Passification-based adaptive control: uncertain input and output delays

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Abstract

For a class of uncertain systems we analyze passification-based adaptive controller in the presence of small, unavoidable input and output time-varying delays as may be present in controller implementation. We derive upper bounds for time delays such that in some domain of initial conditions the states of the closed-loop system tend to zero, whereas an adaptive controller gain tends to a constant value. The results are semi-global, that is the domain of initial conditions is bounded but can be made arbitrary large by tuning an appropriate controller parameter. For the first time, we apply an adaptive controller to linear uncertain networked control systems, where sensors, controllers, and actuators exchange their information through communication networks. The efficiency of the results is demonstrated by the example of adaptive network-based control of an aircraft.

Key words: Adaptive control; passification method; input and output delay; networked control systems.

1 Introduction

In this paper we consider passification-based adaptive controller, which proved to be efficient for stabilization of delay-free systems. As it has been shown in [5], any hyper-minimum-phase linear time-invariant system can be stabilized by a static output feedback $u(t) = -ky(t)$ if k is large enough (for more established description see [1]). For the case of uncertain systems an adaptive version of this controller has been derived via the speed gradient method [7].

While applying adaptive controller it is important to take into account unknown unavoidable input/output delays, which is a challenging problem [24,15]. Most of the existing works on adaptive control deal only with state delays, e.g. [17,18,3,25] to name a few. Adaptive controllers for linear systems with full state measurements and a *constant* input delay have been proposed and analyzed in [23,4]. Passification-based adaptive *output*-feedaback controller with a constant input delay has been studied in [19].

Note that for linear time-invariant systems with constant time-delays there is almost no difference between an input and output delay, since the transfer function is the same. A more challenging problem is adaptive stabilization with *time-varying* delays, where input and output delays should be treated separately. A possible way to approach this problem is to assume that the difference between current and delayed signal is small enough [2,20], but this assumption is restrictive and difficult to verify.

In the present paper we suggest a simple adaptive output-feedback controller that stabilizes hyper-minimum-phase systems with input and output time-varying delays. Namely, we derive upper bounds on the time-delays such that in a given domain of initial conditions the states of the closed-loop system tend to zero, whereas an adaptive controller gain tends to a constant value. By changing a particular controller parameter the domain of acceptable initial conditions can be made arbitrary large leading to semi-global stability (see Remark 2). Moreover, we consider fast-varying delays (without any constraints on the delay-derivatives). This allows to apply, for the first time, an adaptive controller to linear uncertain networked control systems, where variable sampling intervals and communication delays

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are taken into account (see Section 4). Some preliminary results (without input delays) have been presented in [22].

Notations: Throughout the paper the superscript "T" stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, the notation $P>0$ for $P\in\mathbb{R}^{n\times n}$ means that P is symmetric and positive definite, $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ stand for the minimum and maximum eigenvalues of the matrix P, respectively. The symmetric elements of the symmetric matrix will be denoted by $*$. The set $\{0,1,2,\dots\}$ is denoted by $\mathbb{Z}_+$.

2 Preliminaries and problem formulation

2.1 Preliminaries: Passification method

For non-delay linear time-invariant systems passification method and the corresponding design of an adaptive controller are based on Passification lemma that we state below.

Definition 1 *For given matrices $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times 1}$, $C\in\mathbb{R}^{l\times n}$, $g\in\mathbb{R}^{l\times 1}$ a transfer function $g^TW(s)=g^TC(sI-A)^{-1}B$ is called* hyper-minimum-phase (HMP) *if the polynomial $\varphi(s)=g^TW(s)\det(sI-A)$ is Hurwitz and g^TCB is a positive number.*

Lemma 1 (Passification lemma) *[6] Let the matrices $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times 1}$, $C\in\mathbb{R}^{l\times n}$, $g\in\mathbb{R}^{l\times 1}$ be given. Then for existence of $P\in\mathbb{R}^{n\times n}$ and $k_*\in\mathbb{R}$ such that*

$$P>0,\quad PA_*+A_*^TP<0,\quad PB=C^Tg, \tag{1}$$

where $A_*=A-Bk_*g^TC$, it is necessary and sufficient that the function $g^TW(s)=g^TC(sI-A)^{-1}B$ is HMP.

An appropriate value for k_* in Lemma 1 is any positive number such that

$$k_*>-\inf_{\omega\in\mathbb{R}}\operatorname{Re}\left\{\left(g^TW(i\omega)\right)^{-1}\right\}. \tag{2}$$

See [1] for more details on Passification method.

2.2 Problem formulation

Consider an uncertain linear system

$$\begin{aligned}&\dot x(t)=A_\xi x(t)+Bu(t-r_1(t)),\quad x(0)=x_0,\\&y(t)=Cx(t-r_2(t)),\end{aligned} \tag{3}$$

where $x\in\mathbb{R}^n$ is the state, $u\in\mathbb{R}$ is the control input, $y\in\mathbb{R}^l$ is the measurable output; A_ξ is an uncertain matrix that resides in the polytope

$$A_\xi=\sum_{i=1}^N\xi_iA_i,\quad 0\le\xi_i\le 1,\quad \sum_{i=1}^N\xi_i=1. \tag{4}$$

The delays $r_1(t)$, $r_2(t)$ are supposed to be unknown and bounded:

$$0\le r_1(t)\le h_1,\quad 0\le r_2(t)\le h_2.$$

We set $x(t)=0$ for $t<0$. This does not affect the solution $x(t)$ and implies that $y(t)=0$ if $t-r_2(t)<0$.

Denote

$$r(t)=r_1(t)+r_2(t-r_1(t)). \tag{5}$$

The quantity $r(t)$ is the overall delay of the closed-loop system. Clearly

$$r(t)\le h_1+h_2\triangleq h.$$

If $t-r(t)<0$ then the system (3) is in the open-loop since it has not received a signal from the controller. Therefore, a special analysis is needed on the intervals where $t-r(t)<0$. Following [16] we assume

Assumption 1 *There exists a unique $t_*>0$ such that*

$$\begin{cases}t-r(t)<0, & t<t_*,\\ t-r(t)\ge 0, & t\ge t_*.\end{cases}$$

Assumption 1 has a simple physical meaning: the system (3) starts to receive signals from the controller at time t_*. It is clear that $t_*\le h$. Assumption 1 is always satisfied for slowly-varying delays with $\dot r(t)\le 1$ (since $t-r(t)$ is increasing) and for networked control systems as considered in Section 4.

Similar to [1,6] we assume

Assumption 2 *There exists a known $g\in\mathbb{R}^l$ such that $g^TC(sI-A_\xi)^{-1}B$ is HMP for all A_ξ from (4).*

For a given g satisfying Assumption 2 we consider the adaptive controller

$$\begin{aligned}&u(t)=-k(t)g^Ty(t),\\&\dot k(t)=\gamma^{-2}\left(g^Ty(t)\right)^2,\end{aligned} \tag{6}$$

where $k,\gamma\in\mathbb{R}$, $\gamma>0$.

For $r_1(t)=r_2(t)\equiv 0$ under Assumption 2 it has been shown in [1] that solutions of the closed-loop system (3), (6) satisfy the following property: for all $k(0)\in\mathbb{R}$

$$\lim_{t\to\infty}\|x(t)\|=0,\ \lim_{t\to\infty}k(t)=\text{const}. \tag{7}$$

Our objective is to derive conditions ensuring (7) for non-zero delays and for a certain choice of $k(0)$.

3 Main results

The closed-loop system (3), (6) can be presented in the form

$$\begin{aligned} \dot{x}(t) &= A_\xi x(t) - k_* B g^T C x(t-r(t)) \\ &\quad + (k_* - k(t-r_1(t))) B g^T C x(t-r(t)), \\ \dot{k}(t) &= \gamma^{-2} \left(g^T C x(t-r_2(t))\right)^2 \end{aligned} \tag{8}$$

with $k(t) = k(0)$ for $t < 0$. Note that here $\dot{x}(0)$ and $\dot{x}(t_*)$ denote right-hand side derivatives.

The idea of passification-based approach is the following. Under Assumption 2 there exist $P > 0$, k_* that satisfy (1). Consider a Lyapunov-like function

$$V_0(x,k) = x^T P x + \gamma^2 (k - k_*)^2.$$

Its derivative along the trajectories of (8) has the form

$$\begin{aligned} \dot{V}_0 &= 2x^T(t) P \left[A_\xi x(t) - k_* B g^T C x(t-r(t))\right] \\ &\quad + 2(k_* - k(t-r_1(t))) x^T(t) P B g^T C x(t-r(t)) \\ &\quad + 2(k(t) - k_*) \left(g^T C x(t-r_2(t))\right)^2. \end{aligned} \tag{9}$$

For $r_1(t) = r_2(t) \equiv 0$ the last two terms can be canceled because $PB = C^T g$. Hence, (1) guarantees that $\dot{V}_0 \le -\varepsilon \|x\|^2$ for some $\varepsilon > 0$. The latter implies (7) (see [1]).

Remark 1 *Note that the above arguments can be easily extended to systems with state delays. Consider the system*

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-r(t)) + B u(t), \\ y(t) &= C x(t). \end{aligned} \tag{10}$$

Here x, u, y are defined as previously and all matrices are constant with appropriate dimensions. Calculating $\dot{V}_0$ we obtain:

$$\begin{aligned} \dot{V}_0 &= 2x^T(t) P \big[A_0 x(t) + A_1 x(t-r(t)) \\ &\quad - k_* B g^T C x(t)\big] + 2(k_* - k(t)) x^T(t) P B g^T C x(t) \\ &\quad + 2(k(t) - k_*)(g^T y(t))^2. \end{aligned}$$

Since $PB = C^T g$, the last two terms can be canceled and the Lyapunov-based analysis of (10) is reduced to the standard one for linear time-delay systems under $u(t) = -k_* g^T y(t)$. In the case of an input/output time-varying delay such a cancelation is not possible since the controller does not measure the current value of the state. Therefore, adaptive control of systems with input/output delays is much more challenging than the one under state delays.

As already mentioned, if $r_1(t) \not\equiv 0$ or $r_2(t) \not\equiv 0$ the cancelation of the last two terms in (9) is not possible because x and k depend on different time instants. Note that the right-hand side of (9) can be considered as a quadratic form in $x(t)$, $x(t-r(t))$, and $x(t-r_i(t))$ $(i = 1, 2)$, where matrices depend on the following time-varying parameters:

$$\begin{aligned} a &= k_* - k(t), \\ b &= k_* - k(t - r_1(t)), \\ c &= k(t) - k(t - r_1(t)). \end{aligned} \tag{11}$$

Consider the following Lyapunov-Krasovskii functional:

$$V(x_t, \dot{x}_t, k) = V_0(x_t, k) + V_S(x_t) + V_R(\dot{x}_t), \tag{12}$$

where $x_t(\theta) = x(t+\theta), \theta \in [-h, 0]$, $h = h_1 + h_2$, and

$$\begin{aligned} V_S(x_t) &= \int_{-h}^{0} x_t^T(s) S x_t(s)\, ds, \quad S > 0, \\ V_R(\dot{x}_t) &= h \int_{-h}^{0} \int_{\theta}^{0} \dot{x}_t^T(s) R \dot{x}_t(s)\, ds\, d\theta, \quad R > 0. \end{aligned}$$

Here V_S and V_R are standard terms for systems with fast-varying delays [11,13]. Our goal is to derive conditions that ensure $\dot{V} \le -\varepsilon \|x(t)\|^2$ for some $\varepsilon > 0$ if

$$|a| \le M_*, \ |b| \le M_*, \ |c| \le M_1 \tag{13}$$

for $t \ge 0$, where M_* and M_1 are some fixed bounds. Further we will show that (13) is valid for $t \ge 0$ if one choose appropriate values of $k(0)$ and γ.

We are in a position to formulate our main result.

Theorem 1 *Let Assumptions 1 and 2 hold. Given $h > 0$ and tuning parameters $M_* > 0$, $M_1 > 0$, $k_* > 0$, let there exist $n \times n$ matrices $P > 0$, $S > 0$, $R > 0$, G_1, G_2, G_3 such that the following relations hold:*

$$\begin{aligned} &\mathcal{H}_i(a,b,c)\Big|_{a \pm M_*, b \pm M_*, c \pm M_1} < 0, \quad i = 1, \dots, N, \\ &PB = C^T g, \quad \begin{pmatrix} R & G_j \\ * & R \end{pmatrix} \ge 0 \ (j = 1,2,3), \end{aligned} \tag{14}$$

where

$$\mathcal{H}_i(a,b,c) = \begin{pmatrix} H_1^i & H_2(c) & 0 & H_3 & H_4(a) & h A_i^T R \\ * & -R & R & H_5(a) & -hG_1 & H_7(b) \\ * & * & -(S+R) & hG_2 & hG_1 & 0 \\ * & * & * & -h^2 R & H_6(a) & 0 \\ * & * & * & * & -h^2 R & 0 \\ * & * & * & * & * & -R \end{pmatrix},$$

$$H_1^i = P[A_i - Bk_* g^T C] + [A_i - Bk_* g^T C]^T P + S,$$
$$H_2(c) = cPBg^T C,$$
$$H_3 = k_* hPBg^T C,$$
$$H_4(a) = k_* hPBg^T C - ahPBg^T C,$$
$$H_5(a) = ahC^T gB^T P - hG_2,$$
$$H_6(a) = ah^2 PBg^T C - h^2 (G_3)^T,$$
$$H_7(b) = hbC^T gB^T R - hk_* C^T gB^T R.$$

Assume additionally that

$$h_1 \leq \frac{M_1 \lambda_{\min}(P)}{M_*^2 \|g^T C\|^2}. \tag{15}$$

Then for any $\delta > 0$ there exists $\gamma > 0$ such that for all initial conditions $x_0, k(0)$ subject to

$$\|x_0\| < \delta, \quad k(0) \in [k_* - M_*, k_*], \tag{16}$$

solutions of the system (3), (6) *satisfy the property* (7).

Proof. See Appendix A.

Remark 2 *Conditions of Theorem 1 ensure semi-global results, where* (7) *is guaranteed for any $\delta > 0$ and x_0 with $\|x_0\| < \delta$. It follows from the proof of Theorem 1 that an appropriate γ can be chosen from the inequality*

$$\delta^2 \leq \gamma^2 \min \left\{ \frac{M_*^2 e^{-2\Lambda_A t_*}}{\lambda_{\min}(P)}, \frac{M_*}{c_1}, \frac{M_*^2 - c_*^2}{c_x} \right\}, \tag{17}$$

where $\Lambda_A = \max_i \|A_i\|$ and c_1, c_*, c_x are given in (A.2), (A.3), (A.6), *correspondingly. If t_* in* (17) *is unknown, one should substitute a known upper bound for t_*.*

Remark 3 *Under Assumptions 1, 2 due to Passification lemma there exist P and k_* that satisfy* (1). *With these P and k_* relations* (14) *are feasible for given $M_1 > 0$ and $M_* > 0$ if h is small enough. Indeed, due to* (A.8) *$H_2 \to 0$ for $h \to 0$. The same is true for $H_3, \ldots, H_7$. Then by Schur complement theorem [12, p.318] it can be shown that $\mathcal{H}_i < 0$ for $R = I$, $S = hI$, $G_j = 0$ ($j = 1, 2, 3$). When $h \to 0$ allowable M_* and M_1 tend to infinity, therefore, our results recover the global results from [1] for delay-free case. Relations* (14) *give acceptable bounds h_1 and h_2 such that* (7) *holds for the closed-loop time-delay system* (3), (6).

Remark 4 *There is a trade-off between enlarging of M_*, M_1 and enlarging of the delay bounds h, h_1. The smaller h is, the larger M_* can be taken such that the LMIs* (14) *are feasible, i.e. a wider choice of adaptive gain is possible. Furthermore, given M_1, k_* such that* (14) *are feasible for M_* they remain feasible with the same decision variables for all $M_*' < M_*$. The latter means that the stability is guaranteed for the same h but for larger h_1 due to* (15).

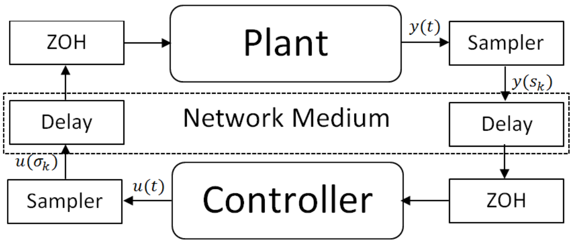

Fig. 1. Networked control system.

4 Network-based adaptive control

4.1 *Case study: adaptive control of networked control systems*

In this section we apply passification-based adaptive control to networked control systems. Consider the uncertain system

$$\begin{aligned} \dot{x}(t) &= A_\xi x(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned} \tag{18}$$

with several nodes (a sensor, a controller, and an actuator) that are connected via two communication networks: a sensor network (from the sensor to the controller) and a control network (from the controller to the actuator). Let s_k be the sequence of sampling instants:

$$0 = s_0 < s_1 < \ldots < s_k < \ldots, \quad k \in \mathbb{N}, \quad \lim_{k \to \infty} s_k = \infty.$$

At each sampling instant s_k the output $y(t)$ is sampled and transmitted via the network to the controller with a variable delay τ_k^{sc}. Therefore, the updating instant time of the controller is $\sigma_k = s_k + \tau_k^{sc}$. For simplicity, we assume that $\sigma_k < \sigma_{k+1}$, that is the old sample cannot get to the destination after the most recent one. Then the controller has the form

$$\begin{aligned} u(t) &= -k(t) g^T y(\sigma_k - \tau_k^{sc}), \\ \dot{k}(t) &= \gamma^{-2} \big(g^T y(\sigma_k - \tau_k^{sc})\big)^2, \end{aligned} \quad t \in [\sigma_k, \sigma_{k+1}). \tag{19}$$

At the sampling instants σ_k the control signal is sampled and transmitted through the network to the Zero-Order Hold (ZOH) with a variable delay τ_k^{ca}. Therefore, the updating instant time of the ZOH is $t_k = s_k + \tau_k^{sc} + \tau_k^{ca}$. We assume that $t_k < t_{k+1}$ and there is a known MAD (maximum allowable delay) such that $\tau_k^{sc} + \tau_k^{ca} \leq \text{MAD}$.

Following the time-delay approach to sampled-data control [10,9], the resulting closed-loop system can be presented in the form

$$\begin{aligned} \dot{x}(t) &= A_\xi x(t) + Bk(t - r_1(t)) g^T Cx(t - r(t)), \\ \dot{k}(t) &= \gamma^{-2} \big(g^T Cx(t - r_2(t))\big)^2, \end{aligned} \tag{20}$$

where

$$\begin{array}{lll} r_1(t) = t - t_k + \tau_k^{ca}, & t \in [t_k, t_{k+1}), & \\ r_2(t) = t - \sigma_k + \tau_k^{sc}, & t \in [\sigma_k, \sigma_{k+1}), & (21) \\ r(t) \ = t - t_k + \tau_k^{sc} + \tau_k^{ca}, & t \in [t_k, t_{k+1}). & \end{array}$$

Here $r(t) = r_1(t) + r_2(t - r_1(t))$. Note that $r(t)$ satisfies Assumption 1 with

$$t_* = t_0 = \tau_0^{sc} + \tau_0^{ca} \leq \text{MAD}.$$

Assume that

$$\begin{array}{l} t_{k+1} - t_k + \tau_k^{ca} \leq h_1, \\ t_{k+1} - t_k + \tau_k^{ca} + \tau_k^{sc} \leq h, \end{array} \quad \forall k \in \mathbb{Z}_+. \qquad (22)$$

Since (20) coincides with (8), the results of Theorem 1 provide bounds for the sampling intervals and network-induced delays. We illustrate this below by an example of network-based adaptive control of an aircraft.

4.2 Example: yaw angle control

As an example we apply our results to the following model of a lateral motion of an aircraft [8]:

$$\begin{aligned} \dot{x}_1(t) &= a_1 x_1(t) + x_2(t) + b_1 u(t - r_1(t)), \\ \dot{x}_2(t) &= a_2 x_1(t) + a_3 x_2(t) + b_2 u(t - r_1(t)), \\ \dot{x}_3(t) &= x_2(t), \\ y_1(t) &= x_2(t - r_2(t)), \\ y_2(t) &= x_3(t - r_2(t)), \end{aligned}$$

where x_3 and x_2 are the yaw angle and the yaw rate, respectively, and x_1 denotes the sideslip angle; u is the rudder angle; y_i are measurable outputs; a_i and b_i denote the aircraft model parameters. We suppose that the aircraft is controlled through a network, that is the closed-loop system has the form (20) with $r_1(t)$, $r_2(t)$ given by (21). Then Assumption 1 is satisfied with $t_* \leq$ MAD. Following [8] we take $a_3 = 1.3$, $b_1 = 19/15$, $b_2 = 19$ and suppose that $a_1 \in [0.1, 1.5]$, $a_2 \in [27, 52]$ are uncertain system parameters. Then for $g = (1, 1)^T$ the transfer function

$$g^T W(s) = \frac{b_2 s^2 + (b_1 a_2 - b_2 a_1 + b_2)s + b_1 a_2 - b_2 a_1}{s(s^2 - (a_1 + a_3)s + a_1 a_3 - a_2)}$$

is HMP, since for all a_1, a_2 from the given sets its numerator is a stable polynomial and $b_2 > 0$. Therefore, Assumption 2 is true. For $M_* = 5$, $M_1 = 0.4$, $k_* = 4.61$ conditions of Theorem 1 are satisfied with $h_1 = 4 \times 10^{-4}$, $h = 10^{-3}$, $\gamma = 25$, $\delta = 20$. To illustrate Remark 4 we take $M'_* = 4 < M_*$ with the same M_1, k_*. This leads to the same h but larger $h_1 = 6.4 \times 10^{-4}$.

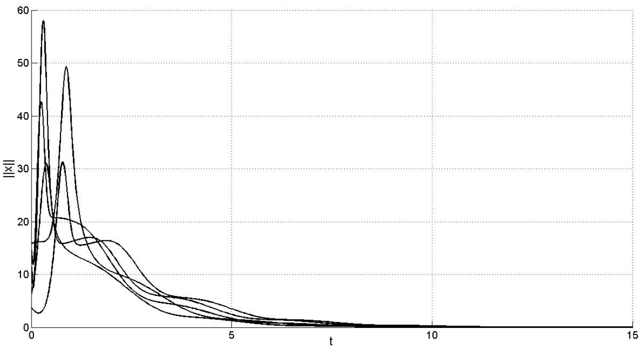

Fig. 2. State norms for 5 different randomly chosen initial conditions.

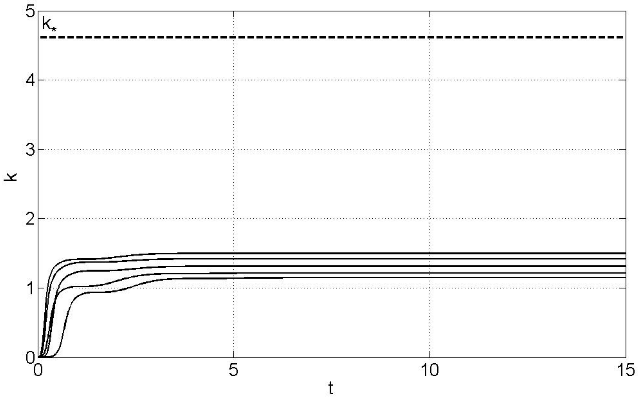

Fig. 3. Evolution of adaptive gains for 5 different randomly chosen initial conditions (solid lines); the value of $k_* = 4.61$ (dashed line).

In Figs. 2, 3 one can see the results of numerical simulations for $a_1 = 0.75$, $a_2 = 33$ and five different randomly chosen initial conditions such that $\|x(0)\| \leq \delta = 20$. We took $t_{k+1} - t_k = 2 \times 10^{-4}$, $\tau_k^{sc} = (1.5 + 0.5(-1)^k)10^{-4}$ and $\tau_k^{sc} = (5.5 + 0.5(-1)^k)10^{-4}$ with MAD $= 8 \times 10^{-4}$. As it follows from Theorem 1, $x(t) \to 0$ and $k(t)$ tend to constant values.

It is easy to prove that k_* is an appropriate gain for the static controller $u(t) = -k_* g^T y(t)$ that stabilizes entire class of uncertain systems. As one can see in Fig. 3, the limit value of $k(t)$ is smaller than $k_* = 4.61$, which shows an advantage of the adaptive control over the static one.

5 Conclusion

For a class of uncertain hyper-minimum-phase systems we analyzed passification-based adaptive controller in the presence of unknown time-varying delays in the measurements and control input. If a delay-free system under the controller is such that the state tends to zero, whereas the adaptive controller gain tends to a constant value, then our results give an acceptable bound for time-delay such that this property is preserved within a given domain of initial conditions. This domain of stability can be made arbitrary large by changing an appropriate parameter in the adaptation law. The results are applicable to networked control systems and provide acceptable bounds for transmission intervals and network-induced delays. This important application was demonstrated by an example of an aircraft that is adaptively controlled through a network. One of the directions for the future research is extension of the obtained results to the adaptive control of networks.

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References

[1] B. R. Andrievskii and A. Fradkov. Method of passification in adaptive control, estimation, and synchronization. *Automation and Remote Control*, 67(11):1699–1731, 2006.

[2] M. Balas and J. P. Nelson. New robustness theorem with application to adaptive control of nonlinear systems with input/output delays. In *IEEE International Conference on Control Applications*, pages 1437–1442, 2011.

[3] R. Ben Yamin, I. Yaesh, and U. Shaked. Robust simple adaptive model following for linear time-delay systems with guaranteed H-infinity performance. *IMA Journal of Mathematical Control and Information*, 27(4):475–491, 2010.

[4] Z. T. Dydek, A. M. Annaswamy, J.-J. E. Slotine, and E. Lavretsky. Composite adaptive posicast control for a class of LTI plants with known delay. *Automatica*, 49(6):1914–1924, 2013.

[5] A. Fradkov. Synthesis of adaptive system of stabilization of linear dynamic plants. *Automation and Remote Control*, (12):96–103, 1974.

[6] A. Fradkov. Quadratic Lyapunov functions in the adaptive stability problem of a linear dynamic target. *Siberian Mathematical Journal*, 17(2):341–348, 1976.

[7] A. Fradkov. Speed-gradient scheme and its application in adaptive control problems. *Automation and Remote Control*, 40(9):1333–1342, 1980.

[8] A. Fradkov and B. R. Andrievsky. Passification-based robust flight control design. *Automatica*, 47(12):2743–2748, 2011.

[9] E. Fridman. *Introduction to Time-Delay Systems: Analysis and Control*. Systems & Control: Foundations & Applications. Springer International Publishing, 2014.

[10] E. Fridman, A. Seuret, and J.-P. Richard. Robust sampled-data stabilization of linear systems: an input delay approach. *Automatica*, 40(8):1441–1446, 2004.

[11] E. Fridman and U. Shaked. Delay-dependent stability and H-infinity control: Constant and time-varying delays. *International Journal of Control*, 76(1):48–60, 2003.

[12] K. Gu, V. L. Kharitonov, and J. Chen. *Stability of Time-Delay Systems*. Birkhauser, Boston, 2003.

[13] Y. He, Q.-G. Wang, C. Lin, and M. Wu. Delay-range-dependent stability for systems with time-varying delay. *Automatica*, 43(2):371–376, 2007.

[14] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 3rd edition, 2002.

[15] M. Krstic. Compensation of Infinite-Dimensional Actuator and Sensor Dynamics. *IEEE Control Systems Magazine*, 30(1):22–41, 2010.

[16] K. Liu and E. Fridman. Delay-dependent methods and the first delay interval. *Systems & Control Letters*, 64:57–63, 2014.

[17] B. Mirkin and P. Gutman. Adaptive control of linear time delay systems. In *American Control Conference*, pages 1241–1246, 2004.

[18] B. Mirkin and P.-O. Gutman. Robust Adaptive Output-Feedback Tracking for a Class of Nonlinear Time-Delayed Plants. *IEEE Transactions on Automatic Control*, 55(10):2418–2424, 2010.

[19] I. Mizumoto. Parallel feedforward compensator design and ASPR based adaptive output feedback control for a time-delay system. In *American Control Conference*, pages 4909–4914, 2013.

[20] J. P. Nelson, M. J. Balas, and R. S. Erwin. Model Reference Adaptive Control of Mildly Non-Linear Systems with Time Varying Input Delays Part I. In *Advances in Aerospace Guidance, Navigation and Control*, pages 49–59. Springer Berlin Heidelberg, 2013.

[21] P. Park, J. W. Ko, and C. Jeong. Reciprocally convex approach to stability of systems with time-varying delays. *Automatica*, 47(1):235–238, 2011.

[22] A. Selivanov, E. Fridman, and A. Fradkov. Adaptive control of systems with fast varying unknown delay in measurements. In *52nd IEEE Conference on Decision and Control*, pages 5583–5587. IEEE, 2013.

[23] M. H. Toodeshki, S. Hosseinnia, and J. Askari. Adaptive robust control of uncertain systems with state and input delay. *International Journal of Control, Automation, and Systems*, 8(6):1212–1220, 2011.

[24] A. Tsykunov. *Adaptive control of systems with aftereffect*. Nauka, Moscow, 1984.

[25] Z. Zhang, S. Xu, and Y. Chu. Adaptive stabilisation for a class of non-linear state time-varying delay systems with unknown time-delay bound. *IET Control Theory & Applications*, 4(10):1905–1913, 2010.

A Proof of Theorem 1

We analyze the dynamics of (8) separately for $t \in [0, t_*]$ (A.1) and $t \in [t_*, \infty)$ (A.2).

A.1 State bound for $t \in [0, t_*]$

Assumption 1 implies $t - r(t) < 0$, therefore $\dot{x}(t) = A_\xi x(t)$. Thus,

$$\|x(t)\| \leq e^{\Lambda_A t} \|x_0\|, \tag{A.1}$$

where $\Lambda_A = \max_\xi \|A_\xi\|$. Note that

$$\|A_\xi\| \le \sum_i \xi_i \|A_i\| \le \max_i \|A_i\|$$

and for appropriate ξ, $\|A_\xi\| = \max_i \|A_i\|$. Therefore,

$$\max_\xi \|A_\xi\| = \max_i \|A_i\|.$$

Note that (A.1) is preserved for $t < 0$, hence $\|x(t - r_2(t))\| \le e^{\Lambda_A(t-r_2(t))}\|x_0\| \le e^{\Lambda_A t}\|x_0\|$. As a result, we have

$$k(t) - k(0) = \gamma^{-2} \int_0^t \left(g^T C x(s - r_2(s))\right)^2 ds \le \frac{c_1\|x_0\|^2}{\gamma^2},$$

where

$$c_1 = \frac{\Lambda_C^2}{2\Lambda_A}\left(e^{2\Lambda_A t_*} - 1\right), \tag{A.2}$$

with $\Lambda_C = \|g^T C\|$. By conditions of the theorem $k(0) \le k_*$, therefore $k(t) - k_* \le k(t) - k(0) \le c_1\gamma^{-2}\|x_0\|^2$. Equation (8) implies $k(t) \ge k(0)$, thus $k_* - k(t) \le k_* - k(0)$. Finally, for $t \in [0, t_*)$ we have

$$|k(t) - k_*| \le c_* = \max\left\{k_* - k(0), c_1\gamma^{-2}\delta^2\right\}, \tag{A.3}$$

where the inequality $\|x(0)\| \le \delta$ was used. Since the right-hand side of (8) is piecewise continuous, functions $x(t)$ and $k(t)$ are continuous for $t > 0$, therefore, (A.1) and (A.3) are valid for $t = t_*$.

A.2 The bound on $\dot V$ for $t \in [t_*, \infty)$ under (13)

Assumption 1 implies $t - r(t) \ge 0$, therefore, $\dot x(t)$ and $\dot k(t)$ do not depend on $x(t)$ with $t < 0$. Thus, we set $x(t) = x_0$ for $t < 0$ and consider V given by (12) (see [16] for details).

Now we calculate the derivative of V along the trajectories of (8) for $t \in [t_*, \infty)$. Denote

$$\mu(t) = \frac{1}{h}\int_{t-r(t)}^{t-r_2(t)} \dot x(s)\,ds,\ \nu(t) = \frac{1}{h}\int_{t-r_2(t)}^{t} \dot x(s)\,ds.$$

Then

$$\begin{aligned}
\dot V_0 &= x^T(t)[PA_* + A_*^T P]x(t)\\
&+ 2x(t)^T k_* PBg^T Ch(\mu(t) + \nu(t))\\
&+ 2\big(k_* - k(t)\big)x^T(t - r_2(t))PBg^T Cx(t - r_2(t))\\
&- 2(k_* - k(t))x^T(t - r_2(t))PBg^T Ch\mu(t)\\
&+ 2(k_* - k(t))h\nu^T(t)PBg^T Cx(t - r(t))\\
&+ 2\big(k(t) - k(t - r_1(t))\big)x^T(t)PBg^T Cx(t - r(t))\\
&+ 2\big(k(t) - k_*\big)\big(g^T Cx(t - r_2(t))\big)^2,
\end{aligned}$$

where $A_* = A_\xi - k_* Bg^T C$. Using the relation $PB = C^T g$ we find that

$$\begin{aligned}
\dot V_0 &= x^T(t)[PA_* + A_*^T P]x(t)\\
&+ 2x^T(t)k_* hPBg^T C(\mu(t) + \nu(t))\\
&- 2(k_* - k(t))hx^T(t - r_2(t))PBg^T C\mu(t)\\
&+ 2(k_* - k(t))h\nu^T(t)PBg^T Cx(t - r(t))\\
&+ 2\big(k(t) - k(t - r_1(t))\big)x^T(t)PBg^T Cx(t - r(t)).
\end{aligned}$$

Further

$$\begin{aligned}
\dot V_S &= x^T(t)Sx(t) - x^T(t - h)Sx(t - h),\\
\dot V_R &= h^2 \dot x^T(t)R\dot x(t) - h\int_{t-h}^{t} \dot x^T(s)R\dot x(s)\,ds.
\end{aligned}$$

Denote

$$\alpha_1 = \frac{h - r(t)}{h}, \quad \alpha_2 = \frac{r(t) - r_2(t)}{h}, \quad \alpha_3 = \frac{r_2(t)}{h},$$

$$f_1(t) = \int_{t-h}^{t-r(t)} \dot x^T(s)\,ds\,R\int_{t-h}^{t-r(t)} \dot x(s)\,ds,$$

$$f_2(t) = h^2\mu^T(t)R\mu(t), \quad f_3(t) = h^2\nu^T(t)R\nu(t).$$

Using Jensen inequality [12, p.322] we have

$$\begin{aligned}
&- h\int_{t-h}^{t} \dot x^T(s)R\dot x(s)\,ds = -h\int_{t-h}^{t-r(t)} \dot x^T(s)R\dot x(s)\,ds\\
&- h\int_{t-r(t)}^{t-r_2(t)} \dot x^T(s)R\dot x(s)\,ds - h\int_{t-r_2(t)}^{t} \dot x^T(s)R\dot x(s)\,ds\\
&\le -\left[\frac{1}{\alpha_1}f_1(t) + \frac{1}{\alpha_2}f_2(t) + \frac{1}{\alpha_3}f_3(t)\right].
\end{aligned}$$

Since G_j $(j = 1, 2, 3)$ are such that

$$\begin{pmatrix} R & G_j \\ * & R \end{pmatrix} \ge 0$$

it follows from Park's theorem [21, Theorem 1] that

$$
\begin{aligned}
&-\left[\frac{1}{\alpha_1}f_1(t)+\frac{1}{\alpha_2}f_2(t)+\frac{1}{\alpha_3}f_3(t)\right]\\
&\quad\le -\left[f_1(t)+f_2(t)+f_3(t)+2g_1(t)+2g_2(t)+2g_3(t)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
g_1(t) &= h\int_{t-h}^{t-r(t)} \dot{x}^T(s)\,ds\,G_1\mu(t),\\
g_2(t) &= h\int_{t-h}^{t-r(t)} \dot{x}^T(s)\,ds\,G_2\nu(t),\\
g_3(t) &= h^2\mu^T(t)G_3\nu(t).
\end{aligned}
$$

Using representation $x(t-r_2(t)) = x(t)-h\nu(t)$ we finally arrive at

$$
\dot{V} \le \eta^T(t)W\eta(t) + h^2\dot{x}^T(t)R\dot{x}(t), \tag{A.4}
$$

where

$$
\eta(t) = (x^T(t), x^T(t-r(t)), x^T(t-h)^T, \nu^T(t), \mu^T(t))^T,
$$

$$
W = \begin{pmatrix}
H_1 & H_2(c) & 0 & H_3 & H_4(a)\\
* & -R & R & H_5(a) & -hG_1\\
* & * & -(S+R) & hG_2 & hG_1\\
* & * & * & -h^2R & H_6(a)\\
* & * & * & * & -h^2R
\end{pmatrix},
$$

with $a = k_*-k(t)$, $c = k(t)-k(t-r_1(t))$, $H_1 = P[A_\xi - Bk_*g^TC] + [A_\xi - Bk_*g^TC]^TP + S$. Substituting right-hand side of (8) instead of $\dot{x}(t)$ into (A.4) and applying Schur complement theorem [12, p.318] we find that, if for $i = 1,\dots,M$

$$
\mathcal{H}_i(a,b,c) < 0, \tag{A.5}
$$

with a, b, c given by (11), then $\exists\,\varepsilon > 0$: $\dot{V}(t) \le -\varepsilon\|\eta(t)\|^2$, where $V(t) = V(x_t,\dot{x}_t,k(t))$.

A.3 Proof of (13) *for* $t \ge 0$

Now we show that $|a| \le M_*$, $|b| \le M_*$, and $|c| \le M_1$, what will guarantee negative definiteness of $\dot{V}$. Using estimates for $|k_* - k(t)|$ and $\|x(t)\|$ on $t \in [0,t_*]$, we calculate

$$
V(t_*) \le c_x\|x_0\|^2 + \gamma^2c_*^2,
$$

where c_* is from (A.3) and

$$
\begin{aligned}
c_x &= \|P\|e^{2\Lambda_At_*} + \|S\|\left(h - t_* + \frac{1}{2\Lambda_A}\left(e^{2\Lambda_At_*}-1\right)\right)\\
&+ \frac{h\Lambda_{ARA}}{2\Lambda_A}\Big[e^{2\Lambda_At_*}t_* + \frac{1}{2\Lambda_A}\left(1-e^{2\Lambda_At_*}\right)\\
&+ (h-t_*)\left(e^{2\Lambda_At_*}-1\right)\Big],
\end{aligned} \tag{A.6}
$$

$$
\Lambda_{ARA} = \max_\xi \|A_\xi^TRA_\xi\|.
$$

By conditions of the theorem $|k_* - k(0)| < M_*$. If γ is large enough then

$$
\|x_0\|^2 < \frac{\gamma^2M_*}{c_1}.
$$

Hence, $c_* < M_*$. By increasing γ one can ensure that

$$
\|x_0\|^2 < \gamma^2(M_*^2 - c_*^2)c_x^{-1},
$$

what will guarantee

$$
V(t_*) < \gamma^2M_*^2.
$$

Now we show that $V(t) < \gamma^2M_*^2$ for $t \in [t_*,\infty)$. Let $t_1 = \min\{t \in [t_*,\infty) | V(t) = \gamma^2M_*^2\}$. Then for $s \in [t_*,t_1]$ we have

$$
V(s) \le \gamma^2M_*^2 \Rightarrow \begin{cases} |k_* - k(s)| \le M_*,\\ \|x(s)\|^2 \le \gamma^2M_*^2\lambda_{\min}^{-1}(P). \end{cases} \tag{A.7}
$$

Since $c_* < M_*$, (A.3) implies $|k_* - k(t)| \le M_*$ for $t \le t_1$. We require γ to be large enough to ensure

$$
\|x_0\|^2 \le \gamma^2M_*^2e^{-2\Lambda_At_*}\lambda_{\min}^{-1}(P).
$$

In this case (A.1) and (A.7) guarantee that $\|x(t)\|^2 \le \gamma^2M_*^2\lambda_{\min}^{-1}(P)$ for $t \le t_1$. Thus, for $t \le t_1$

$$
\begin{aligned}
|c| &= |k(t-r_1(t)) - k(t)|\\
&\le \int_{t-r_1(t)}^{t} \gamma^{-2}(g^TCx(s-r_1(s)))^2\,ds \le h_1\frac{M_*^2\Lambda_C^2}{\lambda_{\min}(P)}.
\end{aligned} \tag{A.8}
$$

As a result, for $t \le t_1$ we have:

$$
\begin{aligned}
&|k_* - k(t)| \le M_*,\\
&|k_* - k(t-r_1(t))| \le M_*,\\
&|k(t-r_1(t)) - k(t)| \le M_1.
\end{aligned} \tag{A.9}
$$

In this case conditions of the theorem guarantee (A.5) for $t \le t_1$. Therefore $\exists\,\varepsilon > 0$: $\dot{V}(t) \le -\varepsilon\|\eta(t)\|^2$. Since $V(t_1) = V(t_*) + \int_{t_*}^{t_1}\dot{V}(s)\,ds$, we have $V(t_1) \le V(t_*) < \gamma^2M_*^2$. The latter contradicts to $V(t_1) = \gamma^2M_*^2$, that is t_1 does not exist and, therefore, $V(t) < \gamma^2M_*^2$ for $t \in [t_*,\infty)$ and (A.9) are valid for $t \ge 0$.

A.4 Proof of (7)

We have proved that $\exists\,\varepsilon > 0$: $\dot{V}(t) \le -\varepsilon\|\eta(t)\|^2$ for $t \ge t_*$. Since $V(t)$ is a nonnegative decreasing function, it has a finite limit: $\lim_{t\to\infty} V(t) < \infty$. Thus,

$$\lim_{t\to\infty} V(t) = V(t_*) + \int_{t_*}^{\infty} \dot{V}(s)\, ds$$
$$\leq V(t_*) - \varepsilon \int_{t_*}^{\infty} \|\eta(s)\|^2\, ds.$$

Therefore, $\varepsilon \int_{t_*}^{\infty} \|\eta(s)\|^2\, ds < \infty$, i.e. $\int_{t_*}^{\infty} \|x(s)\|^2\, ds < \infty$. Boundedness of V implies boundedness of $x(t)$ and $k(t)$. Therefore, $\dot{x}(t)$ given by (8) is bounded and $x(t)$ is uniformly continuous. Then from Barbalat's lemma [14, Lemma 8.2] we have $\|x(t)\| \xrightarrow[t\to\infty]{} 0$. Moreover, $\int_{t_*}^{\infty} \|\eta(s)\|^2\, ds < \infty$ implies that there exists a finite limit

$$\lim_{t\to\infty} k(t) = k(t_*) + \gamma^{-2} \int_{t_*}^{\infty} \left(g^T C x(s - r_2(s))\right)^2\, ds,$$

that is $k(t)$ tends to a constant value.