as possible. Such a property is important for control of ecosystems were possibilities of control are often limited. However no investigation for multispecies models has been done so far.

In mathematical ecology and biophysics the classical Lotka-Volterra model of the population dynamics ("predator-prey" model) and its generalization to the case of N species are well recognized [1,5-7]. Dynamics of the N-species Lotka-Volterra models were considered in detail in [1,10], the special attention was paid to determination and methods of stability analysis within the mathematical models of studying ecosystems. In [6] these models as the thermodynamic systems were investigated, and generalized expressions of entropy-production for the systems and the study of their role in the analysis of ecological stability were derived.

The objective of the present work is to develop the algorithm for control of oscillatory behavior of the multispecies Lotka-Volterra model in order to improve its stability. The proposed control algorithms are based on the Speed-Gradient method that has already demonstrated its efficiency for control of oscillations in a number of problems in physics and engineering [4,11,12].

2. Mathematical Model of Ecosystem

In a class of the ordinary differential equations a generalized Lotka-Volterra model represents the system [1]:

$$\frac{dx_i}{dt} = x_i(t) \cdot \left(k_i + \beta_i^{-1} \sum_{j=1}^{N} a_{ij} \cdot x_j(t) \right), i = 1,2,..N, \tag{1}$$

where k_i is the speed of the natural increase or death rate of the *i*-th kind in the absence of all other species: $k_i < 0$, if the *i*-th species lives at the expense of others and $k_i > 0$ otherwise. The parameter β_i>0 reflects the fact that the appearance of a predator is usually connected with vanishing of one or more preys. Quantities $a_{ij}, i \neq j$ evaluate the type and intensity of the interaction between *i*-th and *j*-th species and form an antisymmetric matrix. The stability of the ecosystem can be interpreted as the special behavior of solutions of (1) when all species stay alive, that is their numbers are always more than zero. Stability of the ecosystem described by (1) means that its solutions do not approach the boundary of the positive ortant.

3. Controlled Model and Problem Statement

In this paper we consider the controlled version of the model (1). Suppose the birth rate of the species x_l, $l = M+1,..N$ can be controlled. Then the interaction between the species is described by the differential system:

$$\begin{cases} \dfrac{dx_i}{dt} = x_i(t)\cdot\left(k_i + \beta_i^{-1}\sum_{j=1}^{N} a_{ij}\cdot x_j(t)\right), i = 1,2,..M, \\ \dfrac{dx_l}{dt} = x_l(t)\cdot\left(k_l + \beta_l^{-1}\sum_{j=1}^{N} a_{lj}\cdot x_j(t) + u_l(t)\right), l = M+1,..N. \end{cases} \tag{2}$$

Assume that there exists at least one positive equilibrium of (1) for some values of the system parameters:

$$n = (n_1, n_2, .., n_N), n_i > 0,\ i = 1,..N, \tag{3}$$

and consider an auxiliary function *W*:

$$W(x) = \sum_{i=1}^{N} \beta_i n_i \left(\frac{x_i}{n_i} - \log\frac{x_i}{n_i}\right). \tag{4}$$

It is well known [8] that if the condition (3) holds, then *W(x)* is constant along trajectories of (1), i.e. *W(x)* is an invariant of (1). Besides, Hessian matrix of *W(x)* is positive definite and, therefore, *W(x)>W(n)* for *x≠n*. Hence *W(x)* can measure the amplitude of oscillations and can be used to achieve the desired amplitude of oscillations.

Introduce the control goal: achievement of the desired level of the quantity *W(x(t)) as t⟶∞*:

$$W(x(t)) \to W^*, t \to \infty. \tag{5}$$

If *W*=W(n)*=min*W(x)*, then the goal (5) means achievement of the equilibrium *x=n*. In the case *W(n)<W*<W(x(0))* achievement of the goal (5) means decrease of the oscillations level. *If W*>W(x(0))*, then achievement of the goal (5) corresponds to the growth of the oscillations intensity. The problem is to find control function *u(t)* in (2), ensuring achievement of the control goal (5).

4. Main results

Apply the speed gradient (SG) method [4] to solve the problem. To this end introduce the so called *goal function Q*:

$$Q(x) = \frac{1}{2}(W(x) - W^*)^2. \tag{6}$$

In order to achieve the goal (5), it is necessary and sufficient that *Q* converges to zero. According to the SG method one needs to evaluate A) derivative (speed of change) of *Q* with respect to the system (2) and B) the gradient of $\dot{Q}$ with respect to u.
Calculation of time derivative of *Q* with respect to the system (2) yields:

$$\dot{Q}(x,u) = (W(x) - W^*)\sum_{l=M+1}^{N}(x_l(t) - n_l)u_l. \tag{7}$$

Partial derivatives with respect to u_l are evaluated as follows:

$$\frac{\partial}{\partial u_l}\dot{Q}(x,u) = (W(x) - W^*)(x_l(t) - n_l),\ \ l = M+1,..N. \tag{8}$$

According to the SG method the control action is chosen as follows:

$$u_l(t) = -\gamma_l (W(x) - W^*)(x_l(t) - n_l), \gamma_l > 0, \;\; l = M+1,..N. \tag{9}$$

The main result of this paper is the following proposition.

Theorem. Assume that there exists an equilibrium in the system (1) such that (3) holds. Then either the algorithm (9) provides the goal (5), or the quantities of the controlled species tend to their equilibrium values.

Proof. Consider the time derivative of the goal function Q (6):

$$\frac{d}{dt} Q(x(t)) = -2\gamma (W(x(t)) - W^*) \sum_{l=M+1}^{N} (x_l - n_l)^2 \le 0. \tag{10}$$

Since Q does not increase, there exists a finite limit of $Q(t)$ as t→∞. Denote it as $\overline{Q}$. Suppose the goal (5) does not hold. Then $\overline{Q} > W^*$. Hence $Q(t) \ge \overline{Q}$ for all $t \ge 0$ and

$$\frac{d}{dt} Q(x(t)) = -2\gamma (\overline{Q} - W^*) \sum_{l=M+1}^{N} (x_l - n_l)^2 \le 0. \tag{11}$$

Integration of (11) yields

$$\begin{aligned} &0 \le Q(x(t)) - W(n) \le Q(x(0)) - W(n) \\ &- 2\gamma (\overline{Q} - W^*) \sum_{l=M+1}^{N} \int_0^t (x_l(s) - n_l(s))^2 ds \le 0. \end{aligned} \tag{12}$$

Therefore

$$\sum_{l=M+1}^{N} \int_0^t (x_l(s) - n_l(s))^2 ds < \infty. \tag{13}$$

The integrand converges to zero according to Barbalat Lemma [4,12], that is

$$x_l(t) \to n_l, t \to \infty, \;\; l = M+1,..N. \tag{14}$$

Thus either the algorithm (9) provides the control goal (5), or the number of the controlled species $x_l(t)$ converges to its equilibrium n_l ■

Remark 1. In Theorem 1 it is supposed that the system (1) has at least one positive equilibrium for some values of its parameters. For a nonsingular matrix composed of a_{ij} we always can choose values of the birth rate k_i such that (3) holds [6]. For a nonsingular matrix composed of a_{ij} positivity conditions depending only on a_{ij} were found in [1]. Finally, for both nonsingular and singular cases positivity conditions were given in [13].

Remark 2. For real world ecological systems it is important that the desired behavior of the system could be achieved with relatively small control intensity. An advantage of the proposed approach is that it allows achieve the control goal with arbitrarily small control by means of appropriate decreasing of the gain γ. Indeed, it follows from (10) that Q(x(t))≤Q(x(0)) for all t≥0. Since the goal function Q(x) tends to infinity as x→∞, the trajectories of the controlled system belong to the bounded set Ω_0={x: Q(x(t))≤Q(x(0))}. Denote u* maximum value of control (9) with γ=1 over the set Ω_0. It is finite since the right hand side of (9) is continuous while the set Ω_0 is compact. Then maximum value of control (9) with any γ will not exceed γu*, i.e. it can be made arbitrarily small by choosing small γ. By similar arguments an arbitrarily small value of the time derivative of the control can be achieved by decreasing the gain gamma (again compactness of Ω_0 is employed).

5. Numerical experiments

We present the results of numerical experiments demonstrating the dynamics of the system controlled by the algorithm (9). Below the behavior of uncontrolled system (1) (Fig.1) and the behavior of the system (2) for the case of controlled the third species (Fig.2, Fig.3, Fig.4) for $N=4$ are shown. We take initial numbers of the species $x_{01} = [4;7;6;5];\ x_{02} = [2;3;5;3]$ and the system parameters

$$\beta_1 = 4;\ \beta_2 = 2;\ \beta_3 = 3;\ \beta_4 = 6;\ k_1 = -9;\ k_2 = -8;\ k_3 = 7;\ k_4 = 6; \\ a_{12} = 2;\ a_{13} = 3;\ a_{14} = 5;\ a_{23} = 4;\ a_{24} = 3.5;\ a_{34} = 2. \tag{15}$$

Three values for the desired level of W are considered:

$$W_1{*} = 52; W_2{*} = 62; W_3{*} = 40.$$

The control gain is taken as $\gamma_3 = 0.2$. The equilibrium of the system (1) for these parameters is $n_1 = 3; n_2 = 5; n_3 = 2; n_4 = 4$, the equilibrium value of the quantity W is $W^e \cong 51.2$.

In Fig.2 and Fig.3 initial sizes of the species have been picked such that $W^e < W^0 < W*$. In this case value of W converge to its desired level rather fast. It is seen that the smaller the value of $W*$ the larger the distance between the trajectories and the boundary of the stability region (coordinate planes or zero sizes of species).

For the initial sizes of the species in Fig.4 the relation $W* < W^e < W^0$ holds. In this case an alternative scenario from the Theorem is realized: the desired level of W has not been achieved, but the sizes of the controlled species converge to their equilibrium value.

In Fig.5 time histories of control for different values of the gain γ_3 are shown. The system parameter values are taken as in (15) while controller parameters are taken as follows: $W* = 61.7$, $x_{01} = [5;5;5;2]$, $\gamma_3^1 = 0.2, \gamma_3^2 = 0.1, \gamma_3^3 = 0.02$. It is seen that decrease of γ_3 implies decrease of intensity of $u(t)$. It is also seen that the smaller control has the longer duration of the transients.

An interesting question is robustness of the closed loop system with respect to the controlled system parameter changes. In Fig.6 the behaviors for three different sets of parameters (see Appendix 1) corresponding to the same equilibrium and the same goal function are shown. It is seen the limit behavior weakly depends on the controlled system parameter values.

6. Conclusion

In this work we have demonstrated the application of the speed-gradient method for solving non-traditional control problems of nonlinear network models, a special case of which is the Lotka-Volterra model of the dynamics of the N species. The SG algorithms for control of oscillations in the multispecies Lotka-Volterra model are proposed for different numbers of species admitting controlled growth coefficients (birth/death rate). It is demonstrated both theoretically and by means of simulations that control of a small level allows one to significantly change oscillations amplitude.

The simulation results have shown that for the smaller value of W the oscillation variations in the number of the species are lower and the distance between the trajectories and the boundary of the stability region is larger. Thus, to improve ecosystem stability it is sufficient to reduce the value of W. The algorithm based on the speed-gradient method can do it with small control signal, which is important in controlling real ecosystems where control action should be sufficiently small.

In turn, it allows to increase stability and robustness of the oscillation behavior by means of increase of the distance from boundaries of population survival area (in the case of the multispecies Lotka-Volterra model it is the N-dimensional positive cone (ortant) in the state space of the system).

Fig.1. Plots of the numbers of the species versus time (top left) and W versus time (top right) and the phase portraits (bottom) of the uncontrolled system (1) for N=4 and initial numbers of the species $x_{01} = [4;7;6;5]$

Fig.2. Plots of the numbers of the species versus time (top left) and W versus time (top right) and the phase portraits (bottom) of the controlled system (2) when controlling the numbers of the third species, for N=4, initial numbers of the species $x_{01} = [4;7;6;5]$ *and desired value W*=52*

Fig.3. Plots of the numbers of the species versus time (top left) and W versus time (top right) and the phase portraits (bottom) of the controlled system (2) when controlling the numbers of the third species, for N=4, initial numbers of the species $x_{01} = [4;7;6;5]$ *and desired value W*=69*

Fig.4. Plots of the numbers of the species versus time (top left) and W versus time (top right) and the phase portraits (bottom) of the controlled system (2) when controlling the numbers of the third species, for N=4, initial numbers of the species $x_{02} = [2;3;5;3]$ *and desired value W*=40*

Fig.5. Plots of the control for different values of the gain γ *when controlling the numbers of the third species, for N=4, initial numbers of the species* $x_{01} = [5;5;5;2]$, $\gamma_3^1 = 0.2, \gamma_3^2 = 0.1, \gamma_3^3 = 0.02$ *and desired value W*=61.7*

Fig.6. The behaviors of the quantity W(x(t)) for three different sets of parameters corresponding to the same equilibrium n = [3;5;2;4] *and the same goal function desired value W*=66 when controlling the numbers of the third species, for N=4*

Appendix 1

Three sets of system parameters corresponding to the same equilibrium *n=[3;5;2;4]* and the same goal function W*=66 when controlling the numbers of the third species were taken:

1) The parameter values (15) with initial numbers of the species $x_{01} = [5;5;5;2]$ and $\gamma_3 = 0.05$.
2) The parameter values

$$\beta_1 = 5;\ \beta_2 = 4;\ \beta_3 = 5;\ \beta_4 = 5;\ k_1 = -7.4;\ k_2 = -4.1;\ k_3 = 4.7;\ k_4 = 7.3;$$
$$a_{12} = 2;\ a_{13} = 3.5;\ a_{14} = 5;\ a_{23} = 4.2;\ a_{24} = 3.5;\ a_{34} = 2.$$

with initial numbers of the species $x_{01} = [2;5;5;5]$ and $\gamma_3 = 0.2$.

3) The parameter values

$$\beta_1 = 5;\ \beta_2 = 2;\ \beta_3 = 6;\ \beta_4 = 5;\ k_1 = -7.48;\ k_2 = -6.7;\ k_3 = 3.6;\ k_4 = 6.8;$$
$$a_{12} = 2.2;\ a_{13} = 3.2;\ a_{14} = 5;\ a_{23} = 4;\ a_{24} = 3;\ a_{34} = 2.$$

with initial numbers of the species $x_{01} = [2;5;2;3]$ and $\gamma_3 = 0.2$.

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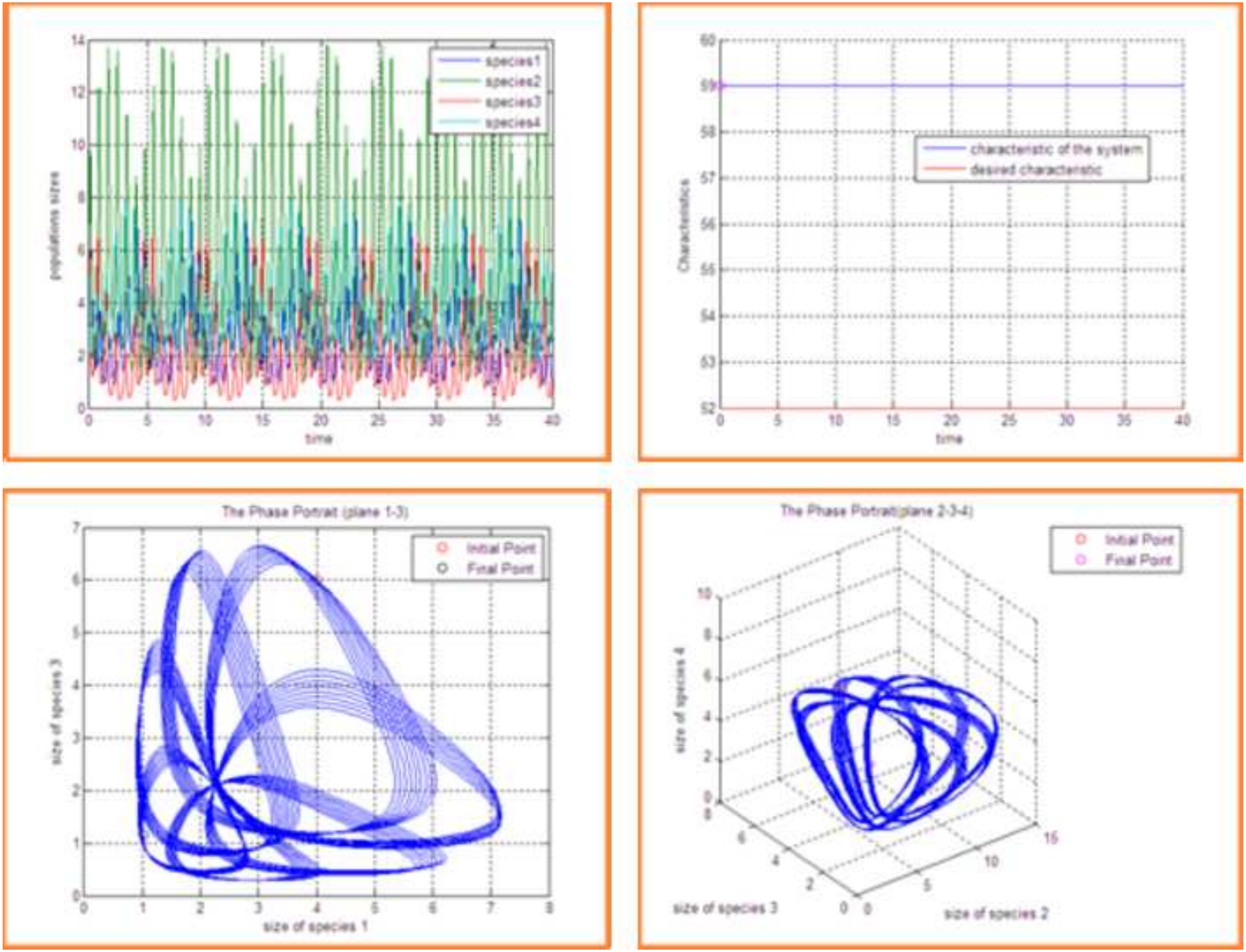

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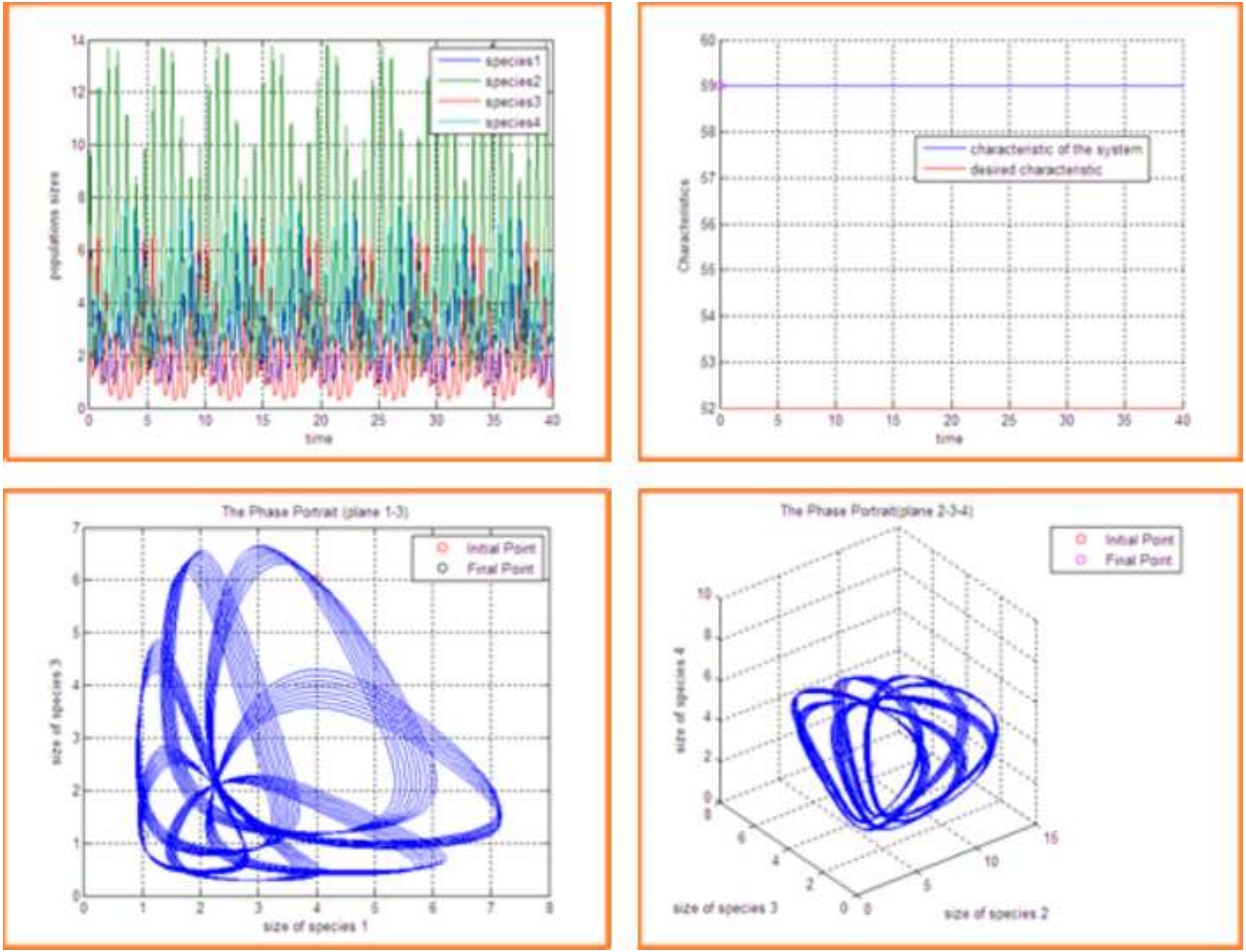

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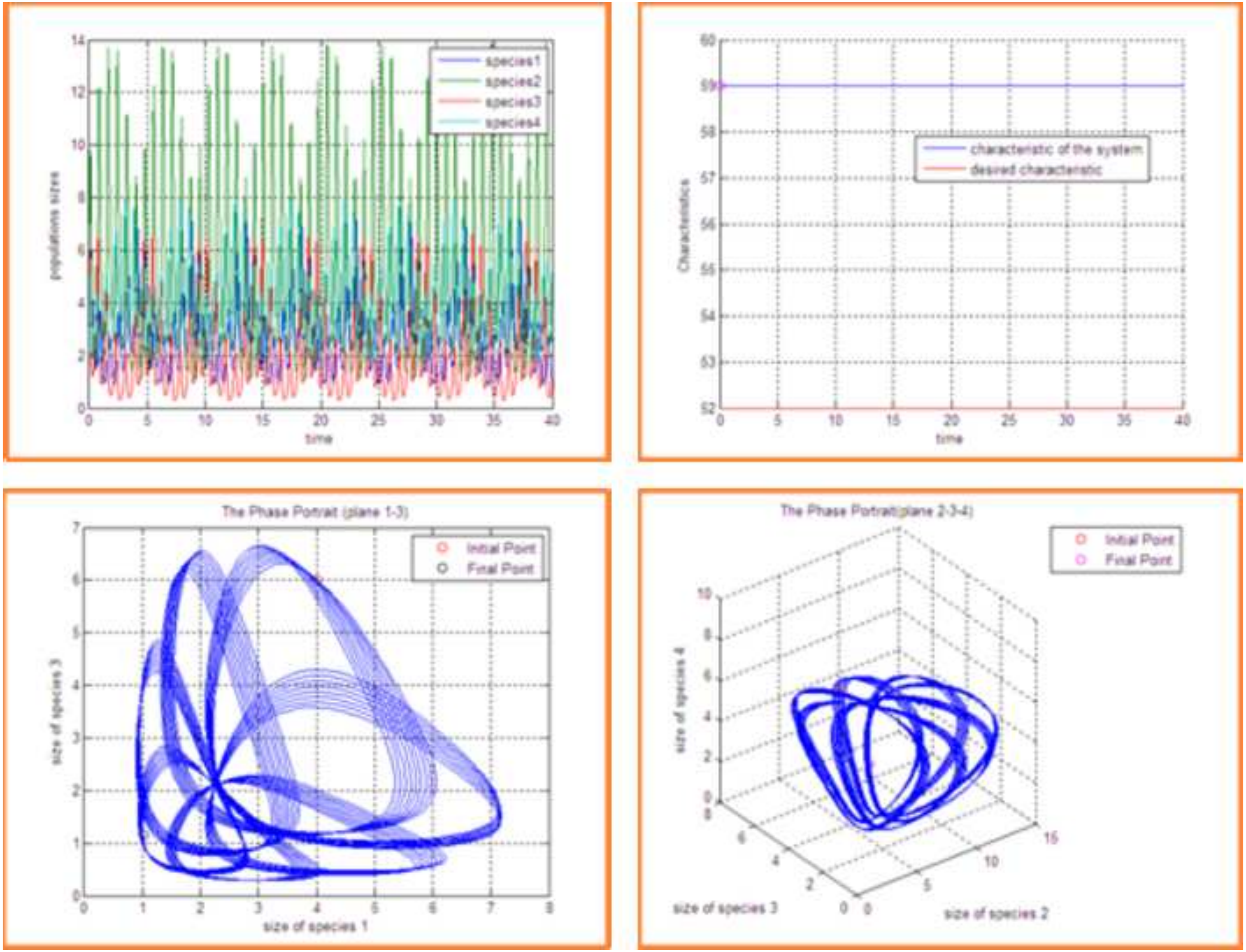

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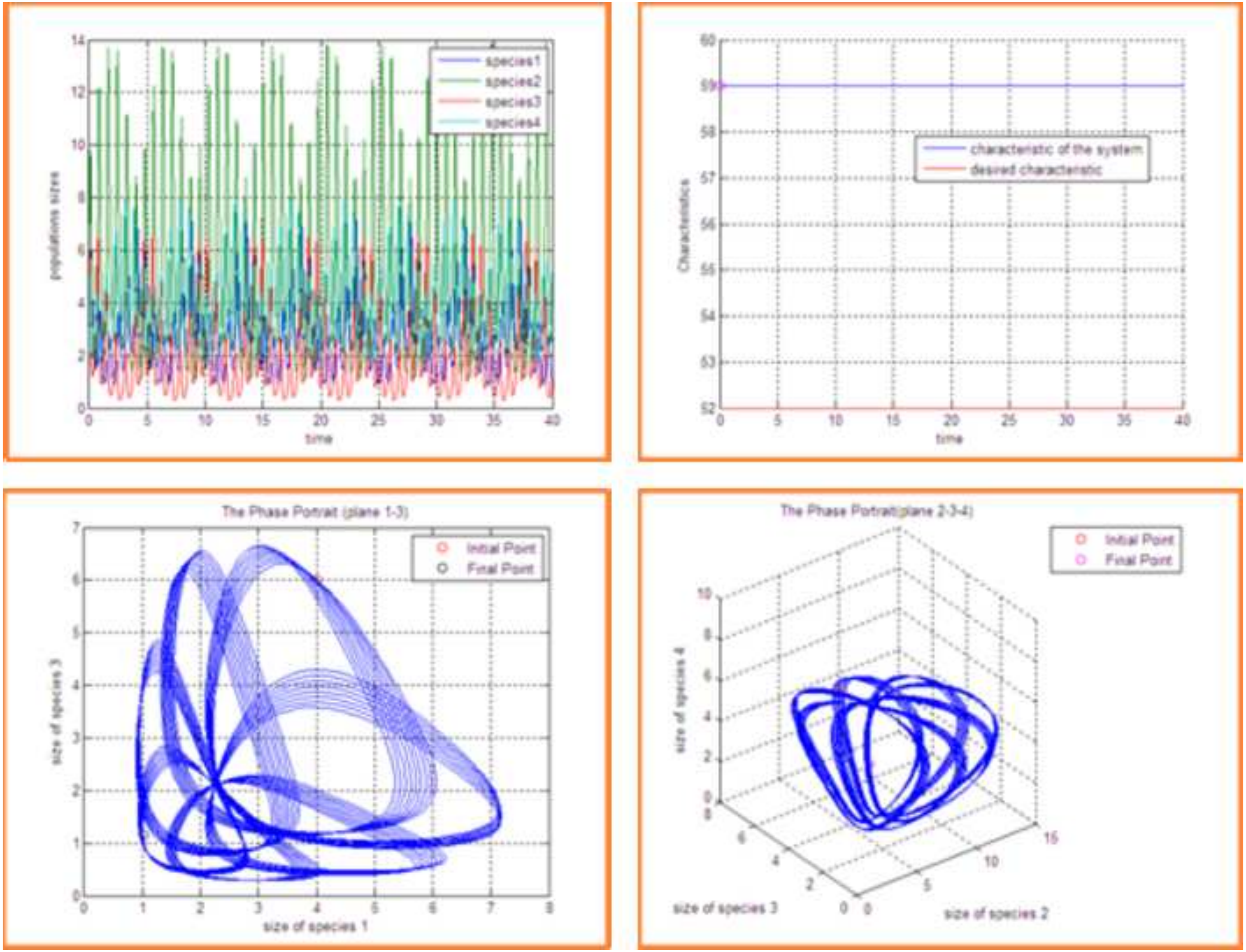

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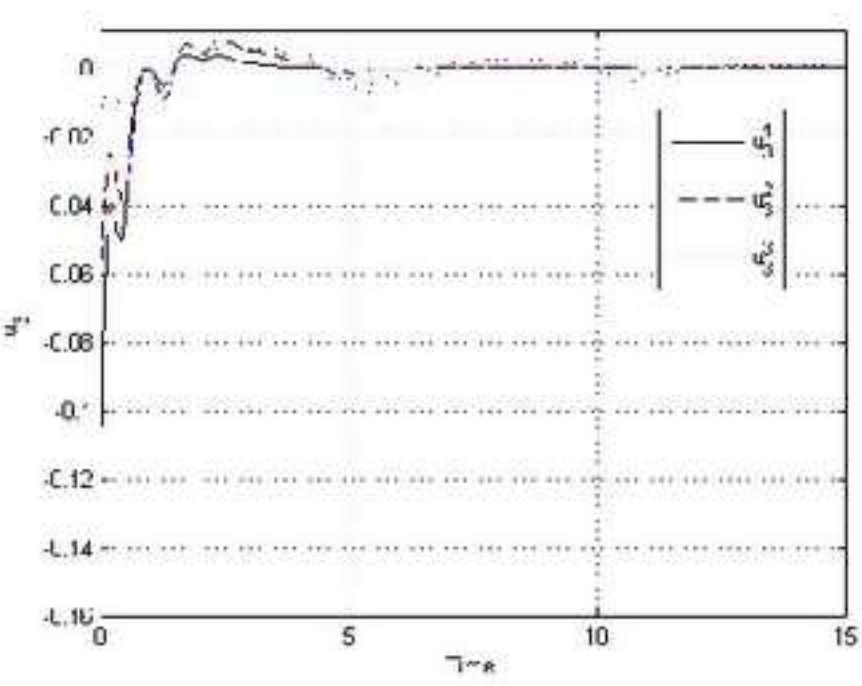

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