Dissipativity of T-Periodic Linear Systems

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***Abstract*—It is proved that the existence of a positive-definite storage function is necessary and sufficient for strict dissipativity of linear systems with periodic coefficients. The connection between strict dissipativity of the system and a nonoscillatory property of an associated Hamiltonian system is established.**

***Index Terms*—Dissipativity, Hamiltonian systems, linear periodic systems.**

I. Introduction

THE concept of dissipativity for linear and nonlinear systems was introduced by Willems in 1972 [22]. It is closely related to hyperstability introduced by Popov [19], [20] and also to absolute stability. An extensive literature is devoted to frequency-domain absolute stability criteria. (Let us mention only the survey [23].) In [8] and [15], the first results on necessity of frequency-domain conditions of absolute stability for systems with several integral constraints were established. Dissipativity is closely related to the energy flow in physical systems and provide physical insight important for study of the systems. Different versions of dissipativity were investigated in detail in [9]–[12].

An interest has been renewed recently in the investigation of systems with integral quadratic constraints which have been studied since the 1960s. Various problems of nonconvex optimization [25], analysis [14], [21], and design [3], [16] of nonlinear uncertain systems can be solved by reduction to problems with integral constraints. Recently, new generalizations of dissipativity have appeared, e.g., structured dissipativity [21] and quasi-dissipativity [18].

The concept of dissipativity is intimately connected to that of a storage function. These functions provide convenient Lyapunov functions in stability analysis of the system. Moreover, it has been shown for linear time-invariant systems [22] that dissipativity is equivalent to existence of a storage function and satisfaction of a frequency-domain condition. This result can be derived from the so called frequency theorem [otherwise called the Kalman–Yakubovich–Popov (KYP) lemma] giving necessary and sufficient conditions for solvability of some matrix inequalities. Conditions for existence of a storage function were extended to time-varying and nonlinear systems [1], [9]. However, some gap between the necessary and sufficient conditions for the time-varying case still remains. Therefore, it is of interest to examine for which classes of systems it is possible to remove this gap.

It has turned out that the gap between the necessary and sufficient conditions of dissipativity can be completely cancelled for T-periodic linear systems. Periodic control systems are of great importance for applications in engineering, physics, and science [17]. Among the classical fields of application, chemical plant control and helicopter modeling are well established; recently, significant applications to aerospace engineering (in particular satellite attitude control) came to the stage. Periodic models are of large interest also in time-series analysis, econometrics and finance, and in all other cases when seasonal phenomena has to be modeled.

In this paper, the necessary and sufficient conditions of dissipativity for T-periodic linear systems are obtained by using an appropriate extension of the KYP lemma [29], [30]. The main result of this paper provides necessary and sufficient conditions for strict dissipativity of T-periodic linear systems. A convenient way of formalizing these conditions is based on the nonoscillatory property of a certain associated Hamiltonian system. (The concept of nonoscillatority for Hamiltonian systems was introduced in [24], while the connection between nonoscillatority and absolute stability for T-periodic systems was established in [27] and [28].) It is shown that, similarly to time-invariant case, strict dissipativity is equivalent to the existence of a strong storage function. Also, it is shown that strict dissipativity is equivalent to the solvability of the corresponding Riccati equation or to the strong nonoscillatory property of an associated Hamiltonian system. The preliminary version of this paper was presented in [33].

It is worth noting that according to the recent tradition in control theory, we use the term "dissipativity" in sense introduced by Willems [22]. However, this term is still being used in the theory of differential equations with a different meaning as all the trajectories falling into some bounded set after some time. This second meaning could be referred to as "dissipativity in sense of Levinson" or "ultimate boundedness"; see, e.g., [4] and [34].

The structure of this paper is as follows. In Section II, the main definitions and the problem formulation are given. In Section III, associated Hamiltonian system is introduced and studied. The main results are formulated in Section IV. Section V is devoted to proofs of the main results.

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II. Problem Statement

Consider a linear periodic controlled system

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t). \tag{2.1}$$

Here, $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$ stand for the state vector and the control input of the system. Hereafter, functions $x(\cdot), u(\cdot)$ are assumed to be locally square integrable. Consider also the associated quadratic form

$$F(t, x, u) = \begin{bmatrix} x \\ u \end{bmatrix}^* \begin{bmatrix} G(t) & g(t) \\ g(t)^* & \Gamma(t) \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{2.2}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. All real matrix-valued functions $A(\cdot)$, $B(\cdot)$, $G(\cdot)$, $g(\cdot)$, and $\Gamma(\cdot)$ are defined on $\mathbb{R}$, bounded and T-periodic, $G(t) = G(t)^*, \Gamma(t) = \Gamma(t)^*$, and

$$\Gamma(t) \geq \gamma_0 I, \qquad \gamma_0 = \text{const} > 0. \tag{2.3}$$

We assume the pair $(A(\cdot), B(\cdot))$ to be controllable, that is, for any t_1 t_2, $a_1 \in \mathbb{R}^n$, and $a_2 \in \mathbb{R}^n$, such that $t_1 < t_2$, there exists a function $u(\cdot)$ such that the solution of (2.1) with $x(t_1) = a_1$ satisfies $x(t_2) = a_2$.

Definition 1: System (2.1) is called dissipative with the supply rate (2.2) if

$$\int_{t_1}^{t_2} F(t, x(t), \qquad u(t))\, dt \geq 0 \tag{2.4}$$

for any $t_1 \leq t_2$ and for any $x(t), u(t)$ such that (2.1) holds and $x(t_1) = 0$.

Definition 2: System (2.1) is called strictly dissipative with the supply rate F if for some $\delta > 0$ it is dissipative with the supply rate $F_\delta = F - \delta(|x|^2 + |u|^2)$.

In [22], the supply rate is defined as a function of input u and an output y. This reduces to the current formulation after substituting for y as a function of x and u.

Definition 3: System (2.1) is called (t_1, T)-dissipative with the supply rate F, if (2.4) is valid for any $t_2 = t_1 + kT$, $k = 0, 1, \ldots$.

It is easy to see that the (t_1, T)-dissipativity definition is equivalent to the following.

Definition 3a: System (2.1) is called (t_1, T)-dissipative with the supply rate F if

$$\int\limits_{t_1 - kT}^{t_1} F(t, x(t), u(t)) dt \geq 0 \tag{2.5}$$

for any $k = 0, 1, 2, \ldots$ and any $x(\cdot), u(\cdot)$ satisfying (2.1) and $x(t_1 - kT) = 0$.

Definition 4: System (2.1) is called strictly (t_1, T)-dissipative with the supply rate F if it is (t_1, T)-dissipative with the supply rate $F_\delta = F - \delta(|x|^2 + |u|^2)$ for some $\delta > 0$.

The problem is to find conditions (necessary and sufficient) for dissipativity, strict dissipativity, and (t_1, T)-dissipativity of system (2.1).

In the following, we study only the cases of strict dissipativity and strict (t_1, T)-dissipativity.

Definition 5: Function $V(t, x)$ is called a storage function for system (2.1) with supply rate $F(t, x, u)$ if $V(t, x) \geq 0$, $V(t, 0) = 0$ and

$$\int_{t'}^{t''} F(s, x(s), u(s)) ds \geq V(t'', x(t'')) - V(t', x(t')) \tag{2.6}$$

for all $t' \leq t''$ and $x(t)$, $u(t)$ satisfying (2.1). $V(t, x)$ is called a strong storage function if, additionally, $V(t, x) > 0$ for $x \neq 0$.

Definition 6: Function $V(t, x)$ is called (t_1, T)-storage function for system (2.1) with supply rate $F(t, x, u)$, if $V(t_1 + kT, x) \geq 0$ for $k = 0, 1, 2, \ldots$ and (2.6) holds. Similarly, $V(t, x)$ is called a strong (t_1, T)-storage function, if $V(t_1 + kT, x) > 0$ for $x \neq 0$, $k = 0, 1, 2, \ldots$ and (2.6) holds.

Obviously, the existence of the storage function [(t_1, T)-storage function] implies the dissipativity [(t_1, T)-dissipativity] of the system (2.1). The opposite, in general, is not true. However, for a T-periodic linear system, the existence of a storage function with supply rate $F_\delta = F - \delta(|x|^2 + |u|^2)$ is necessary and sufficient for strict dissipativity, as follows from the results of this paper.

III. Associated Hamiltonian System and the Frequency Theorem for Periodic Systems

In order to present the main results of this paper, we need to introduce some concepts related to the "frequency theorem" (KYP lemma) for periodically time-varying systems.

We introduce the function

$$\mathcal{H}(t, x, u, \psi) = \psi^*(Ax + Bu) - F(t, x, u) \tag{3.1}$$

where $\psi \in \mathbb{R}^n$, and consider the system

$$\frac{dx}{dt} = \left(\frac{\partial \mathcal{H}}{\partial \psi}\right)^* \quad \frac{d\psi}{dt} = -\left(\frac{\partial \mathcal{H}}{\partial x}\right)^* \quad \frac{\partial \mathcal{H}}{\partial u} = 0. \tag{3.2}$$

Since (2.3) holds, the last (3.2) can be solved for u

$$u = \Gamma^{-1}\left(\frac{1}{2}B^*\psi - g^*x\right) := \mathcal{U}_*(t, x, \psi). \tag{3.3}$$

Denote

$$\mathcal{H}||_{u=\mathcal{U}_*(t,x,\psi)} = \mathcal{H}_0(t, x, \psi). \tag{3.4}$$

Then, system (3.2) can be rewritten in the form

$$\frac{dx}{dt} = \left(\frac{\partial \mathcal{H}_0}{\partial \psi}\right)^* \quad \frac{d\psi}{dt} = -\left(\frac{\partial \mathcal{H}_0}{\partial x}\right)^*. \tag{3.5}$$

The system (3.5) is called an associated Hamiltonian system for the previous problems.

Let

$$z(t) = \begin{bmatrix} x(t) \\ \psi(t) \end{bmatrix} \in \mathbb{R}^{2n} \quad J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$$
$$H(t) = \begin{bmatrix} g(t)\Gamma^{-1}(t)g(t)^* - G(t) & A^*(t) - g(t)\Gamma^{-1}(t)B^*(t) \\ A(t) - B(t)\Gamma^{-1}(t)g^*(t) & B(t)\Gamma^{-1}(t)B^*(t) \end{bmatrix}. \tag{3.6}$$

Then, the system (3.5) takes the form

$$J\frac{dz}{dt} = H(t)z. \tag{3.7}$$

Let $Z(t)$ be the transition matrix of (3.7)

$$J\frac{dZ}{dt} = H(t)Z \quad Z(0) = I_{2n}. \tag{3.8}$$

Introduce the "frequency-domain" condition

$$\det[Z(T) - e^{i\omega}I_{2n}] \neq 0 \qquad \forall\omega \in [0, 2\pi]. \tag{3.9}$$

This means that system (3.7) has no multipliers on the unit circle. Due to symmetry of spectrum $Z(T)$ with respect to the unit circle, condition (3.9) implies that the Hamiltonian (3.7) has $2n$ real linearly independent solutions $z_j^+, z_j^-, j = 1, \dots, n$, such that

$$z_j^+(t) = \begin{bmatrix} x_j^+(t) \\ \psi_j^+(t) \end{bmatrix} \longrightarrow 0, \qquad \text{if} \quad t \to +\infty \tag{3.10}$$

$$z_j^-(t) = \begin{bmatrix} x_j^-(t) \\ \psi_j^-(t) \end{bmatrix} \longrightarrow 0, \qquad \text{if} \quad t \to -\infty. \tag{3.11}$$

Introduce $n \times n$ matrix functions

$$\begin{aligned} X_+(t) &= \left[x_1^+(t), \dots, x_n^+(t)\right] \\ \Psi_+(t) &= \left[\psi_1^+(t), \dots, \psi_n^+(t)\right]. \end{aligned} \tag{3.12}$$

Analogously, let

$$\begin{aligned} X_-(t) &= \left[x_1^-(t), \dots, x_n^-(t)\right] \\ \Psi_-(t) &= \left[\psi_1^-(t), \dots, \psi_n^-(t)\right]. \end{aligned} \tag{3.13}$$

Definition 7: We call the Hamiltonian system (3.7) [or (3.5)] strongly nonoscillatory if the frequency-domain condition (3.9) holds and

$$\det X_+(t) \neq 0, \qquad \text{for} \quad 0 \leq t \leq T. \tag{3.14}$$

We will see in the following that for any strongly nonoscillatory system one has also

$$\det X_-(t) \neq 0, \qquad \text{for} \quad 0 \leq t \leq T. \tag{3.15}$$

Note that (3.14) implies that $\det X^+(t) \neq 0$ for all $t \in \mathbb{R}$.

The following proposition, crucial for the results exposed later, generalizes the well-known KYP lemma for the case of T-periodic systems and in its essential part was obtained in [29] and [30].

Lemma 1: Suppose the pair $(A(\cdot), B(\cdot))$ in (2.1) to be controllable and (2.3) holds. Then, the following assertions are equivalent.

A) Hamiltonian system (3.7) is strongly nonoscillatory.

B) There exists an absolutely continuous T-periodic $n \times n$-matrix valued function $R_+(t) = R_+(t)^*$ such that

$$\begin{aligned} &R_+(t) + R_+(t)A(t) + A(t)^*R_+(t) + G(t) \\ &\quad = [R_+(t)B(t) + g(t)]\Gamma^{-1}(t)[R_+(t)B(t) + g(t)]^* \end{aligned} \tag{3.16}$$

and the system $\dot{x}(t) = [A(t) + B(t)r_+(t)^*]x(t)$, where

$$r_+(t) = -[R_+(t)B(t) + g(t)]\Gamma^{-1}(t) \tag{3.17}$$

is exponentially stable (all multipliers lie in the strict unit circle).

C) There exist a function $V_+(t,x) = x^*R_+(t)x$ with absolutely continuous T-periodic $n \times n$-matrix $R_+(t) = R_+(t)^*$ and T-periodic $n \times m$-matrix function $r_+(t)$ such that

$$\begin{aligned} &\overset{\circ}{V}_+(t,x,u) + F(t,x,u) \\ &\qquad = (u - r_+(t)^*x)^*\Gamma(t)(u - r_+(t)^*x) \end{aligned} \tag{3.18}$$

and the system $\dot{x}(t) = [A(t) + B(t)r(t)^*]x(t)$ is exponentially stable. Here, $\overset{\circ}{V}_+$ stands for the derivative of V_+ due to the system (2.1)

$$\begin{aligned} \overset{\circ}{V}_+(t,x,u) = {}& x^*\dot{R}_+(t)x + x^*R_+(t)(A(t)x + B(t)u) \\ & + (A(t)x + B(t)u)^*R(t)x. \end{aligned}$$

Using (3.18), the functions $r_+(t)$ and $R_+(t)$ are easily shown to satisfy (3.17).

D) There are numbers $\delta > 0$ and $t_0 \in \mathbb{R}$ such that the inequality

$$\int_{t_0}^{+\infty} F(t, x(t), u(t))dt \geq \delta \int_{t_0}^{+\infty} \left(|x(t)|^2 + |u(t)|^2\right) dt \tag{3.19}$$

is valid whenever $x(\cdot)$ and $u(\cdot)$ satisfy (2.1), $x(t_0) = 0$, and $|x(\cdot)|, |u(\cdot)| \in L_2(t_0; +\infty)$.

E) There are numbers $\delta > 0$ and $t_0 \in \mathbb{R}$ such that the inequality

$$\int_{t_0}^{t_0+T} F(t, x(t), u(t))dt \geq \delta \int_{t_0}^{t_0+T} \left(|x(t)|^2 + |u(t)|^2\right) dt \tag{3.20}$$

is valid whenever complex valued functions $x(\cdot)$ and $u(\cdot)$ satisfy (2.1), $|x(\cdot)|, |u(\cdot)| \in L_2(t_0; +\infty)$, and $x(t_0 + T) = \lambda x(t_0)$, where $|\lambda| = 1$. [For complex vectors x and u, operation * in (2.2) stands for the complex-conjugate transposing.]

A') Hamiltonian system (3.7) satisfies (3.9) and $\det X_-(t) \neq 0$ for all $t \in [0; T]$.

B') There exists an absolutely continuous T-periodic $n \times n$-matrix valued function $R_-(t) = R_-(t)^*$ such that

$$\begin{aligned} &\dot{R}_-(t) + R_-(t)A(t) + A(t)^*R_-(t) + G(t) \\ &\quad = [R_-(t)B(t) + g(t)]\Gamma^{-1}(t)[R_-(t)B(t) + g(t)]^* \end{aligned} \tag{3.21}$$

and the system $\dot{x}(t) = [A(t) + B(t)r_-(t)^*]x(t)$, where

$$r_-(t) = -[R_-(t)B(t) + g(t)]\Gamma^{-1}(t) \tag{3.22}$$

is exponentially antistable (all multipliers lie outside the unit circle).

C') There exist a function $V_-(t,x) = x^* R_-(t)x$ with absolutely continuous T-periodic $n \times n$-matrix $R_-(t) = R_-(t)^*$ and T-periodic $n \times m$-matrix function $r_-(t)$ such that

$$\overset{\circ}{V}_-(t,x,u) + F(t,x,u) = (u - r_-(t)^* x)^* \Gamma(t)(u - r_-(t)^* x) \quad (3.23)$$

and the system $\dot{x}(t) = [A(t) + B(t)r_-(t)^*]x(t)$ is exponentially antistable.

D') There are numbers $\delta > 0$ and $t_0 \in \mathbb{R}$ such that the inequality

$$\int_{-\infty}^{t_0} F(t, x(t), u(t))dt \geq \delta \int_{-\infty}^{t_0} (|x(t)|^2 + |u(t)|^2)\, dt \quad (3.24)$$

is valid whenever $x(\cdot)$ and $u(\cdot)$ satisfy (2.1), $x(t_0) = 0$, and $|x(\cdot)|, |u(\cdot)| \in L_2(-\infty; t_0)$.

If these conditions hold, then the functions $R_+(t)$ and $R_-(t)$ in B),C),B'), and C') are unique and are given by

$$R_+(t) = -\Psi_+(t)X_+(t)^{-1} \quad (3.25)$$

where $X(t)$ and $\Psi(t)$ are defined by (3.26) and

$$R_-(t) = -\Psi_-(t)X_-(t)^{-1} \quad (3.26)$$

where $X_-(t)$ and $\Psi_-(t)$ are defined by (3.13). Moreover

$$a^* R_+(t_0)a = \min \int_{t_0}^{+\infty} F(t, x(t), u(t))\, dt \quad (3.27)$$

for all $t_0 \in \mathbb{R}$, where min is taken over the set of all solutions $x(\cdot)$ and $u(\cdot)$ of (2.1), such that $x(0) = a$ and $|x(\cdot)|, |u(\cdot)| \in L_2(t_0; +\infty)$. The optimal process in the specified optimization problem is generated by the feedback $u(t) = r_+(t)x(t)$. Analogously

$$-a^* R_-(t_0)a = \min \int_{-\infty}^{t_0} F(t, x(t), u(t))\, dt \quad (3.28)$$

for any $t_0 \in \mathbb{R}$, where min is taken over the set of all solutions $x(\cdot), u(\cdot)$ of (2.1), such that $x(t_0) = a$ and $|x(\cdot)|, |u(\cdot)| \in L_2(-\infty; t_0)$. The optimal process in the specified optimization problem is generated by the feedback $u(t) = r_-(t)^* x(t)$.

Remark 1: Note that the condition E) is valid for the system (2.1) and the quadratic form (2.1) iff it is valid for the "time-inverted" system

$$\dot{x}(t) = -A(-t)x(t) - B(-t)u(t) \quad (3.29)$$

and the "time inverted" quadratic form $F(-t, x, u)$.

Proof of Lemma 1: The equivalence of conditions A)–E) follows from the results of [30, Th. 2]. However, conditions A')–D') are easily checked to coincide with the conditions A)-D) written for the form $F(-t, x, u)$ and the system (3.29) and, therefore, accordingly to the Remark 1, are equivalent to conditions A)–E). ■

Remark 2: Suppose the system (2.1) and the quadratic form (2.1) to be *time-invariant*. In this case, the "frequency-domain condition" (3.9) means that the constant matrix $J^{-1}H(t) \equiv J^{-1}H$ has no eigenvalues on the imaginary axis. Its characteristic polynomial $\phi(\lambda) = \det(\lambda I - J^{-1}H)$ can be shown to satisfy

$$\phi(i\omega) = (-1)^n (\det \Gamma)^{-1} |\det(i\omega I - A)|^2 \det \Pi(i\omega)$$

where

$$\Pi(i\omega) = \begin{bmatrix} (i\omega I_n - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} G & g \\ g^* & \Gamma \end{bmatrix} \begin{bmatrix} (i\omega I_n - A)^{-1}B \\ I_m \end{bmatrix}$$

for all $\omega \in \mathbb{R}$ such that $\det(i\omega I_n - A) \neq 0$. Thus, the frequency-domain condition (3.9), which can be rewritten as follows:

$$\phi(i\omega) > 0, \qquad \text{for all} \quad \omega \in \mathbb{R}$$

is equivalent to conditions A)–D) in accordance with classical KYP lemma (see [20] and [31]). Particularly, in time-invariant case, conditions (3.14) and (3.15) follow from (3.9). However, as shown in [30], for nonstationary periodic systems, the frequency-domain condition does not longer imply (3.14), and thus, is only necessary, but not sufficient for the validity of assertions A)–E) and A')–D').

Remark 3: The conditions A)–E) remain equivalent if the controllability condition is replaced with a weaker condition of L_2-stabilizability [30]. The pair $(A(\cdot), B(\cdot))$ is said to be L_2-stabilizable, that is, for any $a \in \mathbb{R}^n$ there exist functions $x(\cdot)$ and $u(\cdot)$ such that (2.1) holds, $x(0) = a$, and $|x(\cdot)|, |u(\cdot)| \in L_2(0; +\infty)$. In fact, L_2-stabilizable T-periodic pair turns out to be T-periodically exponentially stabilizable [30]: There exists a feedback of the form $u(t) = h(t)x(t)$ (where $h(t)$ is bounded T-periodic function) rendering the system (2.1) exponentially stable. Analogously, the conditions A')–D') and E) remain equivalent if the pair $(A(\cdot), B(\cdot))$ is L_2-"antistabilizable" in the sense that for any $a \in \mathbb{R}^n$ there exist functions $x(\cdot)$ and $u(\cdot)$ such that (2.1) holds, $x(0) = a$, and $|x(\cdot)|, |u(\cdot)| \in L_2(-\infty; 0)$.

Remark 4: Suppose that for some $n \times n$-matrix valued function $R(t) = R(t)^* = R(t + T)$ the quadratic form $V(t, x) = x^* R(t)x$ satisfies the inequality

$$\overset{\circ}{V}(t, x, u) + F(t, x, u) \geq 0.$$

Then, from (3.27) and (3.28), one can easily see $R_- \leq R \leq R_+$. Particularly, R_- and R_+ are the minimal and the maximal T-periodic solutions of the Riccati equation

$$\dot{R}(t) + R(t)A(t) + A(t)^* R(t) + G(t) = [R(t)B(t) + g(t)]\Gamma^{-1}(t)[R(t)B(t) + g(t)]^*. \quad (3.30)$$

IV. Main Results

In this section, we list the main results of this paper that are the criteria of strong dissipativity.

Theorem 1: The following assertions are equivalent.

1) System (2.1) is strictly dissipative.
2) The associated Hamiltonian system (3.7) is strongly nonoscillatory and for some $\rho > 1$ and any $t_0, t_1 \in \mathbb{R}$, and

$t_0 < t_1$, the matrix $R_+(t)$ defined by (3.25) satisfies the inequality

$$\rho R_+(t_1) \leq \left(\int_{t_0}^{t_1} \Phi_+(t_1;s)B(s)\Gamma^{-1}(s)B^*(s)\Phi_+(t_1;s)^* ds \right)^{-1}. \tag{4.1}$$

Here, $\Phi_+(t;s)$ is the Cauchy transition matrix of the system $\dot{x}(t) = [A(t) + B(t)r_+(t)^*]x(t)$ with $r_+(\cdot)$ given by (3.17).

3) The associated Hamiltonian system (3.7) is strongly nonoscillatory and the matrix $R_-(t)$, defined by (3.26), is negative definite for all $t \in [0;T]$ (or, equivalently, for all $t \in \mathbb{R}$).

If the listed assertions are valid, then the function $V(t,a) = -a^* R_-(t)a$ is a strong storage function, i.e., (2.6) is valid and $V(t,a) > 0$ unless $a \neq 0$.

Remark 5: As the pair $(A(\cdot), B(\cdot))$ is controllable, the pair $(A(\cdot) + B(\cdot)r_+^*(\cdot), B(\cdot))$ is controllable as well. Thus, all inverse matrices in the right side of (4.1) exist.

Theorem 2: The following assertions are equivalent.

1) System (2.1) is strictly (t_1, T)-dissipative.
2) The associated Hamiltonian system (3.7) is strongly nonoscillatory and for some $\rho > 1$ and any $k = 0, 1, 2, \ldots$ the matrix $R_+(t)$ defined by (3.25) satisfies the inequality (4.1) with $t_0 = t_1 - kT$.
3) The associated Hamiltonian system (3.7) is strongly nonoscillatory and $R_-(t_1) < 0$, where $R_-(t)$ is defined by (3.26).

If the listed assertions are valid, then the function $V(t,a) = -a^* R_-(t)a$ is a strong $(t_1; T)$-storage function.

Theorem 3: Let there exist a solution $R(t) = R(t)^* = R(t+T) < 0$ of (3.30), such that for $r(t) = -(R(t)B(t) + g(t))\Gamma(t)^{-1}$ the equation

$$\dot{x}(t) = [A(t) + B(t)r^*(t)]x(t) \tag{4.2}$$

has no multipliers on the unit circle. Then, the system (2.1) is strictly dissipative. Conversely, if the system (2.1) is strictly dissipative, then the Riccati (3.30) has absolutely continuous T-periodic solution $R(t) = R_-(t) = R_-^*(t) < 0$ such that (4.2) is exponentially antistable.

Theorem 4: Let the Riccati (3.30) have solution $R(t) = R(t)^* = R(t+T)$, such that $R(t_1) < 0$ and, for $r(t) = -(R(t)B(t) + g(t))\Gamma(t)^{-1}$, (4.2) has no multipliers on the unit circle. Then, the system (2.1) is strictly (t_1, T)-dissipative. Conversely, if the system (2.1) is strictly (t_1, T)-dissipative, then the Riccati (3.30) has absolutely continuous T-periodic solution $R(t)$ such that $R(t_1) < 0$ and (4.2) is exponentially antistable.

V. Proofs of the Main Results

The proofs of Theorems 1–4 rely heavily on Lemma 1. We need also several propositions that seem to be of independent interest.

Lemma 2: Let the Hamiltonian system (3.7) be strongly nonoscillatory [consequently, matrices $R_+(t), R_-(t)$ in (3.25) and (3.26) are well defined and absolutely continuous]. Then, for $V_+(s,a) = a^* R_+(s)a$, one has

$$V_+(s,a) = \inf_{\substack{u(\cdot)\\ k=1,2,\ldots}} \int_s^{s+kT} F(t,x(t),u(t))\,dt = \inf_{\substack{u(\cdot)\\ \tau>0}} \int_s^{s+\tau} F(t,x(t),u(t))\,dt. \tag{5.3}$$

Here, the first $\inf$ is taken over all $k = 1, 2, \ldots$ and $u(\cdot)$ such that

$$\dot{x}(t) = A(t)x + B(t)u, \quad x(s) = a, \quad x(s+kT) = 0 \tag{5.4}$$

and the second $\inf$ is taken over all $\tau > 0$ and $u(\cdot)$ such that

$$\dot{x}(t) = A(t)x + B(t)u, \quad x(s) = a, \quad x(s+\tau) = 0. \tag{5.5}$$

Analogously, if the Hamiltonian system (3.7) is strongly nonoscillatory, then for $V_-(s,a) = -a^* R_-(s)a$ [with $R_-(t)$ defined by (3.25)] one has

$$V_-(s,a) = \inf_{\substack{u(\cdot)\\ k=1,2,\ldots}} \int_{s-kT}^{s} F(t,x(t),u(t))\,dt = \inf_{\substack{u(\cdot)\\ \tau>0}} \int_{s-\tau}^{s} F(t,x(t),u(t))\,dt. \tag{5.6}$$

Here, the first $\inf$ is taken over all $k = 1, 2, \ldots$ and $u(\cdot)$ such that

$$\dot{x}(t) = A(t)x + B(t)u \quad x(s) = a \quad x(s-kT) = 0 \tag{5.7}$$

and the second $\inf$ is taken over all $\tau > 0$ and $u(\cdot)$ such that

$$\dot{x}(t) = A(t)x + B(t)u \quad x(s) = a \quad x(s-\tau) = 0. \tag{5.8}$$

Proof of Lemma 2: Let I_1 and I_2 be the first and the second infima in (5.3). It is evident that $I_1 \geq I_2$. Due to (3.18) and (5.5) imply that

$$\int_s^{s+\tau} F(t,x(t),u(t))dt = V_+(s,a) + \int_s^{s+\tau} v(t)^* \Gamma(t) v(t) dt \tag{5.9}$$

where $v(t) = u(t) - r_+(t)^* x(t)$ and $r_+(t)$ is given by (3.17). Thus, $I_2 \geq V_+(s,a)$. To prove the first statement, show that $I_1 \leq V_+(s,a)$.

Rewrite system (2.1) in the form

$$\dot{x}(t) = [A(t) + B(t)r_+(t)^*]x(t) + B(t)v(t). \tag{5.10}$$

Let $\Phi_+(t,s)$ be the Cauchy transition matrix of the system $\dot{x}(t) = [A(t) + B(t)r_+(t)^*]x(t)$ and define $v_k(t)$ for

$k = 2, 3, \ldots$ as follows: $v_k(t) = 0$ for $s \le t \le s + (k-1)T$ or $t > s + kT$ and

$$\begin{aligned} v_k(s + (k-1)T + \tau) &= -B(s+\tau)^* \Phi_+(s+\tau; s)^* Y(s), \\ Y(s) &= \left(\int_s^{s+T} \Phi_+(t;s) B(t) B(t)^* \Phi_+(t;s)^* dt \right)^{-1} \\ &\quad \times \Phi_+(s + (k-1)T; s) a. \end{aligned} \tag{5.11}$$

A straightforward computation shows that for $v(t) = v_k(t)$ and $x(s) = a$ in (5.10) one has $x(s + kT) = 0$. By virtue of the exponential stability of the system (5.10) one obtains that $v_k(\cdot) \to 0$ as $k \to +\infty$ in the $L_2(s; +\infty)$ norm. Applying (5.9) to $v(t) = v_k(t)$ and $\tau = kT$, one obtains that $I_1 \le V_+(s, a)$ that proves the first statement. The second one can be proved in the same way. ∎

Remark 6: The infimum in the right-hand side of (5.6) is called the required supply by Willems [22]. The infimum in the right-hand side of (5.3) is called the available storage.

Lemma 3: Consider the following minimization problem. To minimize the functional

$$J = \int_\tau^t u(s)^* \Gamma(s) u(s) ds$$

over the set of all solutions of the system (2.1) such that $x(\tau) = 0, x(t) = a$, where the system is assumed to be controllable and (2.3) holds. The minimum in this problem is given by

$$\min J = a^* \left(\int_\tau^t \Phi(t;s) B(s) \Gamma^{-1}(s) B^*(s) \Phi(t;s)^* ds \right)^{-1} a \tag{5.12}$$

and is achieved at the solution of (2.1) with

$$u(s) = \Gamma^{-1}(s) B^*(s) \Phi^*(t;s) \times \left(\int_\tau^t \Phi(t;s) B(s) \Gamma^{-1}(s) B^*(s) \Phi(t;s)^* ds \right)^{-1} a. \tag{5.13}$$

Here, $\Phi(t;s)$ stands for the Cauchy transition matrix of the system (2.1).

Lemma 3 is a version of standard optimization results for linear time-varying systems over finite time interval; see, e.g., [13].

Lemma 4: For any $a \in \mathbb{R}^n \setminus \{0\}$, one has

$$\inf_{\substack{\tau < t \\ x(\tau)=0, x(t)=a}} \int_\tau^t (|x(s)|^2 + |u(s)|^2)\, ds > 0 \tag{5.14}$$

where infimum is taken over all $\tau, t \in \mathbb{R}, \tau < t$, and all solutions of (2.1), such that $x(\tau) = 0$ and $x(t) = a$.

Proof of Lemma 4: As functions $A(\cdot)$ and $B(\cdot)$ are bounded, there exist constants $C_1 > 0$ and $C_2 > 0$ such that for any $\tau < t$ one has

$$\|\dot{x}\|^2_{L_2(\tau;t)} \le C_1 \|x\|^2_{L_2(\tau;t)} + C_2 \|u\|^2_{L_2(\tau;t)}.$$

Let $x(\tau) = 0$ and $x(t) = a$; then

$$\begin{aligned} |a|^2 = 2 \int_\tau^t x(s)^* \dot{x}(s) ds &\le \|\dot{x}\|^2_{L_2(\tau;t)} + \|x\|^2_{L_2(\tau;t)} \\ &\le (C_1 + 1) \|x\|^2_{L_2(\tau;t)} + C_2 \|u\|^2_{L_2(\tau;t)}. \end{aligned}$$

Thus, the infimum in the left-hand side of (5.14) is greater than $(C_1 + C_2 + 1)^{-1} |a|^2 > 0$. ∎

Now we turn to proofs of Theorems 1,2.

Proof of Theorem 1: First, we prove implication 1 $\Longrightarrow$ 2. Suppose that the system (2.1) is strictly dissipative. Then, the condition D) of Lemma 1 holds with arbitrary t_0. This is easily seen by substituting $F_\delta(t, x, u) = F(t, x, u) - \delta(|x|^2 + |u|^2)$ (where $\delta > 0$ is sufficiently small) into the (2.4) and passing to the limits as $t_1 \to +\infty$. By Lemma 1, this proves that the Hamiltonian system (3.7) is strongly nonoscillatory. Thus, $R_+(t)$, given by (3.25), is absolutely continuous and satisfies (3.18). Integrating (3.18) and using strict dissipativity, one obtains that for some $\delta_0 > 0$

$$\begin{aligned} \int_{t_0}^{t_1} F(t, x(t), u(t)) dt &= -x(t_1)^* R_+(t_1) x(t_1) \\ &\quad + \int_{t_0}^{t_1} |\Gamma(t)^{1/2} v(t)|^2 dt \ge \delta_0 \int_{t_0}^{t_1} |\Gamma(t)^{1/2} v(t)|^2 dt \end{aligned} \tag{5.15}$$

whenever $t_0 < t_1$ and $x(t_0) = 0$, where $v(t) = u(t) - r_+(t)^* x(t)$. The system (2.1) is rewritten as (5.10), where $A_+(t) = A(t) + B(t) r_+(t)^*$. Taking $\rho = (1 - \delta_0)^{-1}$, one obtains that

$$\rho a^* R_+(t_1) a \le \int_{t_0}^{t_1} |\Gamma^{1/2}(t) v(t)|^2 dt \tag{5.16}$$

if $x(t_0) = 0, x(t_1) = a$, and (5.10) holds. Applying Lemma 3 to the system (5.10), one obtains (4.1).

Proof of the implication 2 $\Longrightarrow$ 1. Let $\delta_0 = 1 - \rho^{-1}$, where ρ is from (4.1). As (4.1) can be rewritten as (5.15), one obtains that (5.10) holds. Since (5.10) is exponentially stable system (all multipliers of $\Phi_+(T;0)$ are inside the unit circle), there exist constants $C_1, C_2 > 0$, determined by the function $\Phi_+(\cdot,\cdot)$ only, such that for any t_0 and t_1 and any functions $x(\cdot)$ and $u(\cdot)$, satisfying (2.1) and $x(t_0) = 0$, one has

$$\begin{aligned} \int_{t_0}^{t_1} |x(t)|^2 dt &\le C_1 \int_{t_0}^{t_1} |v(t)|^2\, dt \\ \int_{t_0}^{t_1} |u(t)|^2 dt &= \int_{t_0}^{t_1} |v(t) + r(t)^* x(t)|^2 dt \le C_2 \int_{t_0}^{t_1} |v(t)|^2\, dt. \end{aligned}$$

Thus, for sufficiently small $\delta > 0$, one has

$$\int_{t_0}^{t_1} F(t, x(t), u(t)) dt \ge \delta \int_{t_0}^{t_1} (|x(t)|^2 + |u(t)|^2) dt \tag{5.17}$$

if $x(t_0) = 0$ and (2.1) holds, that shows the system (2.1) to be strictly dissipative.

Let us prove $1 \Longrightarrow 3$. The strong nonoscillatority was derived from Assertion 1 while proving $1 \Longrightarrow 2$. Lemmas 1 and 2 imply that, for any $a \in \mathbb{R}^n$, one has

$$\begin{aligned} &- a^* R_-(t_1) a \\ &= \inf_{x(t_0)=0, x(t_1)=a} \int_{t_0}^{t_1} F(s, x(s), u(s))\, ds \\ &\geq \delta \inf_{x(t_0)=0, x(t_1)=a} \int_{t_0}^{t_1} (|x(s)|^2 + |u(s)|^2)\, ds \quad (5.18) \end{aligned}$$

for sufficiently small $\delta > 0$, where infima are taken over all $t_0 < t_1$ and all pairs $x(\cdot)$ and $u(\cdot)$, such that (2.1) holds, $x(t_0) = 0$, and $x(t_1) = a$. From Lemma 4, one obtains that $R_-(t_1) < 0$ for all $t_1 \in \mathbb{R}$.

Now, we turn to the proofs of the last implication $3 \Longrightarrow 1$ and the statement on storage functions. From (3.23), one obtains that

$$\int_{t_0}^{t_1} F(t, x(t), u(t)) dt = -x(t_1)^* R_-(t_1) x(t_1) + \int_{t_0}^{t_1} |\Gamma(t)^{1/2} v_-(t)|^2 dt \quad (5.19)$$

for $t_0 < t_1$ and $x(t_0) = 0$, where $v_-(t) = u(t) - r_-(t)^* x(t)$. The system (2.1) is then equivalent to the system

$$\dot{x}(t) = [A(t) + B(t) r_-(t)^*] x(t) + B(t) v_-(t). \quad (5.20)$$

As the system (5.20) is antistable, then there exists constants C_1 and C_2 such that for all $t_0, t_1 \in \mathbb{R}$, $t_0 < t$, one has

$$\begin{aligned} \int_{t_0}^{t_1} |x(t)|^2 dt &\leq C_1 \left(\int_{t_0}^{t_1} |v_-(t)|^2 dt + |x(t_1)|^2 \right) \\ \int_{t_0}^{t_1} |u(t)|^2 dt &\leq C_2 \left(\int_{t_0}^{t_1} |v_-(t)|^2 dt + |x(t_1)|^2 \right). \end{aligned}$$

As $-R_-(t) > 0$ for all t_1, then, for sufficiently small $\delta > 0$, one has (5.17).

From (3.23), its evident that $V(t, a) = -a^* R_-(t) a$ is a storage function. Condition 3 implies that $R_-(t) < 0$ and thus V is in fact a strong storage function. ∎

Proof of Theorem 2: Let $x(t)$ and $u(t)$ be a solution of (2.1) with $x(\cdot)$ and $u(\cdot)$ from $L_2(t_1; +\infty)$ and $x(t_0) = 0$. Passing to the limit as $k \to +\infty$ in the inequality

$$\int_{t_1}^{t_1+kT} F(t, x(t), u(t)) dt \geq \delta \int_{t_1}^{t_1+kT} (|x(t)|^2 + |u(t)|^2)\, dt$$

one obtains (3.19) (with $t_0 = t_1$). Thus, by Lemma 1, the strict (t_1, T)-dissipativity implies the strong nonoscillatority of the associated Hamiltonian system (3.7).

Further arguments replicate the proof of Theorem 1 with the only difference that instead of arbitrary t_0 and t_1 one should pick up the specified t_1 and $t_0 = t_1 - kT$, where k is a positive integer. Respectively, instead of the dissipativity inequality (2.4), the inequality (2.5) should be used throughout the proof. ∎

Proof of Theorem 3: The "converse" part of the theorem follows immediately from Theorem 1 and Lemma 1. By Theorem 1, the system (3.7) is strongly nonoscillatory and the matrix $R_-(t)$ given by (3.26) is negative definite. Lemma 1 implies the function $R_-(t)$ to satisfy (3.21) and the system $\dot{x}(t) = [A(t) + B(t) r_-(t)^*] x(t)$, where r_- is given by (3.22), is antistable.

Assume that the function $R(t)$ exists with the properties specified in the theorem. We show that the condition E) from Lemma 1 holds. Let $t_0 \in \mathbb{R}$ be chosen arbitrary. Let the pair $(x(\cdot), u(\cdot))$ be solution of (2.1) such that $x(t_0) = \lambda x(t_0 + T)$, where $|\lambda| = 1$. Denote $v(t) = u(t) - r(t)^* x(t)$ [where $r(t) = -(R(t) B(t) + g(t)) \Gamma(t)^{-1}$]. Then, one has

$$\dot{x}(t) = [A(t) + B(t) r(t)^*] x(t) + B(t) v(t). \quad (5.21)$$

Let $\Phi_T = \Phi(t_0 + T; t_0)$, where Φ is the Cauchy transition matrix of the system (4.2). One has

$$\lambda^{-1} x(t_0) = x(t_0 + T) = \Phi_T x(t_0) + \int_{t_0}^{t_0+T} \Phi(t_0 + T; s) B(s) v(s)\, ds$$

or, equivalently

$$x(t_0) = (\lambda^{-1} I_n - \Phi_T)^{-1} \int_{t_0}^{t_0+T} \Phi(t_0 + T; s) B(s) v(s)\, ds.$$

By assumption, the function $(\lambda^{-1} I_n - \Phi_T)^{-1}$ is continuous (and thus bounded) for $|\lambda| = 1$, so there are constants $C_1, C_2 > 0$, determined by the function $\Phi(\cdot, \cdot)$ only, such that

$$\int_{t_0}^{t_0+T} |x(t)|^2 dt \leq C_1 \int_{t_0}^{t_0+T} |v(t)|^2\, dt, \quad \int_{t_0}^{t_0+T} |u(t)|^2 dt$$

$$\int_{t_0}^{t_0+T} |v(t) + r(t)^* x(t)|^2 dt \leq C_2 \int_{t_0}^{t_0+T} |v(t)|^2 dt \quad (5.22)$$

whenever (2.1) holds, $v(t) = u(t) - r(t)^* x(t)$, and $x(t_0) = \lambda x(t_0 + T)$, $|\lambda| = 1$. On the other hand, from the Riccati equation, one obtains in a standard way that

$$\int_{t_0}^{t_0+T} F(t, x(t), u(t)) dt \geq x(t_0)^* R(t_0) x(t_0) - x(t_0 + T)^* R(t_0 + T) x(t_0 + T) + \int_{t_0}^{t_0+T} |v(t)|^2\, dt$$

so by (5.22) one obtains that for sufficiently small $\delta > 0$ the inequality (3.20) is valid if $x(t_0) = \lambda x(t_0 + T)$, $|\lambda| = 1$. Thus, by Lemma 1, the Hamiltonian system (3.7) is strongly nonoscillatory. Define R_- by (3.26). According to Remark 3, $R_-(t) \leq R(t)$, thus $R_-(t) < 0$. From Theorem 1, one obtains the strict dissipativity. ∎

Proof of Theorem 4: Proof of Theorem 4 is analogous to that of Theorem 3. The "converse" part follows immediately from Theorem 2 and Lemma 1. By Theorem 1, the system (3.7) is strongly nonoscillatory and for the matrix $R_{-}(t)$ given by (3.26) one has $R_{-}(t_1) < 0$. Lemma 1 implies the function $R_{-}(t)$ to satisfy (3.21) and the system $\dot{x}(t) = [A(t) + B(t)r_{-}(t)^{*}]x(t)$, where r_{-} is given by (3.22), is antistable.

We have shown (see proof of Theorem 3) that if the function $R(t)$ with mentioned properties exists, then the Hamiltonian system (3.7) is strongly nonoscillatory. Define R_{-} by (3.26). Accordingly to Remark 3, $R_{-}(t) \leq R(t)$, thus $R_{-}(t_1) < 0$. From Theorem 2, one obtains the strict dissipativity. ■

VI. Conclusion

In this paper, the results on dissipativity criteria for T-periodic linear systems are presented. The concept of (t_1, T)-dissipativity is introduced as dissipativity over intervals of length kT, $k = 1, 2, \ldots$. It is shown that dissipativity and (t_1, T)-dissipativity are related to nonoscillatority of Hamiltonian systems and to existence of storage functions with particular positivity properties.

The results can be applied for feasibility and optimality analysis of periodic control systems in many areas of science and technology.

An open question is: Is it possible to extend the dissipativity results of this paper to time-varying nonperiodic systems? In this respect, the results of [5]–[7] may be helpful.

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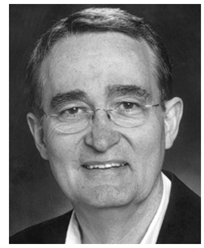

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