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On self-synchronization and controlled synchronization[1]

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Abstract

An attempt is made to give a general formalism for synchronization in dynamical systems encompassing most of the known definitions and applications. The proposed set-up describes synchronization of interconnected systems with respect to a set of functionals and captures peculiarities of both self-synchronization and controlled synchronization. Various illustrative examples are given. © 1997 Elsevier Science B.V.

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1. Introduction

Starting with the work of Huygens [13] synchronization phenomena attracted attention of many researchers. The development of small parameter and averaging methods by Poincaré [22], Van der Pol [24], Bogolyubov [7] in the first-half of the 20th century allowed for a better understanding and theoretical explanation of the mechanism of self-synchronization [2, 3], a phenomenon which has numerous applications, see e.g. [3,15]. Motivated by the study of chaotic phenomena (see e.g. [23, 16]) recent years have exhibited an increase in the interest in synchronization. Synchronization in chaotic systems was discussed, for instance in [1, 21, 8, 6]. In [8, 6] and some other recent work on chaotic synchronization, the synchronization was understood as the asymptotic coincidence of the state vectors of two (or more) systems or of some parts of the state vectors. In [1,21] two different definitions of synchronization were suggested based on relations between attractors of interacting systems. Both of them differ from the traditional definition for deterministic systems with periodic solutions [2,3,12].

Recently, specialists from (nonlinear) control theory turned attention to the study of (controlled) synchronization. Incomplete information about the system parameters has been taken into account (adaptive and robust synchronization [9, 10, 17]), as well as incomplete information of the state of the system (observer-based synchronization [18, 19]). The problems of how to cope with uncertainties and incomplete measurements are traditional in control theory. However, experts from different fields understand synchronization in different ways which in turn requires additional efforts to apply conventional control methods. Therefore, there is still a strong need for unified definitions of synchronization which would capture peculiarities of both self-synchronization and controlled synchronization and which would also allow to rigorously pose and systematically solve various synchronization problems.

In this paper general definitions of synchronization are introduced and discussed in Section 2. Two large classes of synchronization problems are extracted,

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namely, frequency synchronization (Section 3) and coordinate synchronization (Section 4). A number of examples demonstrating the potential of the introduced definitions is given.

2. Definitions of synchronization

Synchronization in its most general interpretation means correlated or corresponding in-time behavior of two or more processes. According to [25]: "to synchronize" means to concur or agree in time, to proceed or to operate at exactly the same rate, to happen at the same time. Below we formalize the above description and also formulate a "controlled" version.

To this end consider k dynamical systems

$$\Sigma_i = \{T, U_i, X_i, Y_i, \phi_i, h_i\}, \quad i = 1, \ldots, k,$$

where T is the common set of time instances, U_i, X_i, Y_i are sets of inputs, states and outputs, respectively; $\phi_i : T \times X_i \times U_i \to X_i$ are transition maps, $h_i : T \times X_i \times U_i \to Y_i$ are output maps. (We use here one of the standard definition of dynamical system, see e.g. [20, 14].)

First, consider the case when all U_i are just singletons, i.e. inputs are not present and may be omitted in the formulation.

Suppose l functionals $g_j : \mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_k \times T \to \mathbb{R}^1$, $j = 1, \ldots, l$, are given. Here $\mathcal{Y}_i$ are the sets of all functions from T into Y_i, i.e. $\mathcal{Y}_i = \{y : T \to Y_i\}$.

In the sequel, we take as time set T either $T = \mathbb{R} \geqslant 0$ (continuous time) or $T = \mathbb{Z} \geqslant 0$ (discrete time). For any $\tau \in T$ we then define σ_τ as the *shift operator*, i.e. $\sigma_\tau : \mathcal{Y}_i \to \mathcal{Y}_i$ is given as $(\sigma_\tau y)(t) = y(t + \tau)$ for all $y \in \mathcal{Y}_i$ and all $t \in T$.

Definition 2.1. We call the solutions $x_1(\cdot), \ldots, x_k(\cdot)$ of the systems $\Sigma_1, \ldots, \Sigma_k$ with initial conditions $x_1(0), \ldots, x_k(0)$ *synchronized with respect to the functionals* $g_1, \ldots, g_l$ if

$$g_j(\sigma_{\tau_1} y_1(\cdot), \ldots, \sigma_{\tau_k} y_k(\cdot), t) \equiv 0, \quad j = 1, \ldots, l \tag{1}$$

is valid for all $t \in T$ and some $\tau_1, \ldots, \tau_k \in T$, where $y_i(\cdot)$ denotes the output function of the system Σ_i: $y_i(t) = h(x_i(t), t)$, $t \in T$, $i = 1, \ldots, k$.

We say that solutions $x_1(\cdot), \ldots, x_k(\cdot)$ of the systems $\Sigma_1, \ldots, \Sigma_k$ with initial conditions $x_1(0), \ldots, x_k(0)$ are *approximately synchronized with respect to the functionals* $g_1, \ldots, g_l$, if there are an $\varepsilon > 0$ and $\tau_1, \ldots, \tau_k \in T$ such that

$$|g_j(\sigma_{\tau_1} y_1(\cdot), \ldots, \sigma_{\tau_k} y_k(\cdot), t)| \leqslant \varepsilon, \quad j = 1, \ldots, l \tag{2}$$

for all $t \in T$.

The solutions $x_1(\cdot), \ldots, x_k(\cdot)$ of the systems $\Sigma_1, \ldots, \Sigma_k$ with initial conditions $x_1(0), \ldots, x_k(0)$ are *asymptotically synchronized with respect to the functionals* $g_1, \ldots, g_l$, if for some $\tau_1, \ldots, \tau_k \in T$

$$\lim_{t \to \infty} g_j(\sigma_{\tau_1} y_1(\cdot), \ldots, \sigma_{\tau_k} y_k(\cdot), t)) = 0, \quad j = 1, \ldots, l. \tag{3}$$

If the synchronization phenomenon is achieved for all initial conditions $x_1(0), \ldots, x_k(0)$ it is possible to say that the systems $\Sigma_1, \ldots, \Sigma_k$ are synchronized (in the appropriate sense with respect to the given functionals). In the case of asymptotic synchronization it is also possible to define the basins of the initial conditions which yield synchronization. In the sequel, we will only consider the case when synchronization is achieved for all initial conditions.

Although this definition is rather general, it can be further generalized. For example, in many practical problems the time shifts τ_i, $i = 1, \ldots, k$ are not constant but tend to constant values, the so-called "asymptotic phases". In this case, instead of the shift operator for each output function $y_i(\cdot)$, it is convenient to consider the time-varying shift operator defined as follows:

$$(\sigma_{\tau_i}) y(t) = y(t_i'(t)),$$

where $t_i' : T \to T$, $i = 1, \ldots, k$ are homeomorphisms (continuous functions having continuous inverses) such that

$$\lim_{t \to \infty} (t_i'(t) - t) = \tau_i. \tag{4}$$

In [1], instead of (4), the milder condition $\lim_{t \to \infty} t_i'(t)/t = 1$ is proposed which, however, allows for infinitely large phase shifts.

In many practical synchronization problems the spaces $\mathcal{Y}_i$ are identical $\mathcal{Y}_i = \mathcal{Y}$ and the functionals $\{g_{jsr}\}$ are chosen to compare similar characteristics of different systems, e.g.

$$g_{jsr}(y_s(\cdot), y_r(\cdot)) = \mathrm{dist}(J_j(\sigma_{\tau_s} y_s(\cdot)), J_j(\sigma_{\tau_r} y_r(\cdot))),$$

where $r, s = 1, \ldots, k$, $j = 1, \ldots, l$ and $J_j : \mathcal{Y} \times \mathcal{Y} \to \mathcal{J}_j$, are some mappings (synchronization indices) which map the (output) trajectory $y_i(\cdot)$ of each system $\Sigma_1, \ldots, \Sigma_k$, into some metric space $\mathcal{J}_j$. This will be refered as *synchronization with respect to the indices* $\{J_j\}$. The specific choice of the synchronization indices depends on the essence of the mathematical, physical or engineering problem. The same is valid for the phase shifts τ_i which may be fixed in some problems and may be arbitrary in others. Naturally, the possibility of efficient solutions of the synchronization problems crucially depends on the chosen functionals and/or indices.

Remark 1. Note that instead of a set of the functionals it is always possible to take one functional which expresses the same synchronization phenomenon. For example, one can take the functional G as follows:

$$G(y_1(\cdot),\dots,y_k(\cdot),t)=\sum_{j=1}^{l} g_j^2(y_1(\cdot),\dots,y_k(\cdot),t). \quad (5)$$

Remark 2. In applications of synchronization it is important to require that the conditions (1)–(3) are not violated (or not significantly violated) when some parameters of the systems are varied in some range. In other words, the properties (1)–(3) should be robust but in this case the phase shifts may not be constant and even not tend to constant values; however, the following condition:

$$\overline{\lim_{t\to\infty}}|t_i'(t)-t|\leqslant\tau_i$$

may be imposed instead of (4).

In many cases the sets U_i, X_i, Y_i are finite-dimensional vector spaces and the systems Σ_i can be described by ordinary differential equations. First consider the simplest case of disconnected systems without inputs

$$\Sigma_i\colon \frac{dx_i}{dt}=F_i(x_i,t), \quad (6)$$

where F_i, $i=1,\dots,k$ are some time-dependent vector fields. Sometimes synchronization may occur in disconnected systems (6) (e.g. all precise clocks are synchronized in the frequency sense). This case will be refered to as *natural synchronization*. A more interesting and important case, however, seems synchronization of interconnected systems. In this case the system models are augmented with interconnections and look as follows:

$$\begin{aligned}\frac{dx_i}{dt}&=F_i(x_i,t)+\tilde F_i(x_0,x_1,\dots,x_k,t), \quad i=1,\dots,k,\\ \frac{dx_0}{dt}&=F_0(x_0,x_1,\dots,x_k,t),\end{aligned} \quad (7)$$

where the vector field F_0 describes the dynamics of the interconnection system, $\tilde F_i$ are vector fields describing the interconnections. For example, in the synchronization of generators of a power station this interconnection is caused by a common electrical load. The model (7) can formally not be considered within the given definition. To include the case of synchronization of interconnected systems we should introduce a dynamical system which describes interconnections between the systems. Recall that in the previous definition we supposed that the sets of inputs of each system Σ_i, $i=1,\dots,k$ are just singletons. To describe the possible interconnections we suppose now that the input of each system Σ_i, $i=1,\dots,k$ can be composed from the output of the interconnection system $\Sigma_0=\{T,U_0,X_0,Y_0,\phi_0,h_0\}$ where the transition and output maps are given by $\phi_0: T\times X_0\times U_0\to X_0$ and $h_0: T\times X_0\times U_0\to Y_0$ with $U_0=Y_1\times Y_2\times\cdots\times Y_k$ and $Y_0=U_1\times U_2\times\cdots\times U_k$. Now, it is possible to define synchronization of interconnected systems.

Definition 2.2. We call the solutions $x_0(\cdot),\dots,x_k(\cdot)$ of the systems $\Sigma_1,\dots,\Sigma_k$ and interconnection system Σ_0 with initial conditions $x_0(0),\dots,x_k(0)$ *synchronized with respect to the functionals* $g_1,\dots,g_l$ if

$$g_j(\sigma_{\tau_0}y_0(\cdot),\dots,\sigma_{\tau_k}y_k(\cdot),t)\equiv 0, \quad j=1,\dots,l \quad (8)$$

is valid for all $t\in T$ and some $\tau_0,\dots,\tau_k\in T$, where $y_i(\cdot)$ denotes the output function of the system Σ_i: $y_i(t)=h(x_i(t),t)$, $t\in T$, $i=0,\dots,k$.

We say that solutions $x_0(\cdot),\dots,x_k(\cdot)$ of the systems $\Sigma_1,\dots,\Sigma_k$ and interconnection system Σ_0 with initial conditions $x_0(0),\dots,x_k(0)$ are *approximately synchronized with respect to the functionals* $g_1,\dots,g_l$, if there are an $\varepsilon>0$ and $\tau_0,\dots,\tau_k\in T$ such that

$$|g_j(\sigma_{\tau_0}y_0(\cdot),\dots,\sigma_{\tau_k}y_k(\cdot),t)|\leqslant\varepsilon, \quad j=1,\dots,l \quad (9)$$

for all $t\in T$.

The solutions $x_0(\cdot),\dots,x_k(\cdot)$ of the systems $\Sigma_1,\dots,\Sigma_k$ and interconnection system Σ_0 with initial conditions $x_0(0),\dots,x_k(0)$ are *asymptotically synchronized with respect to the functionals* $g_1,\dots,g_l$, if for some $\tau_0,\dots,\tau_k \in T$

$$\lim_{t\to\infty} g_j(\sigma_{\tau_0} y_0(\cdot),\dots,\sigma_{\tau_k} y_k(\cdot),t)=0, \quad j=1,\dots,l. \tag{10}$$

A remarkable and widely used observation is that the synchronization may exist, i.e. identity (8) may be valid in the interconnected system without any artificially introduced external action, i.e. when the interconnection system Σ_0 is given. In this case the system (7) can be called *self-synchronized with respect to the functionals* $g_1,\dots,g_l$, or with respect to the indices $\mathcal{J}_1,\dots,\mathcal{J}_k$. Similar definitions can be introduced for approximate and asymptotic self-synchronization.

In cases important for applications the interconnections between the systems $\Sigma_1,\dots,\Sigma_k$ are weak, for instance when (7) can be represented as follows:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = F_i(x_i,t) + \mu\tilde{F}_i(x_0,x_1,\dots,x_k,t),$$

$$i=1,\dots,k,$$

$$\frac{\mathrm{d}x_0}{\mathrm{d}t} = F_0(x_0,x_1,\dots,x_k,t), \tag{11}$$

where μ is a small parameter. Therefore, finding conditions for self-synchronization in systems with small interactions is of special interest. Such conditions were found for a large class of dynamical systems (11) in particular, with time-periodic vector fields F_i, [2, 3]. However, in many cases self-synchronization is not observed and the question arises: is it possible to affect, i.e. to *control* the systems in such a way that the goal (2) or (3) can be achieved?

The above definitions do not yet include the possibility of controlling the system. Assume for simplicity that all Σ_i, $i=0,\dots,k$ are smooth finite-dimensional systems, described by differential equations with a finite-dimensional input, i.e.

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = F_i(x_i,t) + \tilde{F}_i(x_0,x_1,\dots,x_k,u,t),$$

$$i=1,\dots,k,$$

$$\frac{\mathrm{d}x_0}{\mathrm{d}t} = F_0(x_0,x_1,\dots,x_k,u,t), \tag{12}$$

where $u=u(t)\in\mathbb{R}^m$ is the input or control variable.

Definition 2.3. The problem of *controlled synchronization with respect to the functionals* g_j, $j=1,\dots,l$ (respectively, *controlled asymptotic synchronization with respect to the functionals* g_j, $j=1,\dots,l$) is to find a control u as a feedback function of the states $x_0,x_1,\dots,x_n$ and time providing that (1) (respectively, (2), (3)) holds for the closed-loop system.

The problem of controlled synchronization with respect to indices $\mathcal{J}_1,\dots,\mathcal{J}_k$ is formulated similarly.

Sometimes the goal can be ensured without measuring any variables of the systems, for instance, by a time-periodic forcing. In this case the control function u does not depend on system states and the problem of finding such a control is called an *open-loop-controlled (asymptotic) synchronization problem.* However, a more powerful approach assumes the possibility of measuring the states or some function of the system variables. Finding a control function in this case is called a *closed-loop* or *(asymptotic) feedback synchronization problem.*

The simplest form of feedback is *static-state feedback* where the controller equation is as follows:

$$u(t)=\mathcal{U}(x_0,x_1,\dots,x_k,t) \tag{13}$$

for some function $\mathcal{U}:\mathbb{R}^{n_0}\times\mathbb{R}^{n_1},\dots,\mathbb{R}^{n_k}\times T\to\mathbb{R}^m$.

A more general form is *dynamic-state feedback*

$$\frac{\mathrm{d}w}{\mathrm{d}t} = W(x_0,x_1,\dots,x_k,w,t), \tag{14}$$

$$u(t)=\mathcal{U}(x_0,x_1,\dots,x_k,w,t) \tag{15}$$

with $w\in\mathbb{R}^\nu$, $W:\mathbb{R}^{n_0}\times\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}\times\mathbb{R}^\nu\times T\to\mathbb{R}^\nu$, $\mathcal{U}:\mathbb{R}^{n_0}\times\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}\times\mathbb{R}^\nu\times T\to\mathbb{R}^m$.

Now, the problem of *state feedback synchronization* consists of finding a control law (13), (or (14), (15)) ensuring the asymptotic synchronization (3) in the closed-loop system (12), (13) (or respectively, (12), (14), (15)).

Remark 3. Controlled synchronization becomes relevant only in cases when self-synchronization (1) does not occur and the inclusion of a static or dynamic-state feedback (13) or (14), (15) will only lead to (1) after some transient behavior. Therefore, we will only be concerned with asymptotic feedback synchronization (3).

In a variety of problems complete information about the states of the systems $\Sigma_0,\Sigma_1,\dots,\Sigma_k$ is not available and only some *output variables* $\tilde{y}_s$, $s=1,\dots,r$, with $\tilde{y}_s$ output functions of the interconnected system, so $\tilde{y}_s=\tilde{h}_s(x_0,x_1,\dots,x_k,t)$, are available for use in the control law. The problem of *output feedback synchronization* can be posed as follows: find controller equations in the form of static output feedback

$$u(t) = \mathscr{U}(\tilde{y}_1, \ldots, \tilde{y}_r, t) \tag{16}$$

or in the form of dynamic output feedback

$$\frac{dw}{dt} = W(\tilde{y}_1, \ldots, \tilde{y}_r, w, t), \tag{17}$$

$$u(t) = \mathscr{U}(\tilde{y}_1, \ldots, \tilde{y}_r, w, t) \tag{18}$$

with $w \in \mathbb{R}^v$, $y_s \in \mathbb{R}^{p_s}$, $W : \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_s} \times \mathbb{R}^v \times T \to \mathbb{R}^v$ and $\mathscr{U} : \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_s} \times \mathbb{R}^v \times T \to \mathbb{R}^m$, are smooth parametrized vectorfields resp. functions, such that the goal (3) in system (12), (16) (or (12), (17), (18)) is achieved.

To illustrate the given definitions we will discuss some special cases.

3. Frequency (Huygens) synchronization

A frequency synchronization property, or Huygens synchronization property may be defined for periodic (oscillatory or rotational) motions with frequencies $\omega_1, \ldots, \omega_k$. The frequency synchronization is understood (see [2–6]) as a coincidence or, more generally, commensurability of ω_i, i.e. the following relations should be fulfilled:

$$\omega_i = n_i \omega_*, \quad i = 1, \ldots, k \tag{19}$$

for some integers n_i where $\omega_* > 0$ is the so-called synchronous frequency. In this case a single synchronization index is introduced as

$$J(y_i(\cdot)) = \omega_i,$$

while the functionals can be chosen as

$$g_{sr}(y_s(\cdot), y_r(\cdot)) = \frac{\omega_s}{n_s} - \frac{\omega_r}{n_r}.$$

This version of synchronization can be extended to nonperiodic motions if some kind of average frequencies ω_i can be defined. Note that the case when the relations (19) hold, is usually refered in the celestial mechanics to as the *resonance*, or *commensurability* case when speaking about orbital or rotational motions of celestial bodies.

Also the "piecewise-periodic" case can be considered. In this case the set of all time instances is splitted into disjoint intervals $\Delta_q = [t_q, t_{q+1})$, $q = 1, 2, \ldots$ such that all motions $y_i(\cdot)$ are periodic on each interval Δ_q with frequencies $\omega_i(t)$ which are piecewise constant functions.

An extended version of Huygens synchronization arises if we replace the requirement of coincidence of the average frequencies by that of agreement of spectra in the following sense: Introduce positive spectra scaling functions $\alpha_i(\omega), \beta_i(\omega)$ for each system Σ_i, $i = 1, \ldots, k$, and define the family of synchronization indices J_ω as follows:

$$J_\omega(y_i(\cdot)) = \alpha_i(\omega) S_i(\beta_i(\omega)\omega), \tag{20}$$

where S_i is the spectral density of the output signal $y_i(t)$ which is supposed to be well defined.

The agreement of spectra can be understood as synchronization with respect to the family of functionals

$$g_{sr}(y_s(\cdot), y_r(\cdot)) = \|J_\omega(y_s(\cdot)) - J_\omega(y_r(\cdot))\|$$

for some appropriate norm $\|\cdot\|$, e.g. $\mathscr{L}_2$-norm.

A good example of such kind of synchronization is provided by a color music system. A possible description is as follows. The color system device modulates the light sources by sound signal. Human acoustic and optical analyzers evaluate the power spectra $S_{\text{sound}}(\omega)$ and $S_{\text{color}}(\omega)$. Then the human brain evaluates some measure of difference

$$g = \|S_{\text{sound}}(\omega) - \alpha(\omega) S_{\text{color}}(\beta(\omega)\omega)\|,$$

where $\alpha(\omega), \beta(\omega)$ are the scaling multipliers. The feeling of synchronization is determined by a weighted average of $g(\omega)$ over the audio frequency band. Note that the spectra of real (e.g. audio) signals change with time. Therefore, in practice, the spectra of the processes should be evaluated over intervals $[t_q, t_{q+1}]$ for some sequence $\{t_q\}$ to estimate the spectral density on this interval. Synchronization may only occur for some time intervals.

4. Coordinate synchronization

This type of synchronization occurs when outputs or some phase coordinates of the system Σ_i coincide with corresponding coordinates of the other systems for all $t \geqslant 0$, or asymptotically for $t \to \infty$ (may be with some time shifts τ_i). In this case, one can introduce synchronization indices as the respective outputs of the systems: $J(y_i(\cdot)) = y_i(t)$ with the corresponding set of functionals: $g_{sr}(y_s(\cdot), y_r(\cdot)) = \|J(\sigma_{\tau_s} y_s(\cdot)) - J(\sigma_{\tau_r} y_r(\cdot))\|$. Notice that, in particular, this type of synchronization occurs when the overall system consisting of all interconnected systems possesses an asymptotically stable limit set defined by relations $y_1 = y_2 = \cdots = y_k$.

Sometimes we need to deal with discrete coordinate synchronization when the coincidence of the outputs (or whole-state vectors) only takes place at some discrete set of time instances $\{t_q\}$. In this case, the index $J(y_s(\cdot))$ may be defined as the sequence

$$J(y_s(\cdot)) = \{y_s(t_1), y_s(t_2), \ldots\}$$

while the functionals g_j are chosen using some metric in the space of sequences, e.g. uniform or ℓ^p-metric.

A version of discrete-time coordinate synchronization occurs if t_q is the time instance when some coordinates or outputs $y_i(t)$ approach some prespecified point or cross some given surface or level. Also t_q may be defined as the time of achieving the qth local extremum of the signal. This kind of coordinate synchronization can be described similarly to the definition of the Poincaré map. Assume that at some time instances t_{iq}, $i = 1, \ldots, k$; $q = 1, 2, \ldots$ solutions of each system satisfy the condition $\varphi_i(y_i(t_{iq})) = 0$ (i.e. the phase curve of the ith system intersects the Poincaré cross section at the qth time). If for any given q and for all $1 \leqslant i \leqslant k$ the time instances t_{iq} coincide then we may say that the systems Σ_i synchronize. In this case we may introduce infinite number of indices $J_q(y_i(\cdot)) = t_{iq}$, $q = 1, 2, \ldots$, i.e. J_q is the time of qth crossing of the surface. However, it will require an infinite number of functionals g_q. Alternatively, we define the index $J(y_i(\cdot))$ as the infinite sequence $J(y_i(\cdot)) = \{t_{iq}\}_{q=1}^{\infty}$ and use some norm in the space of sequences as a single functional (as in the previous case).

We have demonstrated how to formalize problems of synchronization in sense of closeness of either values of signals in some specific time instances or those time instances themselves. However, the above formulations are not suitable to express the phenomenon of asymptotic synchronization because the introduced functionals do not depend explicitly on time. To capture asymptotic synchronization we may introduce the "current" indices

$$J_i(y_i(\cdot), t) = \inf_{v \geqslant t} \{v : \varphi_i(y_i(v)) = 0\}$$

or (a "causal" version)

$$J_i(y_i(\cdot), t) = \sup_{0 \leqslant v \leqslant t} \{v : \varphi_i(y_i(v)) = 0\}$$

and "current" functionals as before

$$g_{sr}(y_r(\cdot), y_s(\cdot)) = \|J_r(\sigma_{\tau_r} y_r(\cdot), t) - J_s(\sigma_{\tau_s} y_s(\cdot), t)\|.$$

Of course, additional conditions should be imposed guaranteeing that all the introduced quantities are well defined, e.g. each trajectory crosses the section for arbitrarily large $t \geqslant 0$.

5. Conclusions

An attempt is made to give a fairly general definition of synchronization corresponding to intuition, encompassing most of the known definitions and applications, and capturing peculiarities of both self-synchronization and controlled synchronization. Synchronization in a general sense can be defined as the coincidence of any scalar characteristics of the subsystems: amplitudes, frequencies, energies, powers, fractal dimensions [6], etc. Naturally, we may use several synchronization functionals and/or indices of different kind and thus, obtain a great variety of combined synchronization problems. The general definition was illustrated by a number of examples. The key point of our approach is the understanding that the synchronization as a phenomenon should be understood with respect to some condition which defines the presence of synchronism in the particular problem, i.e. the systems may be in synchronous motion from one point of view and may be asynchronous from another. To give a general definition we introduced the concept of synchronization with respect to the given functional (or to a set of functionals). To capture the physical peculiarities when dealing with the problem it is also convenient to use the concept of synchronization index which allows to understand better the physical meaning of the problem.

The concept of frequency synchronization extending the classical definition of Huygens is introduced and discussed as well as that of coordinate synchronization.

Based on the introduced definitions a practical problem of synchronization of vibrating actuators is described in [5, 4].

The formulation of the controlled synchronization problem allows to address and to solve the control design problem for various dynamical systems. We hope to provide a basis for the development of new controlled synchronization methods. For example, taking the combined functional G in (5) as a goal functional we may apply the speed-gradient method for a synchronization algorithm design (see [9–11]). In cases when the system states are not available for measurement the observer-based methods [18, 19] may be applied. A robustness analysis in the spirit of [17] is also possible. However, many problems of controlled synchronization remain unsolved. For example, still no general methods are available for controlled frequency synchronization, for synchronization with respect to times of crossings of some surfaces and for chaotic systems in the sense of fractal dimensions. The last problem seems especially challenging because it corresponds to a nonlocal in-time-property and even involves noncausal functionals.

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