Robust nonlinear sampled-data system analysis based on Fridman's method and S-procedure

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Abstract

This paper is devoted to the evaluation of sampling interval providing robust exponential stability of nonlinear system with sector bounded nonlinearities. It extends our previous results (R.E. Seifullaev, A.L. Fradkov. Sampled-Data Control of Nonlinear Oscillations Based on LMIs and Fridmans Method. In 5th IFAC International Workshop on Periodic Control Systems, 95-100. Caen, France. 2013). The proposed approach exploats E.Fridmans method for linear systems based on a general time-dependent Lyapunov-Krasovskii functional. With classical results of V.A. Yakubovich about S-procedure the problem is reduced to feasibility analysis of linear matrix inequalities. The results are illustrated by example: the pendulum system with friction and sector bounded multiple nonlinearities.

1 Introduction

A long standing problem in computer controlled systems analysis and design is evaluation of admissible sampling period. Since the 1950s a number of approaches to this problem were proposed [1, 2, 3]. However conservativeness reduction of the sampling period estimates is still of interest even for linear systems. The problem has become still more important after broad propagation of networked control. E.g. in [18] a solution to the approximate tracking problem of sampled-data systems with uncertain, time-varying sampling intervals and delays is presented and sufficient conditions for the input-to-state stability of the tracking error dynamics with respect to this perturbation are given.

An efficient approach to estimation of the admissible sampling period is being developed for more than two decades by Emilia Fridman with coauthors. It is based on the interpretation of a networked control system as a continuous-time delayed system with time-varying (sawtooth) delay ("Input-delay method") [4, 5, 6, 7, 8, 9, 10]. Early results [4, 5] were significantly extended and powered with the so called descriptor method [6], proposed by E.Fridman. In [7] the input delay approach was extended to robust stabilization. It was further refined in [8] with the novel time-dependent Lyapunov-Krasovskii functional and efficient LMI solvers this method has become significantly less conservative. In this paper we will use the version of the input delay approach formulated and justified in [8]. This version will be called "Fridman's method" throughout this paper.

Until recent the existing results on Fridman's method and its extensions (e.g. [19]) were applied only to sampled-data linear systems and switched linear systems([13, 14, 15, 16, 10]), up to authors' knowledge. Some results were extended to nonlinear Lurie systems in [12]. This paper is devoted to further extension of this method to include both robust and nonlinear settings. Namely the structured uncertainty in plant parameters and sector bounded nonlinearities are considered.

There exist a number of papers related to evaluation of sampling period for nonlinear systems. For example, in [24] the analytic conditions for the delay size and the maximum sampling interval ensuring stability of affine system with delay and sampling in feedback are given. In paper [25] the existence of a nonzero sampling interval for ISS nonlinear systems is shown. In [23] the stabilization of nonlinear system with delayed and sampled-data control is studied, where it is shown that sampled- data feedback laws with a predictor-based delay compensation can guarantee global asymptotic stability for the closed-loop system with no restrictions for the magnitude of the delays and arbitrarily long sampling period. Paper [20] investigates the stabilization problem of the nonlinear networked control systems with drops and variable delays. In [26] the synchronization algorithm for chaotic Lurie systems using sampled-data control is proposed.

In this paper we study the simple for implementation zero-order-hold linear controller for robust stabilization of nonlinear Lurie systems with sector bounded nonlinearities. The contribution of this paper is the condition for sampling interval size providing exponential stability of sample-data system. To verify our condition one needs to check feasibility of a number of new linear matrix inequalities. The key tools to obtain the results is application of Fridman's method and Yakubovich's S-procedure. In the paper we also examine an example: the pendulum system with friction and sector bounded multiple nonlinearities. The system is closed by sampled-time linear state feedback. Our aim is to evaluate the upper bound of the sampling interval below which the system is absolutely stable. Evaluation of the maximum sampling period ensuring exponential stability is performed for several cases: with known parameters, with unknown friction, with unknown mass, with unknown length and with unknown all those parameters at once. Comparison with Matlab simulation results demonstrates good quality of the estimates.

2 Problem formulation

Consider the uncertain nonlinear system

$$\begin{aligned}
\dot{x}(t) &= (A + \Delta A)\, x(t) + \sum_{i=1}^{k_1} (\tilde{q}_i + \Delta\tilde{q}_i)\, \tilde{\xi}_i(t) + \Big(B + \Delta B \tilde{\xi}_0(t)\Big)\, u(t), \\
\tilde{\sigma}_0(t) &= \tilde{r}_0^T x(t), \quad \tilde{\xi}_0(t) = \tilde{\varphi}_0(\tilde{\sigma}_0(t), t), \\
\tilde{\sigma}_i(t) &= \tilde{r}_i^T x(t), \quad \tilde{\xi}_i(t) = \tilde{\varphi}_i(\tilde{\sigma}_i(t), t), \quad i = 1, \ldots, k_1,
\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control function, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are constant matrices, $\tilde{q}_i \in \mathbb{R}^n, \tilde{r}_i \in \mathbb{R}^n, \tilde{r}_0 \in \mathbb{R}^n$ are constant vectors.

Assume that $\tilde{\xi}_i(t) = \tilde{\varphi}_i(\tilde{\sigma}_i(t), t)$ are nonlinear scalar functions (see Fig.1) satisfying

$$\tilde{\mu}_i^- \, \tilde{\sigma}_i^2 \leqslant \tilde{\sigma}_i \, \tilde{\xi}_i \leqslant \tilde{\mu}_i^+ \, \tilde{\sigma}_i^2, \quad i = 1, \dots, k_1, \tag{2}$$

for all $t \geqslant 0$, where $\tilde{\mu}_i^- \leqslant \tilde{\mu}_i^+$ are real numbers. Let scalar nonlinear function $\tilde{\xi}_0(t) = \tilde{\varphi}_0(\tilde{\sigma}_0(t), t)$ be bounded for all $t \geqslant 0$

$$\tilde{\varphi}_0^- \leqslant \tilde{\xi}_0(t) \leqslant \tilde{\varphi}_0^+, \tag{3}$$

where $\tilde{\varphi}_0^- \leqslant \tilde{\varphi}_0^+$ are real numbers.

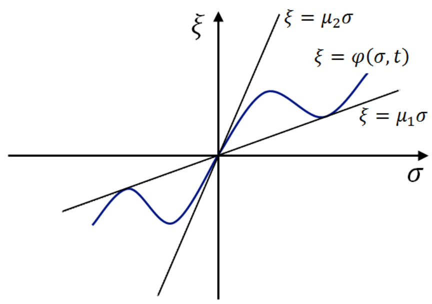

Figure 1: Sector bounded nonlinearity

Assume that uncertainties $\Delta A, \Delta\tilde{q}_i, \Delta B$ are structured as follows:

$$\begin{aligned} \Delta A &= \sum_{l=1}^{k_2} \bar{q}_l \, a_l \, \bar{r}_l^T, \\ \Delta \tilde{q}_i &= \sum_{j=1}^{k_3} \bar{q}_{ij} \, a_{ij}, \quad i = 1, \dots, k_1, \\ \Delta B &= B_0 \, b, \end{aligned} \tag{4}$$

where $\bar{q}_l \in \mathbb{R}^n, \bar{r}_l \in \mathbb{R}^n (l = 1, \dots, k_2)$, $\bar{q}_{ij} \in \mathbb{R}^n, (i = 1, \dots, k_1, \quad j = 1, \dots, k_3)$ are known constant vectors, $B_0 \in \mathbb{R}^{n \times m}$ is known constant matrix, and a_l, a_{ij}, b are unknown real numbers satisfying

$$\begin{aligned} 0 < a_l^- &\leqslant a_l \leqslant a_l^+, \\ 0 < a_{ij}^- &\leqslant a_{ij} \leqslant a_{ij}^+, \\ 0 < b^- &\leqslant b \leqslant b^+, \end{aligned} \tag{5}$$

where $a_l^-, a_l^+, a_{ij}^-, a_{ij}^+, b^-, b^+ \, (l = 1, \dots, k_2, \quad i = 1, \dots, k_1, \quad j = 1, \dots, k_3)$ are known positive real numbers.

Given a sequence of sampling times $0 = t_0 < t_1 < \ldots < t_k < \ \ldots$ and a piecewise constant control function

$$u(t) = u_d(t_k), \quad t_k \leqslant t < t_{k+1}, \tag{6}$$

where $\lim\limits_{k\to\infty} t_k = \infty$.

Assume that $h \in \mathbb{R}\,(h > 0)$ and

$$t_{k+1} - t_k \leqslant h, \quad \forall k \geqslant 0 \tag{7}$$

and consider a sampled-time control law

$$u(t) = Kx(t_k), \;\; t_k \leqslant t < t_{k+1}, \tag{8}$$

where $K \in \mathbb{R}^{m\times n}$. The law (8) can be rewritten as follows:

$$u(t) = Kx(t - \tau(t)), \tag{9}$$

where $\tau(t) = t - t_k$, $t_k \leqslant t < t_{k+1}$.

It's required to analyze the influence of the upper bound h of sampling intervals on the closed-loop system exponential stability:

$$\begin{aligned}
\dot{x}(t) &= (A + \Delta A)\, x(t) + \left(B + \Delta B\tilde{\xi}_0(t)\right) Kx(t - \tau(t)) + \sum_{i=1}^{k_1} (\tilde{q}_i + \Delta\tilde{q}_i)\, \tilde{\xi}_i(t),\\
\tilde{\sigma}_0(t) &= \tilde{r}_0^T x(t), \quad \tilde{\xi}_0(t) = \tilde{\varphi}_0(\tilde{\sigma}_0(t), t),\\
\tilde{\sigma}_i(t) &= \tilde{r}_i^T x(t), \quad \tilde{\xi}_i(t) = \tilde{\varphi}_i(\tilde{\sigma}_i(t), t), \quad i = 1, \ldots, k_1\\
\tau(t) &= t - t_k, \;\; t \in [t_k, t_{k+1}).
\end{aligned} \tag{10}$$

Remark 1 *Let us provide some examples of sector bounded nonlinearities, satisfying (2):*

- $\xi = sin(\sigma)$*:* $\mu_1 \approx -0.2173$, $\mu_2 = 1$ *(see. Fig. 2),*
- $\xi = sin(\sigma^2)$*:* $\mu_1 \approx -0.855$, $\mu_2 \approx 0.855$ *(see. Fig. 3),*
- *relay with dead zone, saturation, piecewise-linear function etc. (see [11]).*

Let us give an example of mechanical system described by equation (1). Consider the system, corresponding to the computer controlled pendulum with friction:

$$\begin{aligned}
&\ddot{\varphi}(t) = \frac{g}{l} sin(\varphi(t)) - \frac{\kappa}{l}\dot{\varphi}(t) + \frac{1}{ml^2}u(t),\\
&u(t) = Kx(t - \tau(t)),\\
&\tau(t) = t - t_k, \;\; t \in [t_k, t_{k+1}), \;\; t_{k+1} - t_k = h, \;\; k = 0, 1 \ldots,
\end{aligned} \tag{11}$$

where l and m are the length and the mass of the pendulum respectively, κ is a viscous friction constant, φ is the deviation angle of the pendulum from vertical ($\varphi = 0$ at the upper position), u is the control torque, $x(t) = [\varphi(t),\ \dot{\varphi}(t)]^T$.

Let friction κ be unknown, and

$$0 < \kappa_1 \leqslant \kappa \leqslant \kappa_2.$$

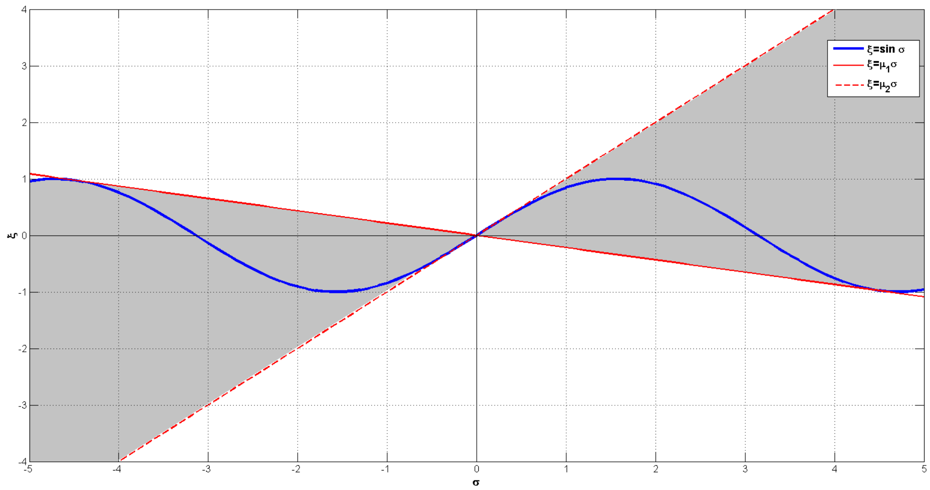

Figure 2: $\xi = sin(\sigma)$

System (11) can be rewritten as follows:

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) + q_1\xi_1(t) + Bu(t), \\ \sigma_1(t) &= r_1^T x(t), \quad \xi_1(t) = \sin\sigma_1(t), \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Delta A = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \kappa \begin{bmatrix} 0 & -\frac{1}{l} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \dfrac{1}{ml^2} \end{bmatrix},$$

$$q_1 = \begin{bmatrix} 0 \\ \dfrac{g}{l} \end{bmatrix}, \quad q_2 = \begin{bmatrix} 0 \\ -\dfrac{1}{l} \end{bmatrix}, \quad r_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

ξ_1 and a for all $t \geqslant 0$ satisfy

$$\mu_1\sigma_1^2 \leqslant \sigma_1\,\xi_1 \leqslant \mu_2\sigma_1^2.$$

3 Preliminaries

Definition 1 *The space of absolutely continuous on* $[-h, 0)$ *functions* $f : [-h, 0] \to I\!R^n$ *having square integrable first-order derivatives is denoted by* W *with the norm* $\|f\|_W = \max\limits_{\theta\in[-h,0]} |f(\theta)| + \left[\int_{-h}^0 |\dot{f}(s)|^2 ds\right]^{\frac{1}{2}}$.

Denote $x_t(\theta) : [-h, 0] \to I\!R^n$ as $x_t(\theta) = x(t + \theta)$, where $x(\theta) \equiv 0$ if $\theta \in [-h, 0)$.

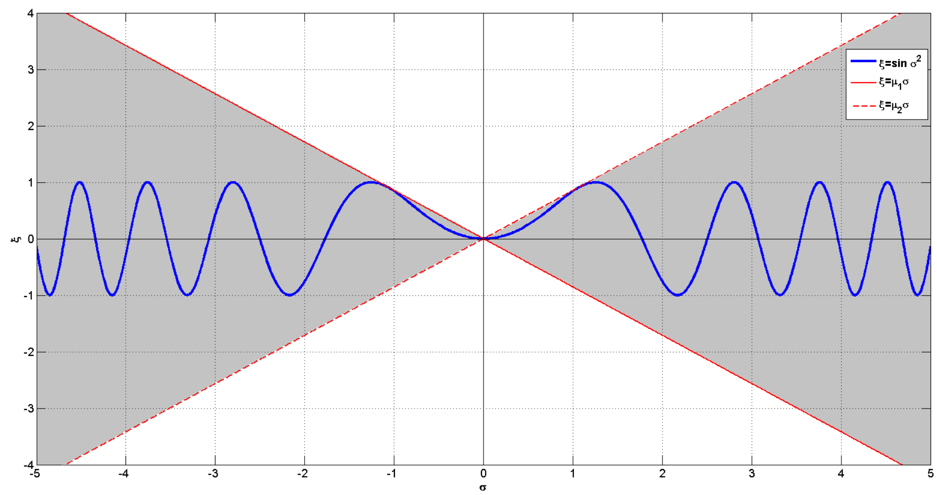

Figure 3: $\xi = sin(\sigma^2)$

Definition 2 *System* (10) *will be called exponentially stable with the decay rate* $\alpha > 0$ *if there exists* $\beta > 0$ *such that for solution* $x(t)$ *of* (10) *with initial condition* x_{t_0} *the following estimate holds*

$$|x(t)|^2 \leqslant \beta e^{-2\alpha(t-t_0)} \|x_{t_0}\|_W^2, \ \forall t \geqslant t_0.$$

The proof of our main result is based on the following auxiliary statement that can be proved along the lines of Lemma 1 in [8].

Lemma 1 *Let there exist positive numbers* β_1, β_2 *and a functional* $V : I\!R \times W \times L_2[-h, 0] \to I\!R$ *such that*

$$\beta_1 |\phi(0)|^2 \leqslant V(t, \phi, \dot{\phi}) \leqslant \beta_2 \|\phi\|_W^2. \tag{12}$$

Let the function $\bar{V}(t) = V(t, x_t, \dot{x}_t)$ *be continuous from the right for* $x(t)$ *satisfying* (10)*, absolutely continuous for* $t \neq t_k$ *and satisfies*

$$\lim_{t \to t_k^-} \bar{V}(t) \geqslant \bar{V}(t_k). \tag{13}$$

Given $\alpha > 0$ *if along* $x(t)$

$$\dot{\bar{V}}(t) + 2\alpha \bar{V}(t) \leqslant 0 \tag{14}$$

almost for all t *then* (10) *is exponentially stable with the decay rate* α.

4 Main result

Introduce new variables as follows:

$$\sigma_0(t) = r_0^T x(t), \quad \xi_0(t) = \varphi_0(\sigma_0(t), t),$$
$$\sigma_i(t) = r_i^T x(t), \quad \xi_i(t) = \varphi_i(\sigma_i(t), t), \quad i = 1, \dots, N,$$

where

$$N = k_1 + k_2 + k_1 k_3, \quad r_0 = \tilde{r}_0, \quad q_i = \tilde{q}_i \quad \text{for} \quad i = 1, \dots, k_1,$$
$$r_i = \bar{r}_{j(i)}, \quad q_i = \bar{q}_{j(i)} \quad \text{for} \quad i = k_1 + 1, \dots, k_1 + k_2, \quad j(i) = i - k_1,$$
$$r_i = \tilde{r}_1, \quad q_i = \bar{q}_{1j(i)} \quad \text{for} \quad i = k_1 + k_2 + 1, \dots, k_1 + k_2 + k_3, \quad j(i) = i - k_1 - k_2,$$
$$r_i = \tilde{r}_2, \quad q_i = \bar{q}_{2j(i)} \quad \text{for} \quad i = k_1 + k_2 + k_3 + 1, \dots, k_1 + k_2 + 2k_3, \quad j(i) = i - k_1 - k_2 - k_3,$$
$$\vdots$$
$$r_i = \tilde{r}_{k_1}, \quad q_i = \bar{q}_{k_1 j(i)} \quad \text{for} \quad i = k_1 + k_2 + (k_1 - 1)\, k_3 + 1, \dots, N, \quad j(i) = i - N + k_3,$$
$$\varphi_0(\sigma_0(t), t) = b\, \tilde{\varphi}_0(\sigma_0(t), t),$$
$$\begin{cases} \varphi_0^- = b^-\, \tilde{\varphi}_0^-, & \varphi_0^+ = b^+\, \tilde{\varphi}_0^+, \quad \text{if} \quad \tilde{\varphi}_0^- \geqslant 0, \\ \varphi_0^- = b^+\, \tilde{\varphi}_0^-, & \varphi_0^+ = b^+\, \tilde{\varphi}_0^+, \quad \text{if} \quad \tilde{\varphi}_0^- < 0,\, \tilde{\varphi}_0^+ > 0, \\ \varphi_0^- = b^+\, \tilde{\varphi}_0^-, & \varphi_0^+ = b^-\, \tilde{\varphi}_0^+, \quad \text{if} \quad \tilde{\varphi}_0^+ \leqslant 0, \end{cases}$$

$$\xi_i(t) = \tilde{\xi}_i(t), \quad \mu_i^- = \tilde{\mu}_i^-, \quad \mu_i^+ = \tilde{\mu}_i^+ \quad \text{for} \quad i = 1, \dots, k_1;$$
$$\xi_i(t) = a_{j(i)} \sigma_i(t), \quad \mu_i^- = a_{j(i)}^-, \quad \mu_i^+ = a_{j(i)}^+$$
$$\text{for} \quad i = k_1 + 1, \dots, k_1 + k_2, \quad j(i) = i - k_1;$$
$$\xi_i(t) = a_{1j(i)} \tilde{\xi}_1(t), \quad \begin{cases} \mu_i^- = a_{1j(i)}^- \tilde{\mu}_1^-, & \mu_i^+ = a_{1j(i)}^+ \tilde{\mu}_1^+, \quad \text{if} \quad \tilde{\mu}_1^- \geqslant 0, \\ \mu_i^- = a_{1j(i)}^+ \tilde{\mu}_1^-, & \mu_i^+ = a_{1j(i)}^+ \tilde{\mu}_1^+, \quad \text{if} \quad \tilde{\mu}_1^- < 0,\, \tilde{\mu}_1^+ > 0, \\ \mu_i^- = a_{1j(i)}^+ \tilde{\mu}_1^-, & \mu_i^+ = a_{1j(i)}^- \tilde{\mu}_1^+, \quad \text{if} \quad \tilde{\mu}_1^+ \leqslant 0, \end{cases}$$
$$\text{for} \quad i = k_1 + k_2 + 1, \dots, k_1 + k_2 + k_3, \quad j(i) = i - k_1 - k_2;$$
$$\xi_i(t) = a_{2j(i)} \tilde{\xi}_2(t), \quad \begin{cases} \mu_i^- = a_{2j(i)}^- \tilde{\mu}_2^-, & \mu_i^+ = a_{2j(i)}^+ \tilde{\mu}_2^+, \quad \text{if} \quad \tilde{\mu}_2^- \geqslant 0, \\ \mu_i^- = a_{2j(i)}^+ \tilde{\mu}_2^-, & \mu_i^+ = a_{2j(i)}^+ \tilde{\mu}_2^+, \quad \text{if} \quad \tilde{\mu}_2^- < 0,\, \tilde{\mu}_2^+ > 0, \\ \mu_i^- = a_{2j(i)}^+ \tilde{\mu}_2^-, & \mu_i^+ = a_{2j(i)}^- \tilde{\mu}_2^+, \quad \text{if} \quad \tilde{\mu}_2^+ \leqslant 0, \end{cases}$$
$$\text{for} \quad i = k_1 + k_2 + k_3 + 1, \dots, k_1 + k_2 + 2k_3, \quad j(i) = i - k_1 - k_2 - k_3;$$
$$\vdots$$
$$\xi_i(t) = a_{k_1 j(i)} \tilde{\xi}_{k_1}(t), \quad \begin{cases} \mu_i^- = a_{k_1 j(i)}^- \tilde{\mu}_{k_1}^-, & \mu_i^+ = a_{k_1 j(i)}^+ \tilde{\mu}_{k_1}^+, \quad \text{if} \quad \tilde{\mu}_{k_1}^- \geqslant 0, \\ \mu_i^- = a_{k_1 j(i)}^+ \tilde{\mu}_{k_1}^-, & \mu_i^+ = a_{k_1 j(i)}^+ \tilde{\mu}_{k_1}^+, \quad \text{if} \quad \tilde{\mu}_{k_1}^- < 0,\, \tilde{\mu}_{k_1}^+ > 0, \\ \mu_i^- = a_{k_1 j(i)}^+ \tilde{\mu}_{k_1}^-, & \mu_i^+ = a_{k_1 j(i)}^- \tilde{\mu}_{k_1}^+, \quad \text{if} \quad \tilde{\mu}_{k_1}^+ \leqslant 0, \end{cases}$$
$$\text{for} \quad i = k_1 + k_2 + (k_1 - 1)k_3 + 1, \dots, N, \quad j(i) = i - N + k_3.$$

From (2), (3), (4) and (5) it follows that

$$\varphi_0^- \leqslant \xi_0(t) \leqslant \varphi_0^+, \tag{15}$$

$$\mu_i^- \, \sigma_i^2 \leqslant \sigma_i \, \xi_i \leqslant \mu_i^+ \, \sigma_i^2, \quad i = 1, \ldots, N. \tag{16}$$

Therefore, system (10) can be rewritten as follows:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + (B + B_0\xi_0(t))\, Kx(t - \tau(t)) + \sum_{i=1}^{N} q_i \xi_i(t), \\ &\sigma_0(t) = r_0^T x(t), \quad \xi_0(t) = \varphi_0(\sigma_0(t), t), \\ &\sigma_i(t) = r_i^T x(t), \quad \xi_i(t) = \varphi_i(\sigma_i(t), t), \quad i = 1, \ldots, N, \\ &\tau(t) = t - t_k, \quad t \in [t_k, t_{k+1}). \end{aligned} \tag{17}$$

Let us start with the special case of the constant sampling intervals $t_{k+1} - t_k = h$, $k = 0, 1, \ldots$ Consider the following functional (introduced in [8]) on $\mathbb{R} \times W \times L_2[-h, 0]$:

$$V(t, x_t, \dot{x}_t) = x_t(0)^T P x_t(0) + (h - \tau(t)) \int_{-\tau(t)}^{0} e^{2\alpha s} \, \dot{x}_t^T(s) \, Q \, \dot{x}_t(s) ds + V_1(t, x_t), \tag{18}$$

where P and Q are symmetric positive definite matrices, and

$$V_1(t, x_t) = (h - \tau(t)) \; \zeta^T \begin{bmatrix} \frac{X + X^T}{2} & -X + X_1 \\ * & -X_1 - X_1^T + \frac{X + X^T}{2} \end{bmatrix} \zeta,$$

and $\zeta = col \left\{ x_t(0), x_{t - \tau(t)}(0) \right\}$, $X \in \mathbb{R}^{n \times n}$, $X_1 \in \mathbb{R}^{n \times n}$.

To formulate the main result of the paper let us check the conditions of Lemma 1.

For fulfillment of (12) it is sufficient that

$$\Theta(h) > 0, \tag{19}$$

where

$$\Theta(h) = \begin{bmatrix} P + h\frac{X + X^T}{2} & hX_1 - hX \\ * & -hX_1 - hX_1^T + h\frac{X + X^T}{2} \end{bmatrix}.$$

Indeed,

$$x_t(0)^T P x_t(0) + V_1(t, x_t) = \frac{h - \tau(t)}{h} \; \zeta^T \, \Theta(h) \, \zeta + \frac{\tau(t)}{h} \; \zeta^T \, \Theta(0) \, \zeta \geqslant \beta_1 \, |x_t(0)|^2, \tag{20}$$

where $\beta_1 = \min(\nu_1, \nu_2)$, ν_1 and ν_2 are minimum eigenvalues of P and $\Theta(h)$ respectively.

Define function $\bar{V}(t) = V(t, x_t, \dot{x}_t)$, i.e.

$$\bar{V}(t) = x(t)^T P x(t) + V_Q(t, \dot{x}(t)) + V_1(t, x(t)), \tag{21}$$

where

$$V_Q(t,\dot{x}(t)) = (h-\tau(t))\int_{-\tau(t)}^{0} e^{2\alpha s}\,\dot{x}^T(t+s)\,Q\,\dot{x}(t+s)ds.$$

Note that

- $V_Q \geqslant 0$ and $\lim\limits_{t\to t_k^+} V_Q(t,\dot{x}(t)) = V_Q(t_k,\dot{x}(t_k)) = 0$ because $\tau(t)_{|t=t_k} = 0$,
- $\lim\limits_{t\to t_k^-} V_1(t,x_t) = V_1(t_k,x_{t_k}) = \lim\limits_{t\to t_k^+} V_1(t,x_t) = 0$ (i. e. $\tau(t) = h$ at $t\to t_k^-$ and $\tau(t) = 0$ at $t \to t_k^+$, hence, $x(t) = x(t-\tau(t))$).

Therefore, $\bar{V}(t)$ is continuous from the right and condition (13) holds.

Evaluate the left hand side of (14). Since $\frac{d}{dt}x(t-\tau(t)) = (1-\dot{\tau}(t))\dot{x}(t-\tau(t)) = 0$, we obtain

$$\begin{aligned}
&\dot{\bar{V}}(t) + 2\alpha\bar{V}(t) \leqslant 2x^T(t)P\dot{x}(t) + 2\alpha x^T(t)Px(t) + (h-\tau(t))\dot{x}^T(t)Q\dot{x}(t)\\
&-e^{-2\alpha h}\int_{-\tau(t)}^{0}\dot{x}^T(t+s)Q\dot{x}(t+s)ds - \zeta^T(t)\begin{bmatrix}\frac{X+X^T}{2} & -X+X_1\\ * & -X_1-X_1^T+\frac{X+X^T}{2}\end{bmatrix}\zeta(t)\\
&+(h-\tau(t))\left(\dot{x}^T(t)(X+X^T)x(t) + 2\dot{x}^T(t)(-X+X_1)x(t-\tau(t))\right) + 2\alpha V_1(t,x(t)).
\end{aligned} \tag{22}$$

Denote

$$v_1(t) = \frac{1}{\tau(t)}\int_{-\tau(t)}^{0}\dot{x}(t+s)ds, \tag{23}$$

where right hand side of (23) for $\tau(t) = 0$ is understood as $\lim\limits_{\tau(t)\to 0} v_1 = \dot{x}(t)$.

From the Jensen's inequality [22] we have

$$\int_{-\tau(t)}^{0}\dot{x}^T(t+s)\,Q\,\dot{x}(t+s)ds \geqslant \tau(t)\,v_1^TQv_1. \tag{24}$$

Denote for brevity $\mathcal{B}(t) = B + B_0\xi_0(t)$, $\mathcal{B}^- = B + B_0\varphi_0^-$, $\mathcal{B}^+ = B + B_0\varphi_0^+$. If $x(t)$ is the solution of (10), then the following equalities hold

$$\begin{aligned}
0 &= 2\left[-x(t) + x(t-\tau(t)) + \tau(t)v_1\right]\times\\
&\qquad\times\left[x^T(t)Y_1^T + \dot{x}^T(t)Y_2^T + x^T(t-\tau(t))T^T + \sum_{i=1}^{N}\xi_i q_i^T {Y_3^{(i)}}^T\right],\\
0 &= 2\left[x^T(t)P_2^T + \dot{x}^T(t)P_3^T\right]\left[Ax(t) + \mathcal{B}(t)Kx(t-\tau(t)) + \sum_{i=1}^{N}q_i\xi_i(t) - \dot{x}(t)\right],
\end{aligned} \tag{25}$$

where $P_2 \in \mathbb{R}^{n\times n}, P_3 \in \mathbb{R}^{n\times n}$, $Y_1 \in \mathbb{R}^{n\times n}$, $Y_2 \in \mathbb{R}^{n\times n}$, $Y_3^{(i)} \in \mathbb{R}^{n\times n}$ $(i = 1,\ldots,N)$, $T \in \mathbb{R}^{n\times n}$ are some matrices.

Denote $\eta_1(t) = col\left\{x(t),\dot{x}(t),x(t-\tau(t)),\xi_1(t),\ldots,\xi_N(t),v_1(t)\right\},\ \ \eta_1 \in \mathbb{R}^{4n+N}$.

Adding (25) to the right-hand side of (22) and using (24) we obtain

$$\dot{\bar{V}}(t) + 2\alpha \bar{V}(t) \leqslant \eta^T(t)\Psi(t)\,\eta(t), \tag{26}$$

where

$$\Psi(t) = \begin{bmatrix} \Phi_{11}(t) & \Phi_{12}(t) & \Phi_{13}(t) & \Phi_{14}^{(1)} & \dots & \Phi_{14}^{(N)} & \tau(t)Y_1^T \\ * & \Phi_{22}(t) & \Phi_{23}(t) & \Phi_{24}^{(1)} & \dots & \Phi_{24}^{(N)} & \tau(t)Y_2^T \\ * & * & \Phi_{33}(t) & \Phi_{34}^{(1)} & \dots & \Phi_{34}^{(N)} & \tau(t)T^T \\ * & * & * & 0 & \dots & 0 & \tau(t)q_1^T {Y_3^{(1)}}^T \\ * & * & * & \vdots & \ddots & \vdots & \vdots \\ * & * & * & 0 & \dots & 0 & \tau(t)q_N^T {Y_3^{(N)}}^T \\ * & * & * & * & * & * & -\tau(t)Qe^{-2\alpha h} \end{bmatrix}. \tag{27}$$

where " $*$ " stands for corresponding block of the symmetric matrix and

$$\begin{aligned} \Phi_{11}(t) &= A^T P_2 + P_2^T A + 2\alpha P - Y_1 - Y_1^T - (1 - 2\alpha(h - \tau(t)))\frac{X + X^T}{2}, \\ \Phi_{12}(t) &= P - P_2^T + A^T P_3 - Y_2 + (h - \tau(t))\frac{X + X^T}{2}, \\ \Phi_{13}(t) &= Y_1^T + P_2^T \mathcal{B}(t)K - T + (1 - 2\alpha(h - \tau(t)))(X - X_1), \\ \Phi_{22}(t) &= -P_3 - P_3^T + (h - \tau(t))Q, \\ \Phi_{23}(t) &= Y_2^T + P_3^T \mathcal{B}(t)K - (h - \tau(t))(X - X_1), \\ \Phi_{33}(t) &= T + T^T - (1 - 2\alpha(h - \tau(t)))\frac{X + X^T - 2X_1 - 2X_1^T}{2}, \\ \Phi_{14}^{(i)} &= P_2^T q_i - Y_3^{(i)} q_i, \quad \Phi_{24}^{(i)} = P_3^T q_i, \quad \Phi_{34}^{(i)} = Y_3^{(i)} q_i, \quad i = 1, \dots, N. \end{aligned}$$

Thus, to check condition (14) it is sufficient to verify that the matrix $\Psi(t)$ is nonpositive for all $t \geqslant 0$. Consider the following linear matrix inequalities:

$$\Psi_0^- = \begin{bmatrix} \Phi_{11|\tau(t)=0} & \Phi_{12|\tau(t)=0} & \Phi^-_{13|\tau(t)=0} & \Phi_{14}^{(1)} & \dots & \Phi_{14}^{(N)} \\ * & \Phi_{22|\tau(t)=0} & \Phi^-_{23|\tau(t)=0} & \Phi_{24}^{(1)} & \dots & \Phi_{24}^{(N)} \\ * & * & \Phi_{33|\tau(t)=0} & \Phi_{34}^{(1)} & \dots & \Phi_{34}^{(N)} \\ * & * & * & 0 & \dots & 0 \\ * & * & * & \vdots & \ddots & \vdots \\ * & * & * & 0 & \dots & 0 \end{bmatrix} < 0, \tag{28}$$

$$\Psi_0^+ = \begin{bmatrix} \Phi_{11|\tau(t)=0} & \Phi_{12|\tau(t)=0} & \Phi^+_{13|\tau(t)=0} & \Phi_{14}^{(1)} & \dots & \Phi_{14}^{(N)} \\ * & \Phi_{22|\tau(t)=0} & \Phi^+_{23|\tau(t)=0} & \Phi_{24}^{(1)} & \dots & \Phi_{24}^{(N)} \\ * & * & \Phi_{33|\tau(t)=0} & \Phi_{34}^{(1)} & \dots & \Phi_{34}^{(N)} \\ * & * & * & 0 & \dots & 0 \\ * & * & * & \vdots & \ddots & \vdots \\ * & * & * & 0 & \dots & 0 \end{bmatrix} < 0, \tag{29}$$

$$\Psi_1^- = \begin{bmatrix} \Phi_{11|\tau(t)=h} & \Phi_{12|\tau(t)=h} & \Phi^-_{13|\tau(t)=h} & \Phi_{14}^{(1)} & \dots & \Phi_{14}^{(N)} & hY_1^T \\ * & \Phi_{22|\tau(t)=h} & \Phi^-_{23|\tau(t)=h} & \Phi_{24}^{(1)} & \dots & \Phi_{24}^{(N)} & hY_2^T \\ * & * & \Phi_{33|\tau(t)=h} & \Phi_{34}^{(1)} & \dots & \Phi_{34}^{(N)} & hT^T \\ * & * & * & 0 & \dots & 0 & hq_1^T {Y_3^{(1)}}^T \\ * & * & * & \vdots & \ddots & \vdots & \vdots \\ * & * & * & 0 & \dots & 0 & hq_N^T {Y_3^{(N)}}^T \\ * & * & * & * & * & * & -hQe^{-2\alpha h} \end{bmatrix} < 0, \tag{30}$$

$$\Psi_1^+ = \begin{bmatrix} \Phi_{11|\tau(t)=h} & \Phi_{12|\tau(t)=h} & \Phi^+_{13|\tau(t)=h} & \Phi_{14}^{(1)} & \dots & \Phi_{14}^{(N)} & hY_1^T \\ * & \Phi_{22|\tau(t)=h} & \Phi^+_{23|\tau(t)=h} & \Phi_{24}^{(1)} & \dots & \Phi_{24}^{(N)} & hY_2^T \\ * & * & \Phi_{33|\tau(t)=h} & \Phi_{34}^{(1)} & \dots & \Phi_{34}^{(N)} & hT^T \\ * & * & * & 0 & \dots & 0 & hq_1^T {Y_3^{(1)}}^T \\ * & * & * & \vdots & \ddots & \vdots & \vdots \\ * & * & * & 0 & \dots & 0 & hq_N^T {Y_3^{(N)}}^T \\ * & * & * & * & * & * & -hQe^{-2\alpha h} \end{bmatrix} < 0, \tag{31}$$

where

$$\begin{aligned} \Phi_{13}^-(t) &= Y_1^T + P_2^T \mathcal{B}^- K - T + (1 - 2\alpha(h-\tau(t)))(X - X_1), \\ \Phi_{13}^+(t) &= Y_1^T + P_2^T \mathcal{B}^+ K - T + (1 - 2\alpha(h-\tau(t)))(X - X_1), \\ \Phi_{23}^-(t) &= Y_2^T + P_3^T \mathcal{B}^- K - (h-\tau(t))(X - X_1), \\ \Phi_{23}^+(t) &= Y_2^T + P_3^T \mathcal{B}^+ K - (h-\tau(t))(X - X_1). \end{aligned}$$

Denote $\eta_0(t) = col\,\{x(t), \dot{x}(t), x(t-\tau(t)), \xi_1(t), \dots, \xi_N(t)\}$. Then (28), (29), (30) and (31) imply $\Psi(t) < 0\,\forall t > 0$ because

$$\begin{aligned} &\frac{h-\tau(t)}{h}\,\frac{\varphi_0^+ - \varphi_0(t)}{\varphi_0^+ - \varphi_0^-}\,\eta_0^T\,\Psi_0^-\;\eta_0 + \frac{h-\tau(t)}{h}\,\frac{\varphi_0(t) - \varphi_0^-}{\varphi_0^+ - \varphi_0^-}\,\eta_0^T\,\Psi_0^+\;\eta_0 \\ &+ \frac{\tau(t)}{h}\,\frac{\varphi_0^+ - \varphi_0(t)}{\varphi_0^+ - \varphi_0^-}\,\eta_1^T\,\Psi_1^-\;\eta_1 + \frac{\tau(t)}{h}\,\frac{\varphi_0(t) - \varphi_0^-}{\varphi_0^+ - \varphi_0^-}\,\eta_1^T\,\Psi_1^+\;\eta_1 = \eta_1^T \Psi_F(t)\,\eta_1 < 0, \quad \forall \eta_1 \neq 0. \end{aligned}$$

Denote

$$\begin{aligned} F_0^-(\eta_0) = \eta_0^T\,\Psi_0^-\;\eta_0, \quad F_0^+(\eta_0) = \eta_0^T\,\Psi_0^+\;\eta_0, \\ F_1^-(\eta_1) = \eta_1^T\,\Psi_1^-\;\eta_1, \quad F_1^+(\eta_1) = \eta_1^T\,\Psi_1^+\;\eta_1. \end{aligned} \tag{32}$$

Thus, if

$$\begin{aligned} &F_0^-(\eta_0) < 0, \ \ \forall \eta_0 \neq 0, \\ &F_0^+(\eta_0) < 0, \ \ \forall \eta_0 \neq 0, \\ &F_1^-(\eta_1) < 0, \ \ \forall \eta_1 \neq 0, \\ &F_1^+(\eta_1) < 0, \ \ \forall \eta_1 \neq 0, \end{aligned} \tag{33}$$

then condition (14) of Lemma 1 holds.

Introduce quadratic forms

$$\begin{aligned} G_0^{(i)}(\eta_0) &= (\xi_i - \mu_{1i} r_i^T x)(\mu_{2i} r_i^T x - \xi_i), \quad i = 1, \ldots, N, \\ G_1^{(i)}(\eta_1) &= (\xi_i - \mu_{1i} r_i^T x)(\mu_{2i} r_i^T x - \xi_i), \quad i = 1, \ldots, N. \end{aligned}$$

Note that the forms $G_0^{(i)}(\eta_0)$ and $G_1^{(i)}(\eta_1)$ are defined on different spaces. From (16) the following inequalities holds along trajectories of system (17):

$$G_0^{(i)}(\eta_0) \geqslant 0, \quad G_1^{(i)}(\eta_1) \geqslant 0, \quad i = 1, \ldots, N.$$

Therefore, we can require that the first inequality of (33) holds in the set $G_0^{(i)}(\eta_0) \geqslant 0$ for all $i = 1, \ldots, N$, i. e.

$$F_0^-(\eta_0) < 0 \text{ if } G_0^{(i)}(\eta_0) \geqslant 0, \quad \forall i = 1, \ldots, N, \ \forall \eta_0 \neq 0. \tag{34}$$

Similarly,

$$F_0^+(\eta_0) < 0 \text{ if } G_0^{(i)}(\eta_0) \geqslant 0, \quad \forall i = 1, \ldots, N, \ \forall \eta_0 \neq 0, \tag{35}$$

$$F_1^-(\eta_1) < 0 \text{ if } G_1^{(i)}(\eta_1) \geqslant 0, \quad \forall i = 1, \ldots, N, \ \forall \eta_1 \neq 0, \tag{36}$$

$$F_1^+(\eta_1) < 0 \text{ if } G_1^{(i)}(\eta_1) \geqslant 0, \quad \forall i = 1, \ldots, N, \ \forall \eta_1 \neq 0. \tag{37}$$

Let us transform (34) - (37) using S-procedure [21]. Consider the following forms:

$$\begin{aligned} S_0^-(\eta_0) &= F_0^-(\eta_0) + \sum_{i=1}^{N} \varkappa_{0\ i}^- G_0^{(i)}(\eta_0), \quad S_0^+(\eta_0) = F_0^+(\eta_0) + \sum_{i=1}^{N} \varkappa_{0\ i}^+ G_0^{(i)}(\eta_0), \\ S_1^-(\eta_1) &= F_1^-(\eta_1) + \sum_{i=1}^{N} \varkappa_{1\ i}^- G_1^{(i)}(\eta_1), \quad S_1^-(\eta_1) = F_1^-(\eta_1) + \sum_{i=1}^{N} \varkappa_{1\ i}^- G_1^{(i)}(\eta_1), \end{aligned}$$

and require them to be negative for some non-negative $\{\varkappa_{0\ i}^-\}_{i=1}^N$, $\{\varkappa_{0\ i}^+\}_{i=1}^N$, $\{\varkappa_{1\ i}^-\}_{i=1}^N$ $\{\varkappa_{1\ i}^+\}_{i=1}^N$ respectively:

$$\exists \{\varkappa_{0\ i}^- \geqslant 0\}_{i=1}^N : \ S_0^-(\eta_0) < 0, \ \forall \eta_0 \neq 0, \tag{38}$$

$$\exists\{\varkappa_{0\ i}^{+} \geqslant 0\}_{i=1}^{N}: \ S_0^{+}(\eta_0) < 0, \ \ \forall \eta_0 \neq 0, \tag{39}$$

$$\exists\{\varkappa_{1\ i}^{-} \geqslant 0\}_{i=1}^{N}: \ S_1^{-}(\eta_1) < 0, \ \ \forall \eta_1 \neq 0, \tag{40}$$

$$\exists\{\varkappa_{1\ i}^{+} \geqslant 0\}_{i=1}^{N}: \ S_1^{+}(\eta_1) < 0, \ \ \forall \eta_1 \neq 0. \tag{41}$$

Condition (34) is sufficient for condition (38) (in case $N = 1$ these conditions are equivalent by the theorem about a losslessness of S-procedure [21]). Similarly, (35) is sufficient for (39), (36) is sufficient for (40) and (37) is sufficient for (41). Therefore, if conditions (38) - (41) hold, then (14) is fulfilled. Using (22), (32) we obtain the following inequalities:

$$S_0^{-}(\eta_0) \leqslant \eta_0^T\, \Psi_{S0}^{-}\, \eta_0, \quad S_0^{+}(\eta_0) \leqslant \eta_0^T\, \Psi_{S0}^{+}\, \eta_0,$$
$$S_1^{-}(\eta_1) \leqslant \eta_1^T\, \Psi_{S1}^{-}\, \eta_1, \quad S_1^{+}(\eta_1) \leqslant \eta_1^T\, \Psi_{S1}^{+}\, \eta_1,$$

where

$$\Psi_{S0}^{-} = \begin{bmatrix} \Phi_{S1|\tau(t)=0}^{-} & \Phi_{12|\tau(t)=0} & \Phi_{13|\tau(t)=0}^{-} & \Phi_{S2}^{-\,(1)} & \ldots & \Phi_{S2}^{-\,(N)} \\ * & \Phi_{22|\tau(t)=0} & \Phi_{23|\tau(t)=0}^{-} & \Phi_{24}^{(1)} & \ldots & \Phi_{24}^{(N)} \\ * & * & \Phi_{33|\tau(t)=0} & \Phi_{34}^{(1)} & \ldots & \Phi_{34}^{(N)} \\ * & * & * & \Phi_{S3}^{-\,(1)} & \ldots & 0 \\ * & * & * & \vdots & \ddots & \vdots \\ * & * & * & 0 & \ldots & \Phi_{S3}^{-\,(N)} \end{bmatrix},$$

$$\Psi_{S0}^{+} = \begin{bmatrix} \Phi_{S1|\tau(t)=0}^{+} & \Phi_{12|\tau(t)=0} & \Phi_{13|\tau(t)=0}^{+} & \Phi_{S2}^{+\,(1)} & \ldots & \Phi_{S2}^{+\,(N)} \\ * & \Phi_{22|\tau(t)=0} & \Phi_{23|\tau(t)=0}^{+} & \Phi_{24}^{(1)} & \ldots & \Phi_{24}^{(N)} \\ * & * & \Phi_{33|\tau(t)=0} & \Phi_{34}^{(1)} & \ldots & \Phi_{34}^{(N)} \\ * & * & * & \Phi_{S3}^{+\,(1)} & \ldots & 0 \\ * & * & * & \vdots & \ddots & \vdots \\ * & * & * & 0 & \ldots & \Phi_{S3}^{+\,(N)} \end{bmatrix},$$

$$\Psi_{S1}^{-} = \begin{bmatrix} \Phi_{S4|\tau(t)=h}^{-} & \Phi_{12|\tau(t)=h} & \Phi_{13|\tau(t)=h}^{-} & \Phi_{S5}^{-\,(1)} & \ldots & \Phi_{S5}^{-\,(N)} & hY_1^T \\ * & \Phi_{22|\tau(t)=h} & \Phi_{23|\tau(t)=h}^{-} & \Phi_{24}^{(1)} & \ldots & \Phi_{24}^{(N)} & hY_2^T \\ * & * & \Phi_{33|\tau(t)=h} & \Phi_{34}^{(1)} & \ldots & \Phi_{34}^{(N)} & hT^T \\ * & * & * & \Phi_{S6}^{-\,(1)} & \ldots & 0 & hq_1^T {Y_3^{(1)}}^T \\ * & * & * & \vdots & \ddots & \vdots & \vdots \\ * & * & * & 0 & \ldots & \Phi_{S6}^{-\,(N)} & hq_N^T {Y_3^{(N)}}^T \\ * & * & * & * & * & * & -hQe^{-2\alpha h} \end{bmatrix},$$

$$\Psi_{S1}^{+}=\begin{bmatrix} \Phi_{S4|\tau(t)=h}^{+} & \Phi_{12|\tau(t)=h} & \Phi_{13|\tau(t)=h}^{+} & \Phi_{S5}^{+(1)} & \dots & \Phi_{S5}^{+(N)} & hY_1^T \\ * & \Phi_{22|\tau(t)=h} & \Phi_{23|\tau(t)=h}^{+} & \Phi_{24}^{(1)} & \dots & \Phi_{24}^{(N)} & hY_2^T \\ * & * & \Phi_{33|\tau(t)=h} & \Phi_{34}^{(1)} & \dots & \Phi_{34}^{(N)} & hT^T \\ * & * & * & \Phi_{S6}^{+(1)} & \dots & 0 & hq_1^T {Y_3^{(1)}}^T \\ & * & * & \vdots & \ddots & \vdots & \vdots \\ * & * & * & 0 & \dots & \Phi_{S6}^{+(N)} & hq_N^T {Y_3^{(N)}}^T \\ * & * & * & * & * & * & -hQe^{-2\alpha h} \end{bmatrix},$$

and

$$\begin{aligned}
&\Phi_{S1}^{-}(t)=\Phi_{11}(t)-\sum_{i=1}^{N}\varkappa_{0\ i}^{-}\mu_{1i}\mu_{2i}r_ir_i^T, \quad \Phi_{S1}^{+}(t)=\Phi_{11}(t)-\sum_{i=1}^{N}\varkappa_{0\ i}^{+}\mu_{1i}\mu_{2i}r_ir_i^T,\\
&\Phi_{S2}^{-(i)}=\Phi_{14}^{(i)}+\frac{1}{2}\varkappa_{0\ i}^{-}(\mu_{1i}+\mu_{2i})r_i, \quad \Phi_{H2}^{+(i)}=\Phi_{14}^{(i)}+\frac{1}{2}\varkappa_{0\ i}^{+}(\mu_{1i}+\mu_{2i})r_i,\\
&\Phi_{S3}^{-(i)}=-\varkappa_{0\ i}^{-}, \quad \Phi_{S3}^{+(i)}=-\varkappa_{0\ i}^{+},\\
&\Phi_{S4}^{-}=\Phi_{11}(t)-\sum_{i=1}^{N}\varkappa_{0\ i}^{-}\mu_{1i}\mu_{2i}r_ir_i^T, \quad \Phi_{S4}^{+}=\Phi_{11}(t)-\sum_{i=1}^{N}\varkappa_{0\ i}^{+}\mu_{1i}\mu_{2i}r_ir_i^T,\\
&\Phi_{S5}^{-(i)}=\Phi_{14}^{(i)}+\frac{1}{2}\varkappa_{1\ i}^{-}(\mu_{1i}+\mu_{2i})r_i, \quad \Phi_{S5}^{+(i)}=\Phi_{14}^{(i)}+\frac{1}{2}\varkappa_{1\ i}^{+}(\mu_{1i}+\mu_{2i})r_i,\\
&\Phi_{S6}^{-(i)}=-\varkappa_{1\ i}^{-}, \quad \Phi_{S6}^{+(i)}=-\varkappa_{1\ i}^{+}, \quad i=1,\dots,N.
\end{aligned}$$

Hence, if

$$\Psi_{S0}^{-}<0, \tag{42}$$

$$\Psi_{S0}^{+}<0, \tag{43}$$

$$\Psi_{S1}^{-}<0, \tag{44}$$

$$\Psi_{S1}^{+}<0, \tag{45}$$

then (14) holds.

To formulate our main result we need the following statement that can be proved along the lines of Lemma 2 in [8].

Lemma 2 *LMIs (19), (42) - (45) are convex in h: if they are feasible for h, then they are feasible for all $\bar{h}\in(0,h]$.*

Therefore Lemma 2 is fulfilled in the case of constant sampling: $t_{k+1}-t_k=\bar{h}\leqslant h$. Next we generalize this result to the case of variable sampling: $t_{k+1}-t_k=\bar{h}_k\leqslant h$, $k=0,1,\dots$

Consider the following Lyapunov-Krasovskii functional:

$$V_{var}(t, x_t, \dot{x}_t) = \bar{V}_{var}(t) = x^T(t)Px(t) + (t_{k+1} - t)\int_{t_k}^{t} e^{2\alpha(s-t)}\,\dot{x}^T(s)\,Q\,\dot{x}(s)ds +$$

$$+ (t_{k+1} - t)\ \zeta^T(t)\begin{bmatrix} \frac{X+X^T}{2} & -X + X_1 \\ * & -X_1 - X_1^T + \frac{X+X^T}{2} \end{bmatrix}\zeta(t), \quad t \in [t_k, t_{k+1}), \tag{46}$$

where $\zeta(t) = col\{x(t), x(t_k)\}$. Note that in (46) the second and the third terms are equal to 0 for $t \to t_k^-$ and $t \to t_k^+$. Hence, $\bar{V}_{var}$ is continuous because $\lim\limits_{t\to t_k} \bar{V}_{var}(t) = \bar{V}_{var}(t_k)$. Applying previous arguments to $\bar{V}_{var}(t)$, we arrive at the main result:

Theorem 1 *Given $\alpha > 0$, let there exist matrices $P \in I\!R^{n\times n}$ $(P > 0)$, $Q \in I\!R^{n\times n}$ $(Q > 0)$, $P_2 \in I\!R^{n\times n}$, $P_3 \in I\!R^{n\times n}$, $X \in I\!R^{n\times n}$, $X_1 \in I\!R^{n\times n}$, $T \in I\!R^{n\times n}$, $Y_1 \in I\!R^{n\times n}$, $Y_2 \in I\!R^{n\times n}$, $Y_3^{(i)} \in I\!R^{n\times n}$ (i=1,...,N), and positive real numbers $\{\varkappa_{0\ i}^-\}_{i=1}^N$, $\{\varkappa_{0\ i}^+\}_{i=1}^N$, $\{\varkappa_{1\ i}^-\}_{i=1}^N$ $\{\varkappa_{1\ i}^+\}_{i=1}^N$, such that LMIs (19), (42) - (45) are feasible. Then system (10) is exponentially stable with decay rate α. If LMIs (19), (42) - (45) are feasible for $\alpha = 0$, then (10) is exponentially stable with a small enough decay rate.*

5 Numerical Examples.

5.1 Example 1. Pendulum with friction

Let us illustrate the obtained results by examples. Consider the system, proposed in Section 2:

$$\begin{aligned} &\ddot{\varphi}(t) = \frac{g}{l}sin(\varphi(t)) - \frac{\kappa}{l}\dot{\varphi}(t) + \frac{1}{ml^2}u(t), \\ &u(t) = Kx(t - \tau(t)), \\ &\tau(t) = t - t_k,\ \ t \in [t_k, t_{k+1}),\ \ t_{k+1} - t_k = h,\ \ k = 0, 1 \ldots. \end{aligned} \tag{47}$$

If friction is known, then for the system with the following parameter values

$$l = 2 \text{ m}, \quad m = 1 \text{ kg}, \quad g = 9.8 \text{ m/s}^2, \quad \kappa = 8 \ \text{ N/m}, \quad K = [-20.6, -3]$$

with Theorem 1 the maximum upper bound of sampling was found as 0.95 (for $\alpha = 0$).

5.1.1 Unknown friction.

Let friction κ be unknown, and

$$0 < \kappa_1 \leqslant \kappa \leqslant \kappa_2.$$

System (47) can be rewritten as follows:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + q_1\xi_1(t) + q_2\xi_2(t) + Bu(t),\\ \sigma_1(t) &= r_1^T x(t), \quad \xi_1(t) = \sin\sigma_1(t),\\ \sigma_2(t) &= r_2^T x(t), \quad \xi_2(t) = \kappa\sigma_2,\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \dfrac{1}{ml^2} \end{bmatrix},$$

$$q_1 = \begin{bmatrix} 0 \\ \dfrac{g}{l} \end{bmatrix}, \quad q_2 = \begin{bmatrix} 0 \\ -\dfrac{1}{l} \end{bmatrix}, \quad r_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

ξ_1 and ξ_2 for all $t \geqslant 0$ satisfy

$$\mu_1\sigma_1^2 \leqslant \sigma_1\,\xi_1 \leqslant \mu_2\sigma_1^2, \qquad \kappa_1\,\sigma_2^2 \leqslant \sigma_2\,\xi_2 \leqslant \kappa_2\,\sigma_2^2.$$

The values of maximum upper bound h (for $\kappa_1 = 7.5, \kappa_2 = 8.5$) when (47) is exponentially stable with a small enough decay rate are given in Table 1.

5.1.2 Unknown mass.

Let mass m be unknown, and

$$0 < m_1 \leqslant m \leqslant m_2.$$

System (47) can be rewritten as follows:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + q_1\xi_1(t) + (B + B_0\xi_0(t))\,u(t),\\ \sigma_1(t) &= r_1^T x(t), \quad \xi_1(t) = \sin\sigma_1(t),\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & \dfrac{-\kappa}{l} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ \dfrac{1}{l^2} \end{bmatrix}, \quad q_1 = \begin{bmatrix} 0 \\ \dfrac{g}{l} \end{bmatrix}, \quad r_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

ξ_1 and ξ_0 for all $t \geqslant 0$ satisfy

$$\mu_1\sigma_1^2 \leqslant \sigma_1\,\xi_1 \leqslant \mu_2\sigma_1^2, \qquad \frac{1}{m_2} \leqslant \xi_0 \leqslant \frac{1}{m_1}.$$

The values of maximum upper bound h (for $m_1 = 0.97, m_2 = 1.03$) when (47) is exponentially stable with a small enough decay rate are given in Table 1.

5.1.3 Unknown length.

Let length l be unknown, and

$$0 < l_1 \leqslant l \leqslant l_2.$$

System (47) can be rewritten as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + q_1\xi_1(t) + q_2\xi_2(t) + (B + B_0\xi_0(t))\, u(t), \\ \sigma_1(t) &= r_1^T x(t), \quad \xi_1(t) = \sin\sigma_1(t), \\ \sigma_2(t) &= r_2^T x(t), \quad \xi_2(t) = \frac{1}{l}\sigma_2, \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix},$$

$$q_1 = \begin{bmatrix} 0 \\ g \end{bmatrix}, \quad q_2 = \begin{bmatrix} 0 \\ -\kappa \end{bmatrix}, \quad r_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

ξ_1, ξ_2 and ξ_0 for all $t \geqslant 0$ satisfy

$$\frac{\mu_1}{l_1}\sigma_1^2 \leqslant \sigma_1\,\xi_1 \leqslant \frac{\mu_2}{l_1}\sigma_1^2, \quad \frac{1}{l_2}\,\sigma_2^2 \leqslant \sigma_2\,\xi_2 \leqslant \frac{1}{l_1}\,\sigma_2^2, \quad \frac{1}{l_2^2} \leqslant \xi_0 \leqslant \frac{1}{l_1^2}.$$

The values of maximum upper bound h (for $l_1 = 1.99, l_2 = 2.01$) when (47) is exponentially stable with a small enough decay rate are given in Table 1.

5.1.4 Unknown friction, mass, length.

Let κ, m, l be unknown, and

$$0 < \kappa_1 \leqslant \kappa \leqslant \kappa_2, \quad 0 < m_1 \leqslant m \leqslant m_2, \quad 0 < l_1 \leqslant l \leqslant l_2.$$

System (47) can be rewritten as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + q_1\xi_1(t) + q_2\xi_2(t) + (B + B_0\xi_0(t))\, u(t), \\ \sigma_1(t) &= r_1^T x(t), \quad \xi_1(t) = \sin\sigma_1(t), \\ \sigma_2(t) &= r_2^T x(t), \quad \xi_2(t) = \frac{\kappa}{l}\sigma_2, \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$q_1 = \begin{bmatrix} 0 \\ g \end{bmatrix}, \quad q_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad r_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

ξ_1, ξ_2 and ξ_0 for all $t \geqslant 0$ satisfy

$$\frac{\mu_1}{l_1}\sigma_1^2 \leqslant \sigma_1\,\xi_1 \leqslant \frac{\mu_2}{l_1}\sigma_1^2, \quad \frac{\kappa_1}{l_2}\,\sigma_2^2 \leqslant \sigma_2\,\xi_2 \leqslant \frac{\kappa_2}{l_1}\,\sigma_2^2, \quad \frac{1}{m_2 l_2^2} \leqslant \xi_0 \leqslant \frac{1}{m_1 l_1^2}.$$

The values of maximum upper bound h (for $\kappa_1 = 7.5, \kappa_2 = 8.5, m_1 = 0.97, m_2 = 1.03, l_1 = 1.99, l_2 = 2.01$) when (47) is exponentially stable with a small enough decay rate are given in Table 1.

The estimates obtained by Theorem 1 is compared with those of simulation. LMIs is solved by using Matlab [27] with toolbox Yalmip [28]. The simulation is carry out with Matlab Simulink [27]. System (47) can be considered as a system with polytopic uncertainties as proposed in Remark 2 in [8]. The results obtained with Remark 2 in [8] are also provided in Table 1.

	Remark 2 in [8]	Theorem 1	Simulation	Quality of Estimates
Known parameters	$h = 0.79$	$h = 0.95$	$h = h_*,\ 1.75 < h_* < 1.77$	54%
Unknown friction $7.5 \leqslant \kappa \leqslant 8.5$	$h = 0.73$	$h = 0.86$	$h = h_*,\ 1.69 < h_* < 1.7$	51%
Unknown mass $0.97 \leqslant m \leqslant 1.03$	$h = 0.65$	$h = 0.82$	$h = h_*,\ 1.7 < h_* < 1.71$	48%
Unknown length $1.99 \leqslant l \leqslant 2.01$	$h = 0.74$	$h = 0.88$	$h = h_*,\ 1.74 < h_* < 0.75$	51%
Unknown friction, mass, length $7.5 \leqslant \kappa \leqslant 8.5$ $0.97 \leqslant m \leqslant 1.03$ $1.99 \leqslant l \leqslant 2.01$	$h = 0.31$	$h = 0.36$	$h = h_*,\ 1.62 < h_* < 1.63$	22%

Table 1: Upper bounds for the variable sampling

Note that the estimates obtained with Theorem 1 are more accurate than the estimates from Remark 2 in [8]. Moreover, using S-procedure we increase the size of LMIs without increasing their number unlike using polytopic approach that increases the number of LMIs and, hence, the computational complexity of LMI solving.

5.2 Example 2

Consider the system

$$\dot{x}(t) = \begin{bmatrix} 1 & 0.5 \\ g_1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 + g_2 \\ -1 \end{bmatrix} u(t),$$

where $x(t)$ is the state vector, $u(t)$ is the control input, defined with (7), (8), and $|g_1| \leqslant 0.1$, $|g_2| \leqslant 0.3$. This example coincides exactly with Example 1 in [7], where it was verified that the maximum sampling interval $h \leqslant 0.35$. With Theorem 1 the maximum upper bound was found as 0.65. Hence, the upper bound for the sampling interval obtained by our approach (with S-procedure) is about twice more accurately.

6 Conclusions

With Fridman's method and Yakubovich's S-procedure the problem of estimation of the sampling interval providing exponential stability of the closed loop nonlinear system with parameter uncertainties is reduced to feasibility analysis of linear matrix inequalities The obtained results are illustrated by examples of upper stabilization of simple pendulum with friction, where some system parameters can be uncertain. It is shown that the proposed method provides estimates for sampling interval not less that 54% (for known pendulum parameters) or 22% (for unknown pendulum parameters) of the value evaluated by simulation.

The key idea of the paper is application of the Yakubovich's S-procedure allowing us to extend previous stability criteria and take into account nonlinearities. Its application is demonstrated for "Fridman's method" , but it also can be used with other methods of stability analysis.

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