DETERMINISTIC SYSTEMS

CONTROL OF NONLINEAR VIBRATIONS OF MECHANICAL SYSTEMS VIA THE METHOD OF VELOCITY GRADIENT[1]

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A general approach to problems of control of vibrations based on the method of velocity gradient and energy objective functions is described with applications to control problems of the following mechanical systems: the physical pendulum and a pendulum on a trolley.

1. INTRODUCTION

Problems of regulating and tracking are typical problems of synthesis of control systems. In both cases, the control objective can be described by assigning the desired trajectory of the control object $\mathbf{x}_*(t)$ and by requiring the approximation of the real trajectory of the object $\mathbf{x}(t)$ to the desired one:

$$\|\mathbf{x}(t) - \mathbf{x}_*(t)\| \leqslant \varepsilon \quad \text{for some} \quad \varepsilon > 0.$$

Also, the objectives of the above-mentioned type can be posed in the problems of damping of vibrations or of synchronization (see, for example, [1]). But progress in the field of the study of periodic and chaotic motions has brought about the statement and solution of new classes of problems concerning the stimulation of vibrations with prescribed properties. We can mention, as an example, the well-known problem of the oscillating of a pendulum [2–5]. As usual, solutions of such problems are based on energy considerations [1–3] or on special methods.

In this article, a general approach to the control of vibrations based on the method of velocity gradient [6–9] is described with application to control of the following mechanical systems: the physical pendulum and a pendulum on a trolley.

Section 2 contains a brief description of the method of velocity gradient (VG). In Sec. 3, algorithms of the velocity gradient for Hamiltonian systems and objective functions based on energy are described. In Sec. 4, the proposed method is applied to problems of the oscillation of the physical pendulum and of oscillation with stabilization in the top position of a pendulum on a trolley.

2. ALGORITHMS OF THE VELOCITY GRADIENT

Let us consider in the state space the equation of a controlled object

$$d\mathbf{x}/dt = F(\mathbf{x}, \mathbf{u}, t), \quad t \geqslant 0, \tag{2.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} \in \mathbb{R}^m$ is the input vector, $F(\cdot) : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$ is a vector-valued function continuously differentiable with respect to $\mathbf{x}$, $\mathbf{u}$. The input variables can be of different nature: a control action, parameter estimations, etc.

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Consider the problem of finding of a non-look-ahead control law $\mathbf{u}(t) = U[\mathbf{x}(s), \mathbf{u}(s) : 0 \leqslant s \leqslant t]$ which ensures achievement of the control objective

$$Q_t \to 0 \quad \text{for} \quad t \to \infty, \tag{2.2}$$

where Q_t is some objective functional $Q_t = Q[\mathbf{x}(s), \mathbf{u}(s) : 0 \leqslant s \leqslant t]$. We shall assume that Q_t has the form $Q_t = Q(\mathbf{x}(t), t)$, where $Q(\mathbf{x}, t) \geqslant 0$ is a scalar smooth objective function.

For obtaining the VG algorithm, the function $\omega(\mathbf{x}, \mathbf{u}, t)$ is defined as the velocity of the change of Q_t along trajectories of (2.1): $\omega(\mathbf{x}, \mathbf{u}, t) = (\nabla_{\mathbf{x}} Q)^{\mathrm{T}} F(\mathbf{x}, \mathbf{u}, t)$. The VG algorithm changes the control action in the direction of the gradient of $\omega(\mathbf{x}, \mathbf{u}, t)$ with respect to $\mathbf{u}$. The general combined form has the form [6–9]

$$\frac{d}{dt}[\mathbf{u} + \psi(\mathbf{x}, \mathbf{u}, t)] = -\Gamma \nabla_{\mathbf{u}} \omega(\mathbf{x}, \mathbf{u}, t), \tag{2.3}$$

where $\psi(\cdot)$ satisfies the pseudogradient condition $\psi^{\mathrm{T}} \nabla_{\mathbf{u}} \omega \geqslant 0$, and the symmetric $m \times m$-matrix $\Gamma = \Gamma^{\mathrm{T}} \geqslant 0$ is the matrix of amplification coefficients. The basic special cases of (2.3) are the VG algorithm in the differential form

$$\frac{d}{dt}\mathbf{u} = -\Gamma \nabla_{\mathbf{u}} \omega(\mathbf{x}, \mathbf{u}, t), \tag{2.4}$$

where $\Gamma > 0$, and the VG algorithm in the finite form

$$\mathbf{u} = -\psi(\mathbf{x}, \mathbf{u}, t). \tag{2.5}$$

The linear and relay forms

$$\mathbf{u} = -\Gamma_0 \nabla_{\mathbf{u}} \omega(\mathbf{x}, \mathbf{u}, t), \quad \Gamma_0 > 0, \tag{2.5a}$$
$$\mathbf{u} = -\Gamma_1 \operatorname{sign}\{\nabla_{\mathbf{u}} \omega(\mathbf{x}, \mathbf{u}, t)\}, \quad \Gamma_1 = \operatorname{diag}\{\gamma_i\}, \quad \gamma_i > 0, \tag{2.5b}$$

where the components of the vector-row $\{\mathbf{z}\}$ are signs of the corresponding elements of the vector $\mathbf{z}$, are typical forms of algorithm (2.5). We shall use the following stability theorems for VG systems (2.1), (2.3).

THEOREM 1 (differential form). *Let the right-hand sides of the systems (2.1), (2.3) be smooth with respect to* $\mathbf{x}$, $\mathbf{u}$ *functions bounded together with their derivatives in any domain, where* $Q(\mathbf{x}(t), t)$ *is bounded. Also, let* $\omega(\mathbf{x}, \mathbf{u}, t)$ *be convex with respect to* $\mathbf{u}$, *and let the stabilizability condition*

$$\text{there exists } \mathbf{u}_* \in \mathbb{R}^m \text{ such that } \omega(\mathbf{x}, \mathbf{u}_*, t) \leqslant 0 \text{ for every } \mathbf{x} \in \mathbb{R}^n \tag{2.6}$$

hold.

Then $Q(\mathbf{x}, t)$ *is bounded along each trajectory of (2.1), (2.3). If, moreover, the condition of asymptotic stabilizability*

$$\omega(\mathbf{x}, \mathbf{u}_*, t) \leqslant -\rho(\mathbf{x}, t), \tag{2.7}$$

where $\rho(\mathbf{x}, t)$ *is a uniformly continuous function with* $\rho(\mathbf{x}, t) > 0$ *for* $Q(\mathbf{x}, t) > 0$, *is satisfied, then the objective (2.2) is achieved for all trajectories of the system (2.1), (2.3).*

The proof (see [8]) is based on the use of the Lyapunov function

$$V(\mathbf{x}, \mathbf{u}, t) = Q(\mathbf{x}, t) + \frac{1}{2}(\mathbf{u} - \mathbf{u}_*)^{\mathrm{T}} \Gamma^{-1} (\mathbf{u} - \mathbf{u}_*). \tag{2.8}$$

In the case where it is difficult to establish the existence of an "ideal" control $\mathbf{u}_*$ satisfying (2.6) or (2.7), the VG algorithm in finite form is applied with the following conditions of applicability.

THEOREM 2 (finite form). *Let the function* $Q(\mathbf{x}, t)$ *be smooth, and let the right-hand sides of system (2.1) be smooth with respect to* $\mathbf{x}$, $\mathbf{u}$ *functions which are bounded together with their derivatives in any domain, where* $Q(\mathbf{x}, t)$ *is bounded. Assume that Eq. (2.5) is solvable with respect to* $\mathbf{u}$ *for every* $\mathbf{x} \in \mathbb{R}^n$, *and there exists locally a solution (for example, in the sense of Filippov) of the system (2.1), (2.5) for any* $\mathbf{x}(0) \in \mathbb{R}^n$. *Also, let* $\omega(\mathbf{x}, \mathbf{u}, t)$ *be convex with respect to* $\mathbf{u}$, *and assume that the following strong pseudogradient condition is satisfied: there exist a function* $\beta(\mathbf{x}) > 0$ *and a number* $\delta \geqslant 1$ *such that*

$$\psi^{\mathrm{T}} \nabla_{\mathbf{u}} \omega(\mathbf{x}, \mathbf{u}, t) \geqslant \beta(\mathbf{x}) \|\nabla_{\mathbf{u}} \omega(\mathbf{x}, \mathbf{u}, t)\|^{\delta}. \tag{2.9}$$

Finally, let there exist a function $\mathbf{u}_*(\mathbf{x},t)$ *bounded with respect to* t *for all* $\mathbf{x} \in \mathbb{R}^n$, *and the conditions*

$$\omega(\mathbf{x}, \mathbf{u}_*(\mathbf{x},t), t) \leqslant -\rho(\mathbf{x}), \tag{2.10}$$

where $\rho(\mathbf{x}) \geqslant 0$ *is a continuous function,*

$$\beta(\mathbf{x}) \|\nabla_{\mathbf{u}} \omega(\mathbf{x}, \mathbf{u}, t)\|^{\delta-1} > \|\mathbf{u}_*(\mathbf{x},t)\| \tag{2.11}$$

are satisfied.

Then $Q(\mathbf{x}(t),t)$ *is bounded along each trajectory of (2.1), (2.5), and the following relations hold:*

$$\lim_{t\to\infty} \rho(\mathbf{x}(t)) = 0, \quad \lim_{t\to\infty} \nabla_{\mathbf{u}} \omega(\mathbf{x}(t), \mathbf{u}(t), t) = 0. \tag{2.12}$$

One can show that the objective (2.2) is still achieved under weakened conditions of stabilizability.

THEOREM 3. *Assume that the conditions of Theorem 2 hold, moreover, the conditions (2.6) are satisfied for some bounded* $\mathbf{u}_*(\mathbf{x},t)$ *and with the following stabilizability condition in the integral form:*

there exist a sequence of moments of time $t_k \to \infty$ *and two sequences of nonnegative numbers* $\{\alpha_k\}_1^\infty$, $\{\rho_k\}_1^\infty$ *such that*

$$Q_{k+1} - Q_k \leqslant -\rho_k Q_k + \alpha_k, \quad \sum_{k=1}^{\infty} \rho_k = \infty, \quad \alpha_k/\rho_k \to 0, \tag{2.13}$$

where $Q_k = Q(\mathbf{x}(t_k), t_k)$. *Note that the Lyapunov function which proves Theorem 2 is exactly the objective function* $Q(\mathbf{x},t)$.

The above-mentioned theorems are modifications of the well-known results on the stability of VG systems (see [6–9]). Further facts on properties of VG algorithms are contained in [8].

3. ALGORITHMS OF THE VELOCITY GRADIENT FOR HAMILTONIAN SYSTEMS

Let us consider equations of a controlled object in the Hamiltonian form

$$\dot{\mathbf{p}} = -\left(\frac{\partial H}{\partial \mathbf{q}}\right)^{\mathrm{T}} + B\mathbf{u}, \quad \dot{\mathbf{q}} = \left(\frac{\partial H}{\partial \mathbf{p}}\right)^{\mathrm{T}}, \tag{3.1}$$

where $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ are generalized coordinates and impulses; $H = H(\mathbf{p},\mathbf{q})$ is the Hamiltonian function (full energy of the system); $\mathbf{u} = \mathbf{u}(t)$ is an input (vector of generalized forces), $B \in \mathbb{R}^{m\times n}$, $m \leqslant n$.

The method of velocity gradient can be applied to the control of CO (controlled object) of the form (3.1) if the attaining of a given energy level is the objective of control:

$$S = \{(\mathbf{p},\mathbf{q}) : H(\mathbf{p},\mathbf{q}) = H_*\}. \tag{3.2}$$

Indeed, let us reformulate the objective of control (3.2) in the form

$$H(\mathbf{p}(t), \mathbf{q}(t)) \to H_* \quad \text{as} \quad t \to \infty, \tag{3.3}$$

which corresponds to (2.2) if one introduces $\mathbf{x} = \mathrm{col}(\mathbf{p},\mathbf{q})$ and the objective function

$$Q(\mathbf{x}) = \frac{1}{2}[H(\mathbf{p},\mathbf{q}) - H_*]^2. \tag{3.4}$$

For the construction of the VG algorithm, we compute $\dot{Q}$, the derivative of (3.4) with respect to (3.1):

$$\dot{Q} = (H - H_*)\left(\frac{\partial H}{\partial \mathbf{p}}\right)^{\mathrm{T}} B\mathbf{u}. \tag{3.5}$$

The differential VG algorithm (2.4) takes the form

$$\dot{\mathbf{u}} = -\gamma (H - H_*) B^{\mathrm{T}} \left(\frac{\partial H}{\partial \mathbf{p}}\right)^{\mathrm{T}}, \tag{3.6}$$

where $\gamma > 0$ is an amplification coefficient.

The finite forms (2.5a), (2,5b) are as follows:

$$\mathbf{u} = -\gamma (H - H_*) B^{\mathrm{T}} \left(\frac{\partial H}{\partial \mathbf{p}}\right)^{\mathrm{T}}, \tag{3.7}$$

$$\mathbf{u} = -\gamma \operatorname{sign} \left[(H - H_*) B^{\mathrm{T}} \left(\frac{\partial H}{\partial \mathbf{p}}\right)^{\mathrm{T}} \right]. \tag{3.8}$$

Let us use Theorems 1 and 2 for analyzing of the behavior of systems with algorithms (3.6)–(3.8). Obviously, the differential algorithm satisfies the conditions of Theorem 1 with stabilizability condition of the form (2.6) for the constant $\mathbf{u}_* \equiv 0$. From Theorem 1 it follows that the function $H(\mathbf{p}, \mathbf{q})$ together with $Q(\mathbf{x})$ is bounded along the trajectories of the system (3.1), (3.6). But the theorem does not insure the achievement of the initial objective (3.3). In fact, the objective (3.3), in reality, is not achieved in the system (3.1), (3.6), demonstrating the complicated behavior of the system (see the results of modeling in Sec. 4).

The algorithms (3.7), (3.8) give better convergence. For example, taking $\mathbf{u}_* = -(H - H_*)B^{\mathrm{T}}\dot{\mathbf{q}}$, from (3.5) we obtain $\dot{Q} = -2\gamma Q \left[\dot{\mathbf{q}}^{\mathrm{T}} B B^{\mathrm{T}} \dot{\mathbf{q}}\right]$. This means that condition (2.7) is not valid since $\dot{\mathbf{q}}$ can vanish for some $t_k > 0$, $k = 1, 2, \ldots$. But from the La Salle invariance principle it follows that each trajectory of the system (3.1), (3.7) can converge to (3.2) (i.e., the objective (3.3) is achieved) as well as to the point of equilibrium $\dot{\mathbf{q}} = 0$ (stationary point of H). One can show (see also [14]) that the dimension of the set of initial conditions for which the trajectory converges to saddle points of H is less than n. On the other hand, if the value of H at the point of extremum is different from H_*, then such a point, also, cannot be a limit point for (3.1), (3.7) since, in this case, the stabilizability condition in the integral form (2.13) is satisfied. Thus, the objective (3.3) is achieved for almost all (with respect to the Lebesgue measure) initial conditions.

The approach presented is applicable in the case of more complicated requirements to the desired behavior of the system. For systems with several degrees of freedom composed of several subsystems, the objective function takes the form

$$Q = \alpha_1 Q_1(\mathbf{x}_1) + \ldots + \alpha_p Q_p(\mathbf{x}_p), \tag{3.9}$$

where $\alpha_i \geqslant 0$ is a weight coefficient, while the objective functions $Q_i(\mathbf{x}_i)$ can be given in the form (3.4), as well as in the other forms.

4. EXAMPLES: CONTROL OF VIBRATIONS OF A PENDULUM

Example 1. Let us consider the equation of the controlled physical pendulum

$$J \cdot \ddot{\varphi} + m \cdot g \cdot \ell \cdot \sin\varphi = \mathbf{u}, \tag{4.1}$$

where φ is the angle of deviation of the pendulum from the vertical line ($\varphi = 0$ in the bottom position), u is the control torque; J, m, ℓ is the moment of inertia with respect to the axis of rotation, the mass, and the distance between the axis of rotation and the center of gravity of the pendulum, respectively; g is the acceleration of free incidence. The energy of the pendulum is written in the form

$$H = \frac{1}{2} J \cdot \dot{\varphi}^2 + m \cdot g \cdot \ell \cdot (1 - \cos\varphi). \tag{4.2}$$

Let us consider the problem of oscillation of the pendulum up to the amplitude corresponding to the energy H_*, i.e., let use the objective (3.3). The achievement of the objective (3.3) for $H_* = 0$ means the stabilization of the pendulum at the bottom position, i.e., the damping of vibrations, while this corresponds to the uninterrupted rotation for $H_* > 2mg\ell$.

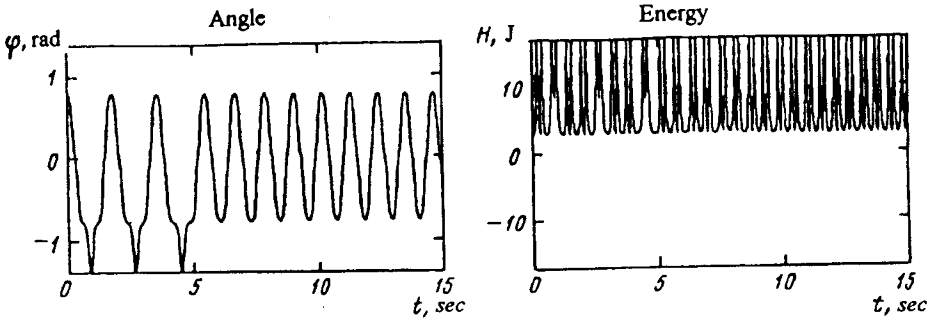

Fig. 1

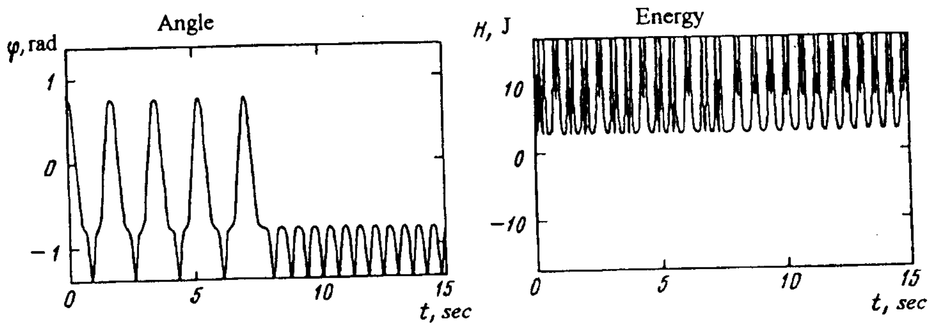

Fig. 2

The VG algorithms in the differential and finite forms are as follows:

$$\dot{u} = -\gamma (H - H_*) \dot{\varphi}, \tag{4.3}$$

$$u = -\gamma (H - H_*) \dot{\varphi}. \tag{4.4}$$

Obviously, the algorithm (4.3) satisfies the conditions of Theorem 1; the stabilizability condition is valid in the form (2.6). Theorem 1 gives the boundedness of the energy, i.e., $\dot{\varphi}(t)$ is bounded. The algorithm (4.4) satisfies Theorem 3 for $\dot{\varphi}(0) \neq 0$ ((2.7) is true, for example, for $u_*(t) = -(H(t) - H_*)\dot{\varphi}(t)$, $\rho(Q,t) = Q^2 [\dot{\varphi}(t)]^2$; the constants in (2.9) are $\beta = 1$, $\delta = 2$).

The results of modeling are shown in Figs. 1–6 for the following values of the parameters: $m = 1$ kg, $\ell = 1$ m, $J = 10\,\text{kgm}^2$, $\dot{\varphi}(0) = 0$, $u(0) = 0$.

Figures 1 and 2 demonstrate the complicated behavior of the system with algorithm (4.3) for $H_* = 10$ corresponding to the amplitude of vibrations $\varphi_{\max} = \pi/2$ for $\varphi(0) = \pi/4$. For Fig. 1, the amplification coefficient $\gamma_1 = 65.2$, for Fig. 2 $\gamma_2 = 65.5$; thus, the bifurcation value lying between γ_1 and γ_2 separates the systems with different limit cycles. Moreover, the objective of control is not achieved.

In Fig. 3, it is shown that algorithm (4.4) oscillates the pendulum up to the required amplitude with small initial conditions and amplification coefficients ($\varphi(0) = 0.5°$, $\gamma = 0.1$). Figure 4 indicates the efficiency of algorithm (4.4) under the oscillation of the pendulum up to rotation with the minimal velocity 2 m/s and with the same initial conditions. The relay algorithm (3.8) for $\varphi_{\max} = \pi/2$ and $\gamma = 0.3$ also oscillates the pendulum up to the required amplitude; after this the control becomes high and unexpectedly oscillating (Fig. 5); this is explained by round-off errors under the computation of $(H - H_*)$. The efficiency of the modified relay algorithm $u = -\gamma (H - H_*) \operatorname{sign} \dot{\varphi}$ is demonstrated in Fig. 6. Finally, in Fig. 7, the behavior of the system is shown when a differentiable filter of the first order is used for the estimation of $\dot{\varphi}$ (in this case, $v_{\min} = 0.02$ m/s, $\gamma = 0.2$, the constant of time $\tau = 0.005$ s).

Example 2. Consider the problem of transforming into the vertical position and of stabilizing an inverted pendulum situated at a trolley. Let the pendulum of mass m with distance ℓ between the axis and the center of gravity, moment of inertia with respect to the center of gravity J, and with angle of deviation φ ($\varphi = 0$ at the bottom position) rotate on the axis fixed on the trolley. The trolley of mass M can move in the horizontal direction along the normal to the axis of rotation of the pendulum under the action of a bounded external force $u(t)$, $|u(t)| \leqslant u_m$. The friction under rotation of the pendulum and under moving of the trolley is not taken into account.

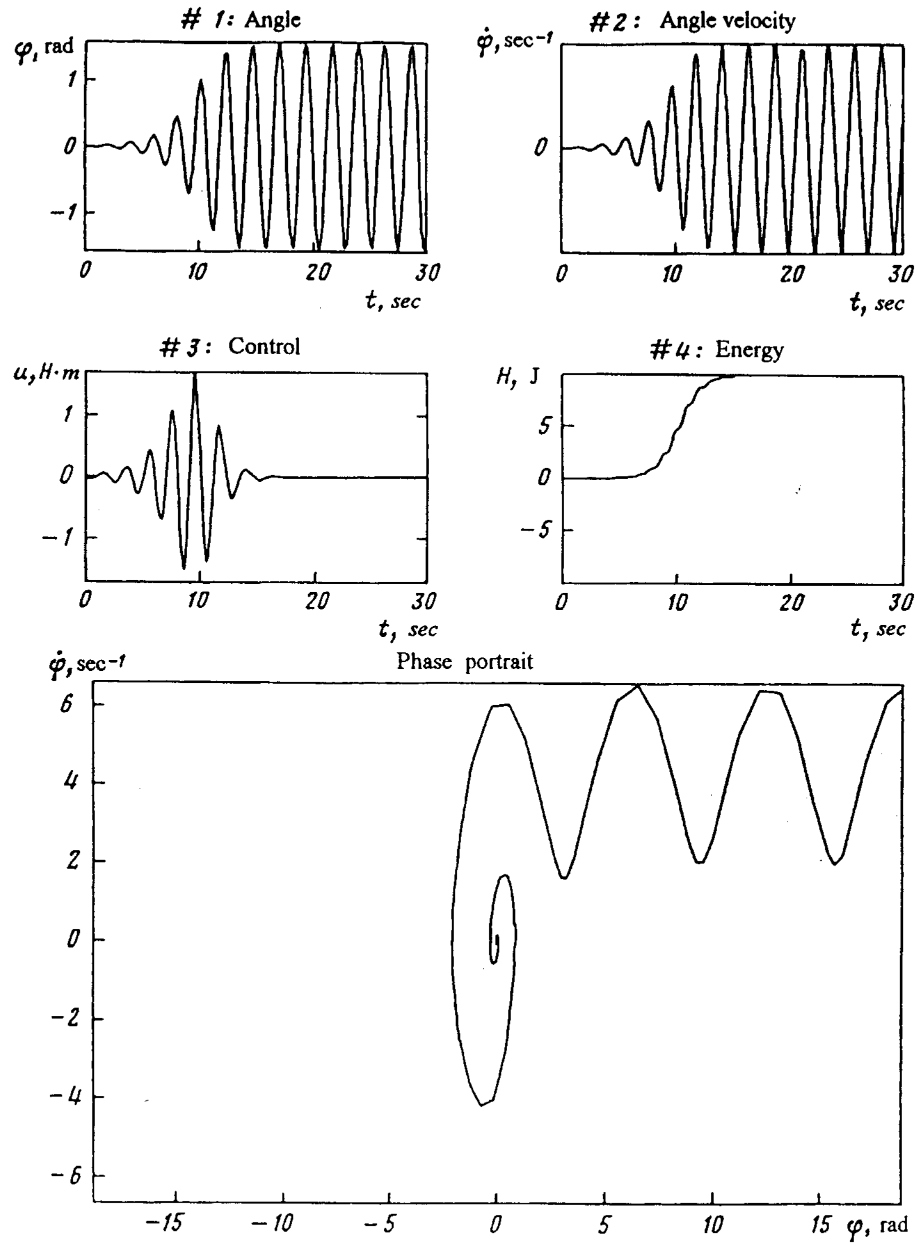

Fig. 3

It is required, by changing $u(t)$, to transfer the pendulum from the initial position ($\varphi = \dot{\varphi} = 0$) to the final one ($\varphi = \pi$) and to stabilize it in this position ensuring simultaneously the stabilization of the position of the trolley at the origin (a similar problem was considered in [2, 11]).

The mathematical model of the system "trolley-pendulum" has the form [12]:

$$\begin{aligned} J\ddot{\varphi} &= \ell v \sin\varphi - \ell h \cos\varphi, \\ M\ddot{s} &= u - h - F\dot{s}, \end{aligned} \tag{4.5}$$

where $h = m\ddot{s} + m\ell\ddot{\varphi}\cos\varphi - m\ell\dot{\varphi}^2\sin\varphi$, $v = mg - m\ell\ddot{\varphi}\sin\varphi - m\ell\dot{\varphi}^2\cos\varphi$ are the horizontal and vertical forces of reaction of the pendulum to the motion of the trolley; F is the friction coefficient of the trolley.

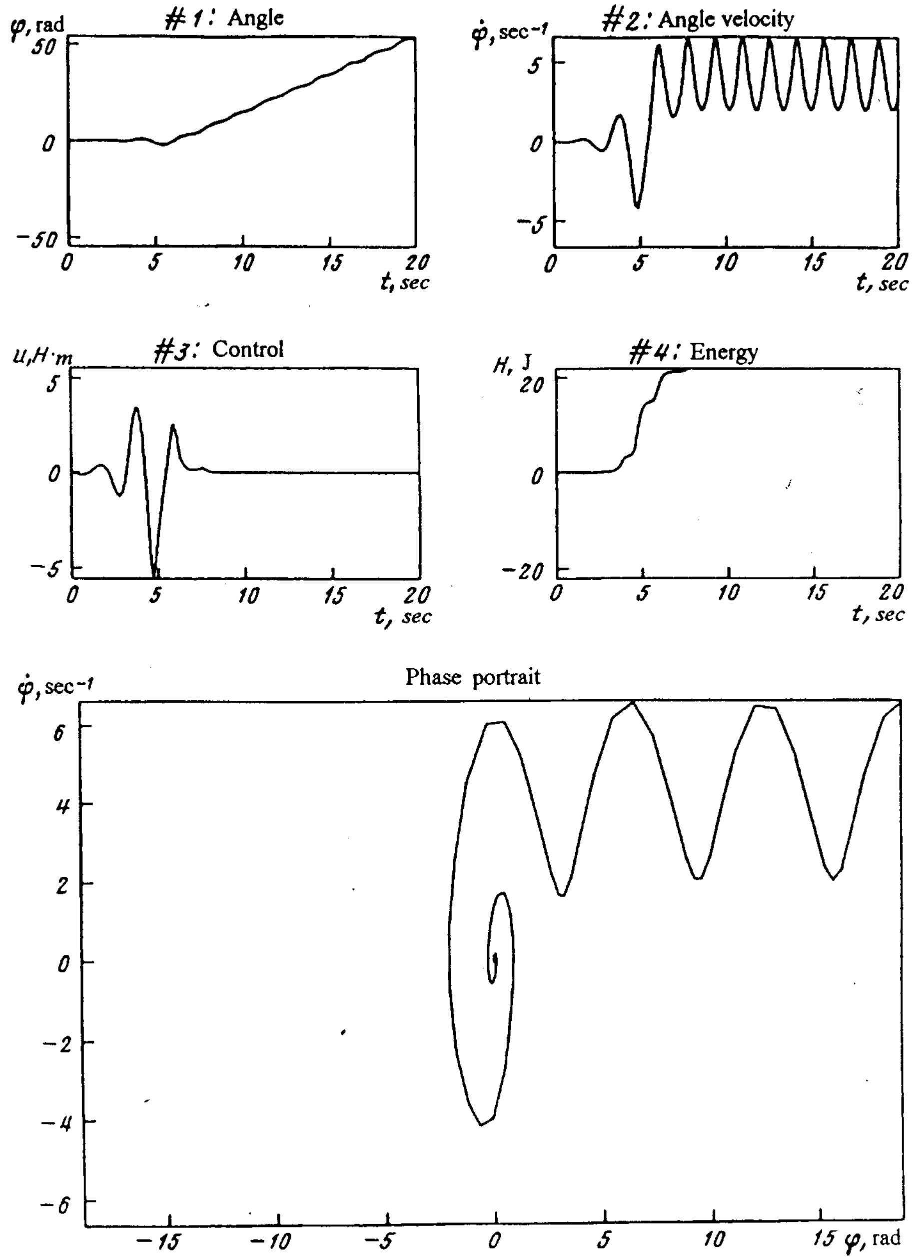

Fig. 4

Setting $M \gg m$ and neglecting the friction of the trolley, we write the approximate equations of the system [12]:

$$\begin{aligned} \ddot{\varphi} &= -\frac{g}{L}\sin\varphi - \frac{\cos\varphi}{M \cdot L}\, u(t), \\ \ddot{s} &= \frac{1}{M}\, u(t), \end{aligned} \tag{4.6}$$

where $L = \dfrac{J + m \cdot \ell^2}{m \cdot \ell}$ is the effective length of the pendulum, $s(t)$ is the displacement of the trolley.

In what follows, we use the state vector $\mathbf{x}\,[\varphi, \dot{\varphi}, s, \dot{s},]^{\mathrm{T}}$. For the solution of the problem posed on the basis of the VG method, we consider two subproblems: oscillation of the pendulum up to the amplitude close to π rad and stabilization of it at this position with the simultaneous stabilization of the trolley.

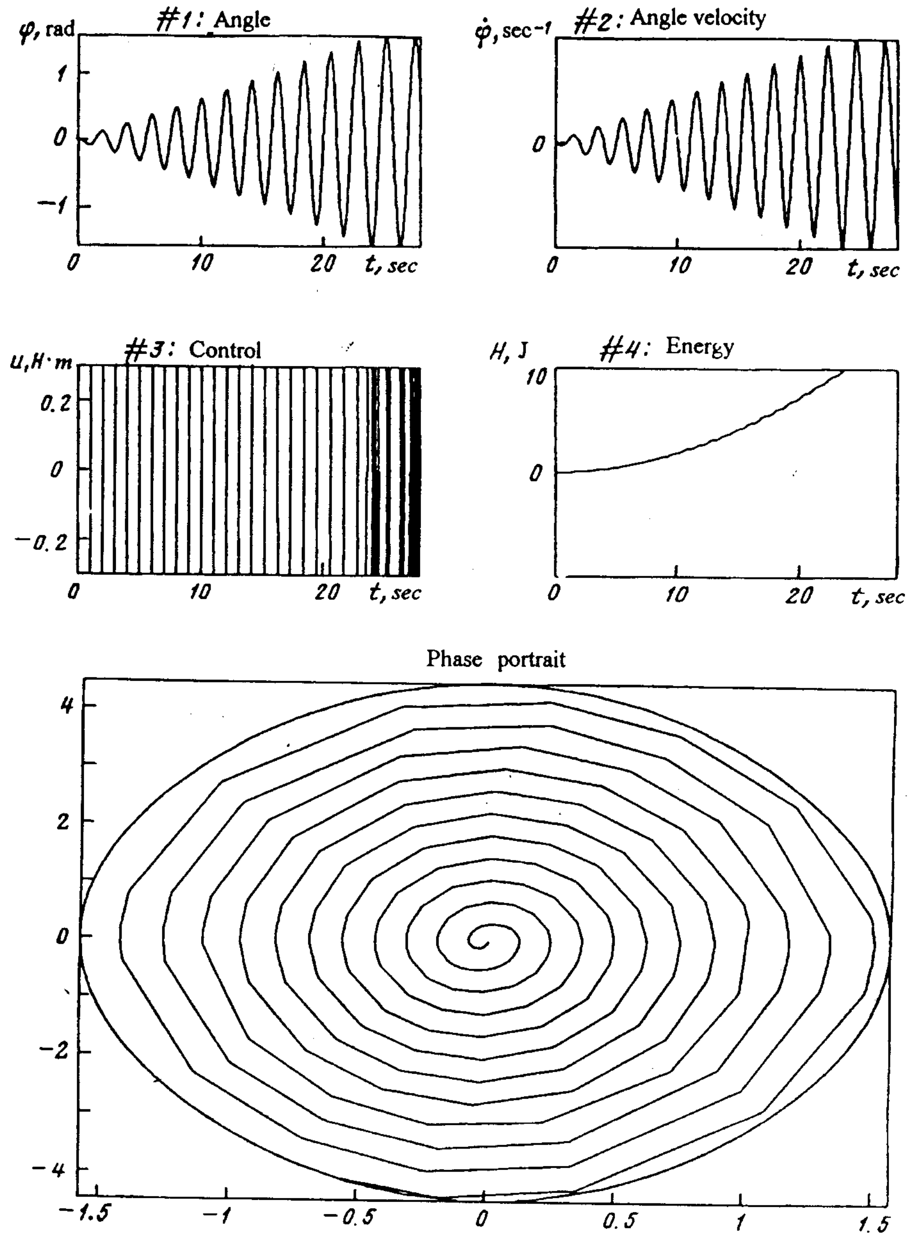

Fig. 5

For each of these subproblems, we introduce their own objective functions $Q_1(\mathbf{x})$ and $Q_2(\mathbf{x})$, which we combine in the common objective function $Q(\mathbf{x}) = \mu(\mathbf{x}) \cdot Q_1(\mathbf{x}) + (1 - \mu(\mathbf{x})) \cdot Q_2(\mathbf{x})$, where $\mu(\mathbf{x}) \in [0, 1]$ varies according to what domain of the state space the depicting point belongs to (the choice of $\mu(\mathbf{x})$ will be made more precise below).

Let us choose the objective functions $Q_1(\mathbf{x})$ and $Q_2(\mathbf{x})$ in the form

$$Q_1(\mathbf{x}) = \alpha_1 \cdot |H - H_*| + \alpha_2 \cdot |\tau \dot{s} + s|, \tag{4.7}$$

$$Q_2(\mathbf{x}) = |\mathbf{c}^{\mathrm{T}} \Delta \mathbf{x}|, \quad \Delta \mathbf{x} = \mathbf{x} - \mathbf{x}^*, \quad \mathbf{x}^* = [\pi, 0, 0, 0]^{\mathrm{T}}. \tag{4.8}$$

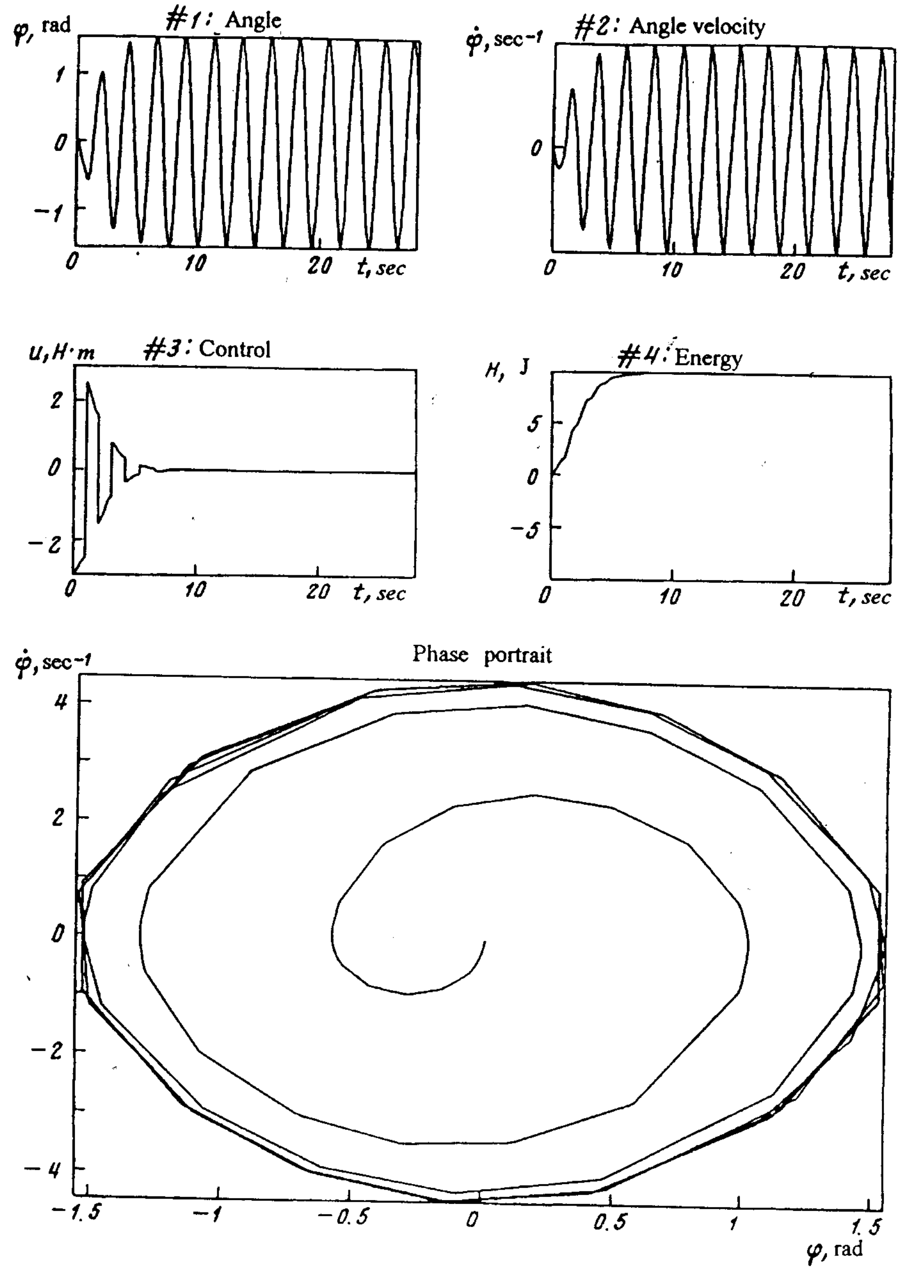

Fig. 6

Here $\alpha_1 > 0$, $\alpha_2 > 0$, $\tau > 0$, $\mathbf{c} \in \mathbb{R}^4$ are parameters determining the level of the control and the character of the process of stabilization. The function $Q_1(\mathbf{x})$ corresponds to the step of oscillation of the pendulum up to the value $H = H_*$ ($H_* = 2 \cdot m \cdot g \cdot L$) and to stabilization of the zero the state of the trolley (the parameter τ is the chosen constant of time); $Q_2(\mathbf{x})$ gives the desired dynamics of the process of stabilization $\Delta\mathbf{x}$ since, for $Q_2(\mathbf{x}) = 0$, the condition $\mathbf{c}^{\mathrm{T}}\Delta\mathbf{x} = 0$ is satisfied. Under the choice of the vector $\mathbf{c}$, the expression $\mathbf{c}^{\mathrm{T}}\Delta\mathbf{x} = 0$ is considered as the equation of motion in the sliding regime [13]. Let us obtain the finite form (2.5) of the VG algorithm on the basis of the objective function $Q(\mathbf{x})$.

First, we consider the domain $\mu(\mathbf{x}) = 1$ in which the pendulum is oscillated up to a position close to the vertical one. If we take $\psi(\mathbf{x}, \mathbf{u}) = \nabla_u \omega(\mathbf{x}, \mathbf{u})$, then we obtain the relay VG algorithm (2.5b) in the form

$$u(t) = u_m \mathrm{sign}\left[\lambda_1 \mathrm{sign}\left(\tau\dot{s} + s\right) - \lambda_2 \cos\varphi \,\mathrm{sign}\left((H - H_*)\dot{\varphi}\right)\right]. \qquad (4.9)$$

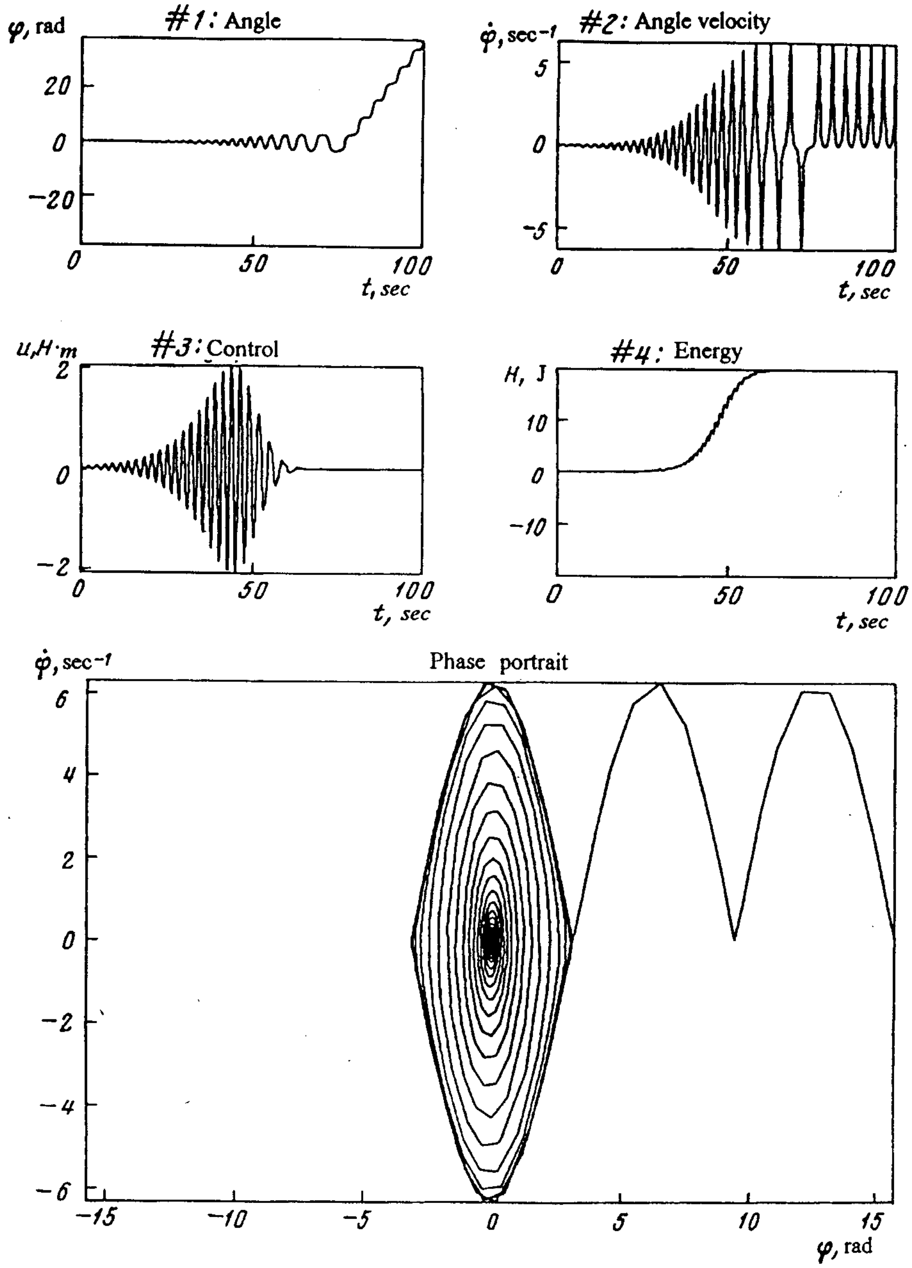

Fig. 7

We have another form of the control law under the choice of $\psi(\mathbf{x}, u)$ in the form

$$\psi(\mathbf{x}, u) = \varkappa \operatorname{sign}(H - H_*) \operatorname{sign}(\cos\varphi) \operatorname{sign}(\dot\varphi) - (1 - \varkappa) \operatorname{sign}(\tau \dot s + s), \tag{4.10}$$

where $\varkappa$ is a parameter, $0 < \varkappa < 1$. It can be shown that, for $\varkappa < \varkappa_*$, the function (4.10) satisfies the pseudogradient condition $\psi^{\mathrm{T}} \nabla_u \omega \geqslant 0$, and the control algorithm takes the form

$$u(t) = u'_m \operatorname{sign}(H - H_*) \operatorname{sign}(\cos\varphi) \operatorname{sign}(\dot\varphi) - u''_m \operatorname{sign}(\tau \dot s + s), \tag{4.11}$$

where $u'_m = \varkappa u_m$, $u''_m = (1 - \varkappa) u_m$.

For $\mu(\mathbf{x}) = 0$ (stabilization of the pendulum and the trolley), the algorithm (2.5b) takes the form

$$u(t) = -u_m \operatorname{sign} \sigma \left(\mathbf{c}_2 - \mathbf{c}_4 \cos\varphi\right), \tag{4.12}$$

where $\sigma(t) = \mathbf{c}^{\mathrm{T}} \Delta \mathbf{x}(t)$ is the signal of the "residual" of the algorithm, $\mathbf{c} \in \mathbb{R}^4$ is the vector of coefficients insuring the desired dynamics of the process of stabilization.

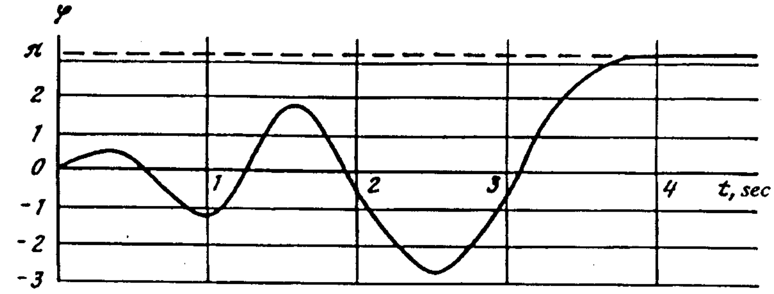

Fig. 8. Process of oscillation and of stabilization of the pendulum.

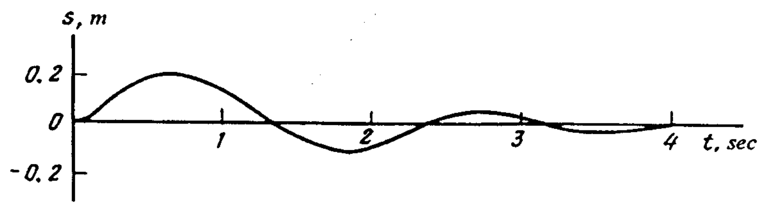

Fig. 9. Process of stabilization of the platform.

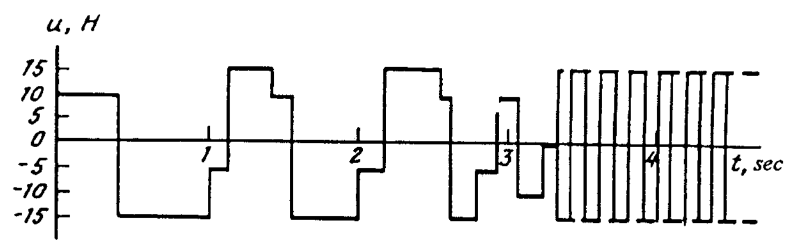

Fig. 10. Control action.

For the choice of these coefficients, we use the notion of an "implicit model" [7, 8] and consider the equatio

$$c^{T}\Delta x(t) = 0 \qquad (4.13$$

as some "standard" equation describing the desired character of the process of stabilization (an analogous approac is used for the synthesis of variable structure systems [13]).

For the choice of c, we linearize (4.6) in a neighborhood of $x^{*} = [\pi, 0, 0, 0]^{T}$:

$$\ddot{\varphi} = \frac{g}{L}\varphi + \frac{1}{M \cdot L}u(t), \qquad \ddot{s} = \frac{1}{M}u(t). \qquad (4.14$$

Setting $c^{T}\Delta x \equiv 0$ and taking (4.14) into account, we find the "standard" characteristic polynomial $G(p) = c^{T}B(p)$, where $B(p)$ is the vector of numerators of the transfer function of system (4.14). Equating the polynomia $G(p)$ to some stable polynomial (for example, to the Batterworth polynomial $G(p) = p^3 + 2\omega_0 p^2 + 2\omega_0^2 p + \omega_0^3$, ω_0 i a parameter), we obtain an expression for determination of the vector c ensuring the desired dynamics.

It is appropriate to take the switching function $\mu(x)$ for the given problem in the form

$$\mu(x) = \text{sign}\left(\left|c^{T}\Delta x\right| - \Delta\right),$$

where the threshold $\Delta > 0$ determines the width of the strip in which the algorithm of stabilization is efficient and is defined in terms of u_m, m, L.

The results of modeling of the system (4.6), (4.11), (4.12) for $\varkappa = 0.66$, $m = 0.5$ kg, $M = 5$ kg, $u_m = 15$ H, $L = 0.3$ m, $\omega_0 = 5\,\text{sec}^{-1}$, $\Delta = 0.1$ are presented in Figs 8–10 in the form of graphs of the transfer processes with respect to $\varphi(t)$, $s(t)$, and $u(t)$. The results obtained justify the efficiency of the proposed algorithm.

5. CONCLUSION

The applicability of the concept of velocity gradient was demonstrated in [6–10] for different problems of stabilization and tracking. In this article, the method of velocity gradient is extended to vibrating systems with the objective function based on energy. The stability theorems as well as the results of modeling justify the efficiency of the proposed control algorithms and also the difference between the properties of algorithms in the differential and finite forms.

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