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Synchronization and phase relations in the motion of two-pendulum system☆

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Abstract

The synchronization phenomenon in non-linear oscillating system is studied by means of examination of the coupled pendulums. Dependence of the phase shift between pendulum states on system parameters and initial conditions is studied both analytically and numerically. The harmonic linearization technique is applied for analytical examinations.

Keywords: Oscillations; Synchronization; Coupled pendulums; Phase shift

1. Introduction. Resonance and synchronization

Non-linear oscillations problem falls within the fields of non-linear mechanics, non-linear physics, as well as non-linear control theory. During the latest years the subject of non-linear oscillations control has attracted growing researchers' attention among the other fields of research where control theory methods were applied to physical phenomena exploration [1].

The resonant property of non-linear systems was intensively studied in the areas of charged-particle acceleration physics [2], plasma physics [3], celestial mechanics [4], and automatic control. Various definitions of the resonance phenomenon have been introduced to describe such properties of the non-linear oscillations as stochastic resonance [5], chaotic resonance [6], autoresonance [2,7], and feedback resonance [8–11].

The resonance property is allied to the synchronization one. Synchronization is usually treated as corresponding in time behavior of two or more processes. General definitions of synchronization were proposed in [12,13]. Starting with the work of Huygens [14], the synchronization phenomena attracted attention of many researchers, see e.g. monographs [15,16]. In his study, Huygens described synchronization of the pair of pendulum clocks weakly coupled one with another by the common base. Huygens had found the pendulum clocks swung in exactly the same frequency and 180° out of phase. After external disturbance was made, the antiphase state was restored within a half of an hour and remained indefinitely. Huygens' synchronization observations have served to inspire study of sympathetic rhythms of interacting non-linear oscillators in many areas of science and technology. The onset of synchronization and the selection of particular phase relations have been studied in the numerous papers and monographs, see e.g. [15,17–20,5]. Exploration of the synchronization phenomena is usually focused on the study of the phase relations between motions of the coherent units. Different studies of the pair of coupled oscillators confirmed an observation that the *asynchronous* mode of motion is a predominant one. At the same time some experiments show that the *synchronous* mode can also be observed. Therefore, the problem of phase relations between coupled oscillators is still opened.

This paper is focused on the study of phase relations between coupled oscillators, excited by means of the feedback control torque. The model of 1-DOF pendulums coupled with a weak spring, "*the Coupled Pendulums of the Kumamoto University*", [21] is considered.

The coupled-pendulums dynamics model is given in Section 2, where some design features of the system are also briefly described. In Section 3, the phase relations are studied

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analytically on the basis of the *harmonic linearization* (HL) method. The case of a single excitation is considered. It is shown that both synchronous and asynchronous modes can appear. For the particular system the computer simulation has been provided for checking up the analytical results and receiving additional information on system behavior in Section 4. Some results of the computer simulations for double-input excitation are presented in Section 5. The summarized information is given in Conclusion.

2. Coupled pendulums model

Let us consider the "Coupled Pendulums of the Kumamoto University" [21]. This system consists of a pair of 1-DOF pendulums, coupled with a weak spring. A motor is attached to the first pendulum. The joint of the second pendulum is passive (Fig. 1).

The system dynamics can be written as [21]

$$\begin{cases} J_1\ddot\varphi_1+\rho_1\dot\varphi_1+m_1gl_1\sin\varphi_1=f(\varphi_1,\varphi_2)\cos\varphi_1+u(t),\\ J_2\ddot\varphi_2+\rho_2\dot\varphi_2+m_2gl_2\sin\varphi_2=-f(\varphi_1,\varphi_2)\cos\varphi_2,\end{cases}$$

$$f(\varphi_1,\varphi_2)=-kl_0^2(\sin\varphi_1-\sin\varphi_2), \tag{1}$$

where u is an external torque (developed by the drive motor); J_i, g, ρ_i, and k represent inertia, gravitational acceleration, frictional coefficient and coupling strength, respectively; m_i denotes mass of the bob; $i=1,2$ is a number of the pendulum. The motor torque $u(t)$ is considered as a *control action*, applied to the system (1). The following linearized model is also used:

$$\begin{cases} J_1\ddot\varphi_1+\rho_1\dot\varphi_1+m_1gl_1\varphi_1+kl_0^2(\varphi_1-\varphi_2)=u(t),\\ J_2\ddot\varphi_2+\rho_2\dot\varphi_2+m_2gl_2\varphi_2+kl_0^2(\varphi_2-\varphi_1)=0.\end{cases} \tag{2}$$

Eqs. (2) are obtained via linearization of (1) at the lower equilibrium point $\varphi_1=\varphi_2=0$.

Let the system be excited by means of the relay feedback control action

$$u=\gamma\,\mathrm{sign}\,\dot\varphi_1, \tag{3}$$

where parameter γ gives the control torque magnitude. Law (3) is a particular case of the speed-gradient control algorithms [9,10,1]. It had been presented and studied in the earlier published papers [8,22,9,23–25].

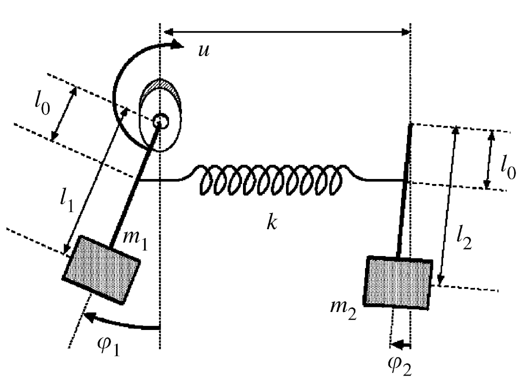

Fig. 1. Coupled pendulums of the Kumamoto University.

3. Examination of synchronous oscillations mode via harmonic linearization method

Let us examine the motion of the system (1) and (3), zeroing in phase shifts between the pendulum deflections in the steady-state mode. The harmonic linearization method is applied to provide the analytical expressions for amplitude and frequency of oscillations and the phase shift between the pendulum states.

3.1. Case of the identical pendulums

At first, consider the simplified case of the identical pendulums. Assume that $J_1=J_2=J$, $m_1=m_2=m$, $l_1=l_2=l$, $\rho_1=\rho_2=\rho$. For that case the transfer function of the "linear part" of the closed-loop system (2), (1) has a simple form and the harmonic balance equation can be easily solved. The transfer function $W(s)$ from the input signal u to the output $\dot\varphi_1$ is as follows:

$$W(s)=\frac{s(Js^2+\rho s+\beta^2+kl_0^2)}{(Js^2+\rho s+\beta^2)(Js^2+\rho s+\beta^2+2kl_0^2)}, \tag{4}$$

where parameter $\beta^2=mgl$ is introduced. The value of $\beta/\sqrt{J}$ can be treated as natural angular frequency of the small oscillations for (each) single pendulum.

The feedback (3) is positive and the harmonic balance equation can be written as $W(j\omega)=1/q(a)$, where $j^2=-1$; $q(a)$ is a *harmonic linearization* gain (also known as a *describing function*). For the considered relay control law (3) this gain has a form: $q(a)=4\gamma/\pi a$, where a denotes the oscillations magnitude of the system's linear part output (i.e. $a=\overline{\lim\limits_{t\to\infty}}\,\dot\varphi_1(t)$). Solutions $\omega=\tilde\omega$, $a=\tilde a$ of the harmonic balance equation are the sought quantities of the frequency and amplitude of oscillations. The simple calculations lead to the following equations:

$$\begin{cases} \pi aJ^2\omega^4+(\pi a(-J\beta^2-\rho^2-J(\beta^2+2kl_0^2))\\ \quad+4\rho\gamma)\omega^2+\pi a\beta^2(\beta^2+2kl_0^2)=0,\\ 2(-\pi a\rho+2\gamma J)\omega^3+(\pi a(\beta^2\rho+(\beta^2+2kl_0^2)\rho\\ \quad-4\gamma(\beta^2+kl_0^2)))\omega=0.\end{cases} \tag{5}$$

Omitting the trivial (and unstable) solution $\tilde\omega=0$, $\tilde a=0$ one may rewrite the second equation in (5) as follows:

$$(-\omega^2J+\beta^2+kl_0^2)(\pi a\rho-2\gamma)=0. \tag{6}$$

Let us consider two cases:

(1) $-\omega^2J+\beta^2+kl_0^2=0$ and
(2) $\pi a\rho-2\gamma=0$.

Both of the above equalities satisfy Eq. (6).

(1) Assume that $\omega^2=(\beta^2+kl_0^2)/J$. Substituting this expression for ω^2 in (5) one gets

$$\tilde a=\frac{4\gamma\rho(\beta^2+kl_0^2)}{\pi(\rho^2(\beta^2+kl_0^2)+k^2l_0^4J)}. \tag{7}$$

Reverting to the expression for ω one obtains

$$\tilde{\omega}_0 = \frac{1}{J}\sqrt{J(\beta^2 + kl_0^2)}. \tag{8}$$

Eqs. (7) and (8) give amplitude and frequency of one of two possible oscillating motions of the system.

(2) Assume now that $\pi a \rho - 2\gamma = 0$. Then it is valid

$$\tilde{a} = \frac{2\gamma}{\rho\pi}. \tag{9}$$

Substituting $\tilde{a}$ given in (9) for a in the first equation of (5) and omitting the negative solution one obtains the following expression for $\tilde{\omega}$:

$$\tilde{\omega}_{1,2} = \frac{1}{2J}\sqrt{\pm\kappa - 2\rho^2 + 4Jkl_0^2 + 4\beta^2 J}, \tag{10}$$

where $\kappa = 2\sqrt{\rho^4 - 4\rho^2 Jkl_0^2 - 4\rho^2\beta^2 J + 4J^2k^2l_0^4}$.

Note that Eqs. (7) and (8) make sense for all (positive) parameter values, while the expression (10) for $\tilde{\omega}_{1,2}$ and, therefore, the formula (9) for $\tilde{a}$ make sense if all radicands in (10) are nonnegative. Let us find the explicit conditions that mark out the region of existence of the solutions $\omega_{1,2}$ at the system parameters space. Regarding κ in (10) as a function of the coupling gain k and supposing that $\kappa \geqslant 0$, one gets the following inequalities with respect to k:

$$k \leqslant \frac{\rho(\rho - 2\beta\sqrt{J})}{2l_0^2 J}, \quad k \geqslant \frac{\rho(\rho + 2\beta\sqrt{J})}{2l_0^2 J}. \tag{11}$$

Examination of the linearized model equation (2) gives that for each uncoupled oscillator it is valid that $\rho < 2\beta\sqrt{J}$. It is natural to assume that the friction parameter ρ of the pendulums is small (so that the pendulums free motion is an *oscillatory* one) and to pick up the second inequality from (11). It can be easily checked that for all positive k, satisfying (11), the radicands in (10) are also positive. Summarizing, note that there exists a "critical" (or a *bifurcation*) value of $k \triangleq k_*$, defined as

$$k_* = \frac{\rho(\rho + 2\beta\sqrt{J})}{2l_0^2 J}, \tag{12}$$

so that for $k > k_*$ there exist solutions with the frequencies $\tilde{\omega}_1$, and $\tilde{\omega}_1$ (see (10)) and the amplitude (9). Evidently, for $k = k_*$ it is valid $\tilde{\omega}_1 = \tilde{\omega}_2$, but these values differ from $\tilde{\omega}$, obtained from (8) for $k = k_*$. It is easy to find that the solutions of (8) and (10) have a common point if

$$k = k_0 = \frac{\rho}{2Jl_0^2}(\rho + \sqrt{\rho^2 + 4\beta^2 J}). \tag{13}$$

It is valid that $k_0 > k_*$, but for the small ρ it can be taken that $k_0 \approx k_*$. For a numerical example let us pick up the system parameters given in [21]: $J = J_1 = 0.0071$ kg m, $m = m_1 = 0.3$ kg, $l = l_1 = 0.150$ m, $l_0 = 0.02$ m, $\rho = \rho_1 = 0.013$ Nm s. Computing from (12), (13) gives $k_* = 286$ N/m, $k_0 = 287$ N/m. Note that such the values of k correspond to a "stiff" spring; in the paper [21] the value $k = 80$–120 N/m is considered.

Let us make some summarizing remarks.

(1) There exists a single *limit cycle* for $k \in [0, k_*]$. It has the frequency $\omega = \tilde{\omega}_0 = J^{-1}\sqrt{J(\beta^2 + kl_0^2)}$ (see Eq. (8)) and the amplitude given by Eq. (7). It can be easily checked that this limit cycle is stable.

(2) If $k > k_*$ then the limit cycles with the frequencies $\tilde{\omega}_1$, $\tilde{\omega}_2$ (see (10)) born. These cycles have the same oscillation amplitudes, see (9). The cycle of item 1 with the frequency $\tilde{\omega}_0$ becomes *unstable*. Accordingly with the HL method, both the limit cycles with the frequencies $\tilde{\omega}_1$, and $\tilde{\omega}_2$ are stable. To divide the system state space on the regions of attraction of each limit cycle some different methods must be used, because the harmonic linearization technique does not give a definite answer in that case.

Note that the formulae (7) and (9) give oscillation amplitudes for the output of the "linear part" of the system. In the considered case this output is $\dot{\varphi}_1$. To find the amplitude of $\varphi_1(t)$ the amplitudes given by the formulae (7) and (9) should be divided by the corresponding frequencies obtained from (8) and (10). Therefore, the amplitude of oscillations with the frequency $\tilde{\omega}_1$ (see (10)) is less that one with the frequency $\tilde{\omega}_2$.

Let us find the *phase shift* between the processes $\varphi_1(t)$ and $\varphi_2(t)$ in the steady-state oscillation mode. For the sake of it let us find the transfer functions of the system (2) from input u to the outputs φ_1 and φ_2

$$W_{\varphi_1}(s) = \frac{Js^2 + \rho s + \beta^2 + kl_0^2}{A(s)}, \tag{14}$$

$$W_{\varphi_2}(s) = \frac{kl_0^2}{A(s)}, \tag{15}$$

where common denominator $A(s)$ is

$$A(s) = (Js^2 + \rho s + \beta^2)(Js^2 + \rho s + \beta^2 + 2kl_0^2)$$

(see also (4)). The phase shifts ψ_1 and ψ_2 between the input $u(t)$ and the output processes $\varphi_1(t)$ and $\varphi_2(t)$ are found, correspondingly, as $\psi_1(\omega) = \arg(-J\omega^2 + \mathrm{j}\rho\omega + \beta^2 + kl_0^2) - \arg\, A(\mathrm{j}\omega)$, $\psi_2(\omega) = -\arg\, A(\mathrm{j}\omega)$. Therefore, the phase shift $\Delta\psi = \psi_1 - \psi_2$ between the pendulum turning angles $\varphi_1(t)$ and $\varphi_2(t)$ depends on oscillations frequency ω and can be found as

$$\begin{aligned}\Delta\psi(\omega) &= \arg(-J\omega^2 + \mathrm{j}\rho\omega + \beta^2 + kl_0^2) \\ &= \begin{cases} \arctan \dfrac{\rho\omega}{U(\omega)} & \text{if } \omega^2 < kl_0^2 + \beta^2, \\ \frac{\pi}{2} & \text{if } \omega^2 = kl_0^2 + \beta^2, \\ \pi + \arctan \dfrac{\rho\omega}{U(\omega)} & \text{if } \omega^2 > kl_0^2 + \beta^2, \end{cases}\end{aligned} \tag{16}$$

where $U(\omega) = kl_0^2 + \beta^2 - J\omega^2$.

Substituting ω_i $(i = 0, 1, 2)$ for ω in (16) one finds the phase shifts $\Delta\psi_i$ for the self-excited oscillations.

In the case of $k < k_*$, the oscillations frequency is defined by (8). Hence it is valid that $\tilde{\omega}^2 = kl_0^2 + \beta^2$ and, accordingly with (16), $\Delta\psi(\tilde{\omega}) = \pi/2$. Thus, if the spring elasticity is "small" (i.e. $k < k_*$) the phase shift between pendulums turning angles is a

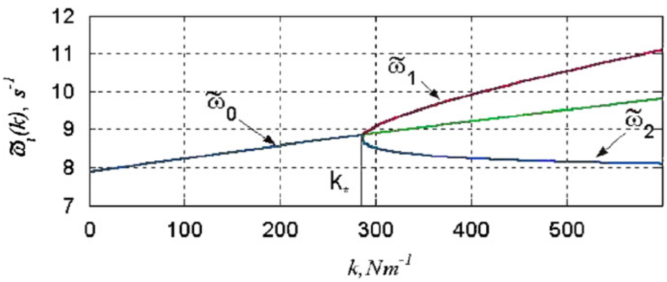

Fig. 2. Oscillation frequencies $\tilde{\omega}_i$ vs k.

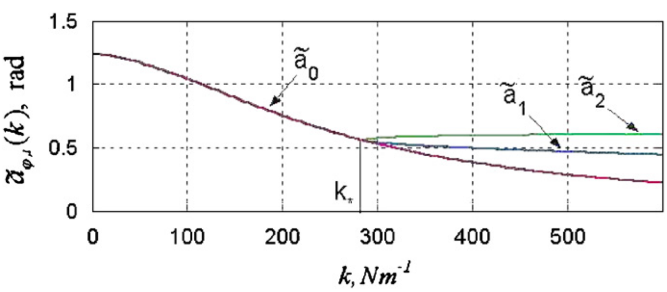

Fig. 3. Oscillation amplitudes $\tilde{a}_{\varphi,i}$ vs k; $\gamma = 0.1$ Nm.

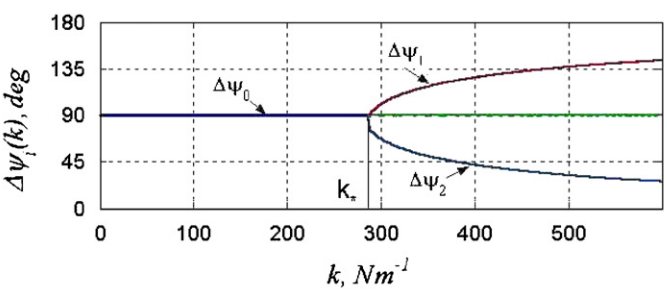

Fig. 4. Phase shifts $\Delta\psi_i$ vs k.

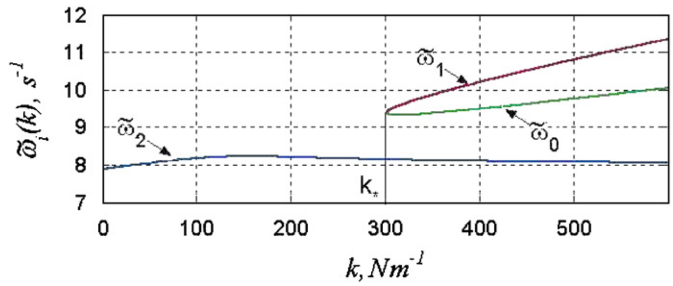

Fig. 5. Oscillation frequencies $\tilde{\omega}_i$ vs k. Non-identical pendulums.

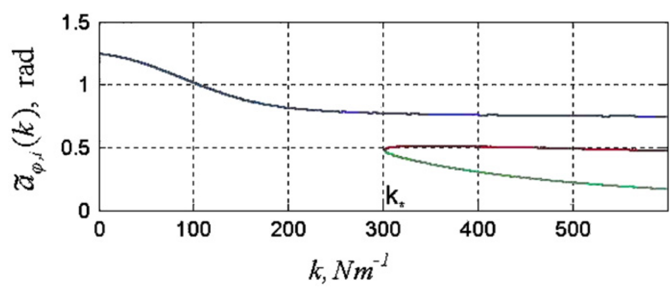

Fig. 6. Oscillation amplitudes $\tilde{a}_{\varphi,i}$ vs k; γ=0.1 Nm. Non-identical pendulums.

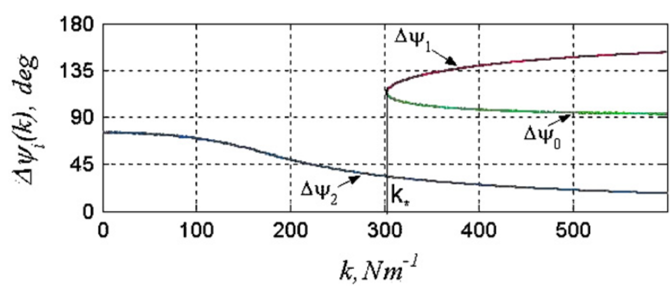

Fig. 7. Phase shifts $\Delta\psi_i$ vs k. Non-identical pendulums.

quarter-period. If k is greater than k_* the oscillation frequencies are $\tilde{\omega}_1$ or $\tilde{\omega}_2$. Substituting $\tilde{\omega}_1$ and $\tilde{\omega}_2$ from (10) for ω in (16) one can find the phase shift as a function of system parameters. Define $\Delta\psi_{k,1} \triangleq \Delta\psi(\omega)|_{\omega=\tilde{\omega}_1(k)}$ and $\Delta\psi_{k,2} \triangleq \Delta\psi(\omega)|_{\omega=\tilde{\omega}_2(k)}$. Evidently, if $k \to \infty$ it is valid that $\Delta\psi_{k,1} \to \pi$, and $\Delta\psi_{k,2} \to 0$. Therefore, for "large" coupling parameter k two stable oscillating modes with different frequencies appear. One of these modes trends toward to the *inphase* motion ($\Delta\psi_k \approx 0$), another one trends to the *antiphase* motion ($\Delta\psi_k \approx \pi$).

Let us make some graphic illustrations. For the above given system parameters the values of $\tilde{\omega}_i$ and $\tilde{a}_{\varphi,i} \equiv \tilde{a}_i/\tilde{\omega}_i$ and $\Delta\psi_i$ are calculated and plotted as the functions of the coupling parameter k in Figs. 2–4. The oscillations amplitudes are found for $\gamma = 0.1$ Nm in (3). The bifurcation value of $k = k_*$ is also marked.

3.2. Case of the non-identical pendulums

Consider now the case of different pendulum parameters. Such a case is more complex that the foregoing one and for the sake of brevity let us avoid any analytical expressions and be confined studying the numerical example. Assume that in (1) the moments of inertia and the frictional coefficients of the pendulums are given as $J_1 = 0.0071$ kg m, $J_2 = 0.0068$ kg m, $\rho_1 = 0.013$ Nm s, $\rho_2 = 0.009$ Nm s, $l_1 = l_2 = 0.150$ m, $m_1 = m_2 = 0.3$ kg.

The plots of the solutions $\tilde{\omega} = \tilde{\omega}(k)$ and $\tilde{a} = \tilde{a}(k)$ of the harmonic balance equation for that case and the corresponding phase shifts $\Delta\psi$ are shown in Figs. 5–7. It is seen that the bifurcation is "hard" for that case: there exists a mode hopping in the bifurcation point $k = k_* \approx 300$ N/m. Accordingly with the HL method it can be easily shown that the oscillations with "intermediate" frequency $\tilde{\omega}_0$ (see Fig. 5) are unstable. The stable oscillations with the both frequencies $\tilde{\omega}_1$ and $\tilde{\omega}_2$ can be engendered in the system if $k > k_*$. Note that the greater frequency $\tilde{\omega}_1$ corresponds to the approximately antiphase motion ($\Delta\psi_1$ plot in Fig. 7), while the frequency $\tilde{\omega}_2$ corresponds to the inphase one ($\Delta\psi_2$ plot). This effect has a lucid physical explanation: in the antiphase mode the spring torque is added to the torque of the gravitational force.

The bifurcation diagram can be elucidated by means of the Nyquist plots of the linear system $W(j\omega)$. For the excitation law (3), the HL of the non-linear element is real and merely amplitude-dependent, therefore the limit cycles corresponds to the positive roots of $V(\omega)$, where it is defined $W(j\omega) \triangleq U(\omega) + jV(\omega)$. Nyquist plots for the cases of $k = 50$, 300, and 600 N/m are shown in Fig. 8. The asterisks mark solutions for stable oscillations, while points correspond to the unstable ones.

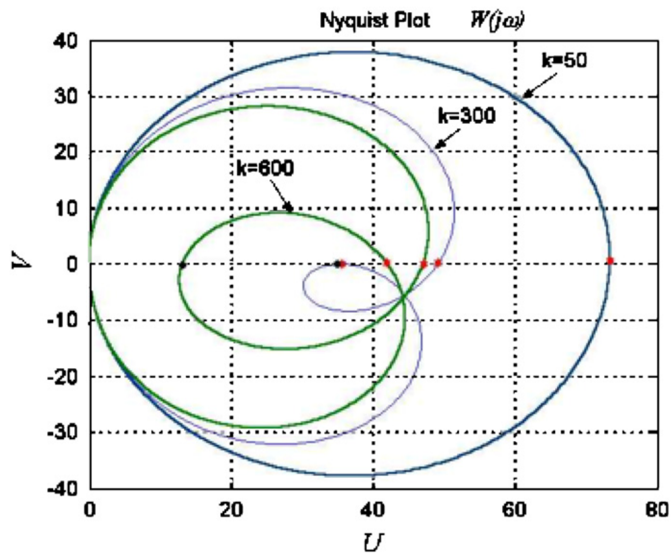

Fig. 8. Nyquist plots for $k = 50$, 300, and 600 N/m.

4. Numerical analyses of the phase relations. The single-input system case

The HL method is the approximate one. Strictly speaking, its application even for oscillation stability analyses is based on heuristic reasons. Besides, we need to get a definite answer about the posed problem of phase relations: what kind of motion (inphase or antiphase) appears in the system? Let us use the computer simulations for solving a problem for the particular set of the system parameters. The values of parameters are given above in Section 3.2.

In our experiments, the closed-loop system (1) and (3) was repeatedly simulated for the different initial conditions $(\varphi_1(0), \varphi_2(0))$ and the coupling parameter k. The simulations was made for time interval $t \in [0, t_{\text{fin}}]$, $t_{\text{fin}} = 50$ s. Time histories of $\varphi_1(t)$ and $\varphi_2(t)$ for the latest τ seconds ($\tau = 15$ s) was processed to find oscillation parameters (frequency Ω, angular amplitudes, and phase shift). Some results are presented in Figs. 9–14.

In Figs. 9 and 10 the 3D-plots of the oscillations frequency and phase shift versus initial value of φ_2 and coupling gain k are pictured. Initial condition on φ_1 is taken equal to one radian. The control magnitude $\gamma = 0.1$ Nm. One can observe that both inphase and antiphase steady-state motions occur if

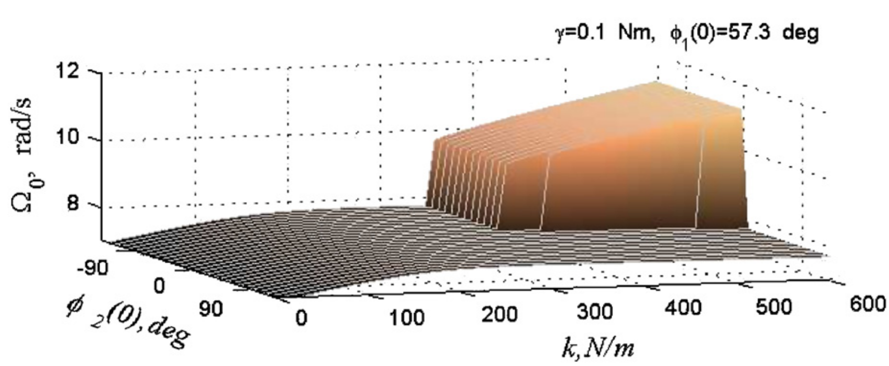

Fig. 9. Oscillation frequency Ω vs k and $\varphi_2(0)$.

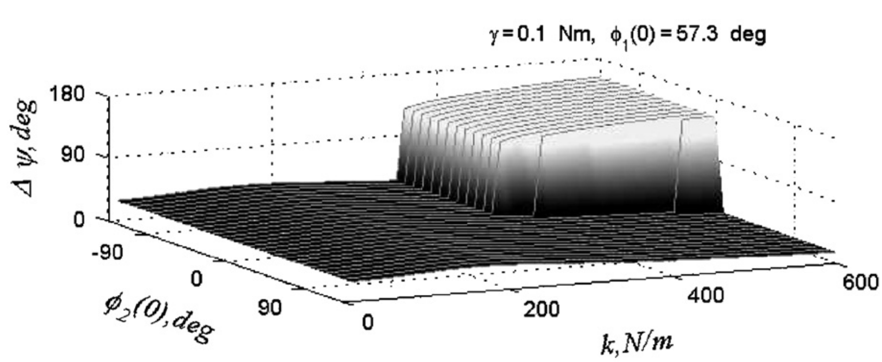

Fig. 10. Phase shift $\Delta\psi$ vs k and $\varphi_2(0)$.

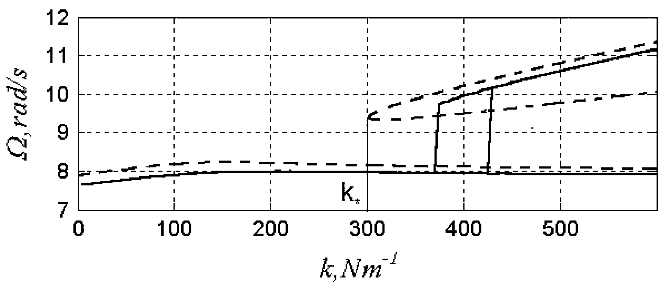

Fig. 11. Oscillation frequencies vs k.

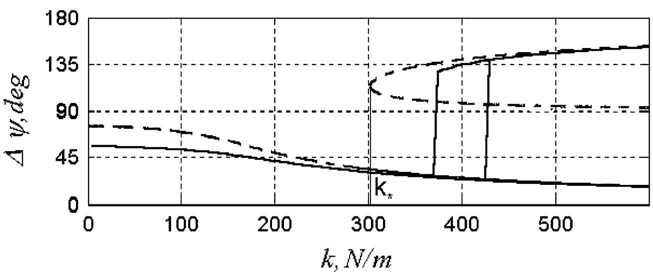

Fig. 12. Phase shifts vs k.

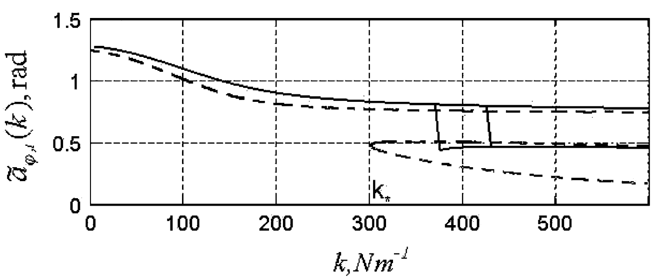

Fig. 13. Oscillation amplitudes vs k.

the coupling gain is sufficiently large. For comparison of the analytical results in Section 3 and the numerical ones let us examine Figs. 11–13. The dashed lines show analytically obtained results, the solid lines correspond to numerical results. The different initial conditions on φ_2 are taken, therefore, the solid curves, in Fig. 11 are projections of the surface shown in Fig. 9 on the (k, Ω)-plane. For more detailed information about phase shift dependence on the initial conditions, let us consider Fig. 14, where the "topographical maps" (contour plots) for the phase shift $\Delta\psi$ on the plane $(\text{ph}_1(0), \varphi_2(0))$ are pictured. As it is seen from the plots, for the small values of k the phase shift is approximately 50° and less. For the values of k greater than the bifurcation one, the phase shift depends also on initial conditions.

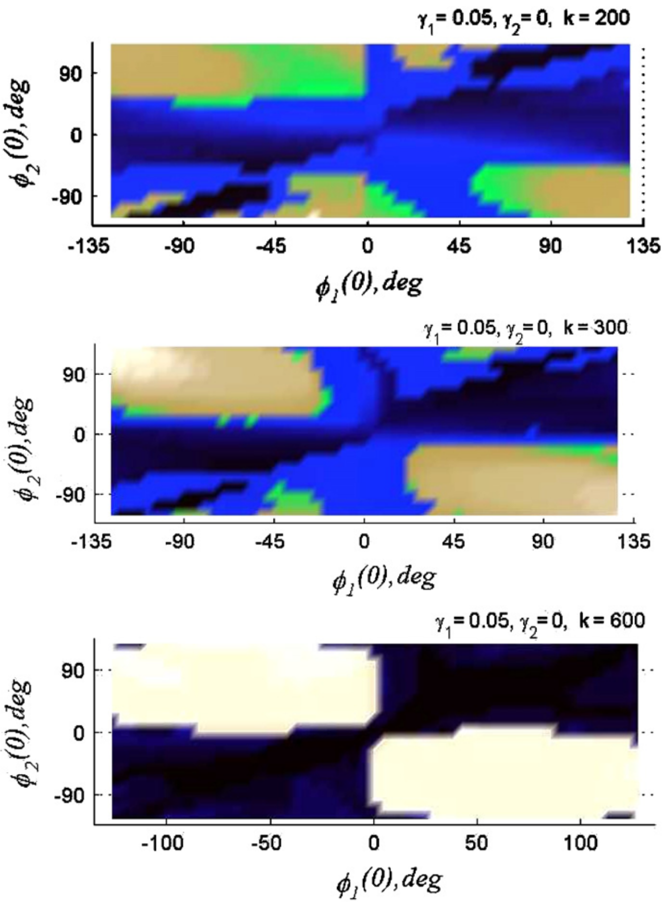

Fig. 14. "Topographical maps" for $\Delta\psi$.

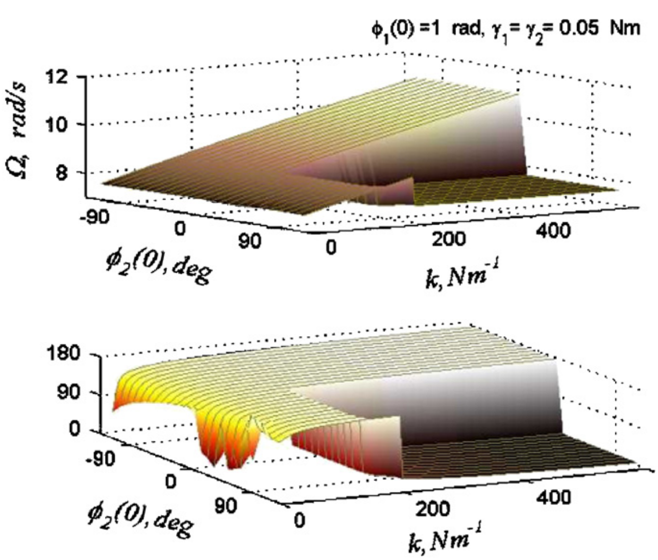

Fig. 15. Oscillation frequency Ω and phase shift $\Delta\psi$ vs $\varphi_2(0)$ and k.

5. Phase relations. Double-input system case

Consider now a model of the pendulum system with two external torques u_1 and u_2:

$$\begin{cases} J_1\ddot{\varphi}_1 + \rho_1\dot{\varphi}_1 + m_1 g l_1 \sin\varphi_1 = f(\varphi_1, \varphi_2)\cos\varphi_1 + u_1(t), \\ J_2\ddot{\varphi}_2 + \rho_2\dot{\varphi}_2 + m_2 g l_2 \sin\varphi_2 = -f(\varphi_1, \varphi_2)\cos\varphi_2 + u_2(t), \end{cases}$$

$$f(\varphi_1, \varphi_2) = -k l_0^2(\sin\varphi_1 - \sin\varphi_2) \tag{17}$$

Similarly to (3) the excitation law is taken as

$$u_1 = \gamma_2 \operatorname{sign}\dot{\varphi}_1, \quad u_2 = \gamma_2 \operatorname{sign}\dot{\varphi}_2. \tag{18}$$

Some numerical results of examination the closed-loop system (17) and (18) are presented in Figs. 15 and 16. It is seen that

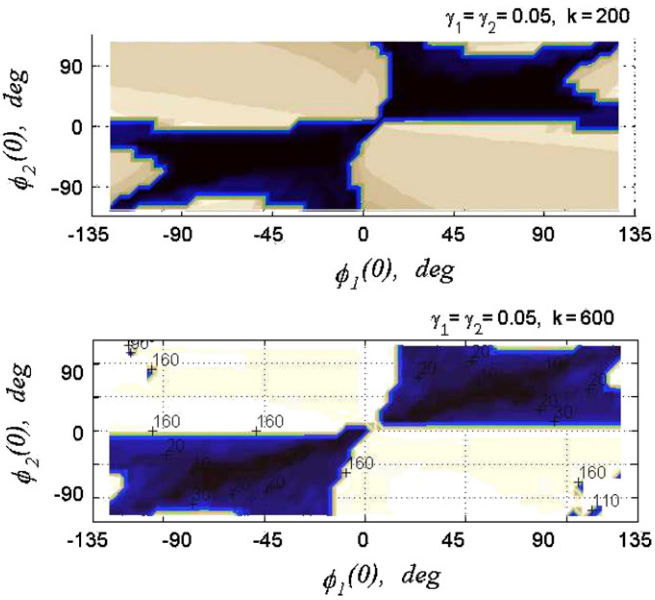

Fig. 16. "Topographical maps" for $\Delta\psi$.

antiphase synchronization occurs for "small" coupling k and for any initial conditions. For greater k the steady-state motion can be inphase (if the initial angles have the same sign), or antiphase (in the case of the opposite signs of the initial angles).

6. Conclusions

In the present work the coupled pendulum system excited via relay differential feedback is studied analytically by means of the HL method, and numerically, by means of the computer simulations. Following conclusions can be drawn:

- there exists frequency synchronization in the steady-state mode. The pendulum deviation angles have the same oscillation frequency and some constant phase shift;
- if only one pendulum is excited, for the small gain k and small angular deflections the phase shift is about $\pi/2$ (a quarter-period). In the same case if the amplitudes are "large" the approximately inphase oscillations occur;
- the steady-state oscillations are antiphase if k is not large enough and both the pendulums are excited;
- for large k both inphase and antiphase motions can occur;
- the phase shift in the last case is determined by means of the initial conditions;
- the frequency of the inphase motions is less that the frequency of the antiphase motion;
- for the case of small damping ρ, the HL method gives the solution that is accurate from the practical viewpoint.

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