Averaged Continuous-time Models in Identification and Control

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In this tutorial paper a brief exposition is made for the research area related to development and justification of the so called method of continuous models (MNM). The essence of the method is in replacement of the analysis or design problem for a discrete stochastic system with a similar problem for its simplified (averaged) continuous-time model. Continuous-time models described by either ordinary differential equations or stochastic differential equations are considered. Relations between MNM and averaging method are demonstrated. Applications to identification and control problems for systems and networks are described.

Keywords: averaging method, fast and slow motions, ODE, SDE, ergodicity, mixing conditions.

I. Introduction

A typical approach to problem solving in applied mathematics is based on decomposition and simplification of the initial problem. In the modern control area such an approach is widely used for large scale control systems design, distributed control, stochastic control, etc. The mathematical models of the processes in the above mentioned areas manifest themselves as dynamical systems described by either differential or difference or combined equations. To apply any simplification method one needs techniques both for creation and for justification of the simplified models. The main requirement needed from any simplification technique is conservation of the main properties of the process under consideration. For continuous time systems an approach based on fast and slow motion separation and averaging of the fast motion is well known [1]–[6]. Such an approach called the *averaging method* is well developed both for deterministic and stochastic systems. Its justification for ordinary differential equations can be found, e.g. in [1], [2]. Averaging effect in such systems may be caused by converging the fast component to a constant or to a periodic oscillation. Applicability of the averaging principle to stochastic differential equations is caused by weak dependence property: the disturbance values at the distant instants are 'almost' independent. Justification of the averaging for stochastic systems can be found, e.g. in [7]–[10].

Averaging may manifest itself also in discrete systems described by stochastic difference equations. Averaging mechanism here is different. It is based on summation of large number of small increments. After summation the trajectory of the system becomes close to the solution of the averaged stochastic difference equation with high probability. It is similar to the law of large numbers. The averaged difference equation can be called *discrete deterministic model* of the initial system. It is easy to show that the order of approximation does not change after the next step: replacing the discrete model with a continuous one by reducing the sampling period to zero. The resulting ordinary differential equation can be called *deterministic differential model* of the initial system.

The deterministic model looks like a model of first approximation. In addition to it one can build a family of higher approximation models described by stochastic differential equations. Such *stochastic continuous models* provide better approximation of the probability distribution of the initial system solutions.

Therefore instead of solving analysis or design problem for initial system one may try to solve a similar problem for its deterministic or stochastic model. Typically this new problem is simpler that the initial one. The above approach was called the *method of continuous models* [12], [13]. Since discrete-time averaged models are also of use the method can be also called the *method of averaged models*.

Since the 1970s hundreds of papers and a number of monographs [22], [14], [58], [59], [67], [26], concerning both application of the machinery and its justification were published. Quite a number of them were published in Russian, i.e. they are not well known in the West. An outstanding impact in the area was made by the celebrated paper by L.Ljung [64] that was later listed among 25 seminal papers of the 20th century in control [32]. Currently the paper [64] has received more than 700 citations and its citing rate is not going to decrease.

In this tutorial talk a brief survey of the method of continuous models is presented. Most material and the bibliography is borrowed from [86] and later papers [87], [88], [89]. Further information and extensions to multi-agent systems and networks are presented in the related tutorial papers [90], [91].

II. Continuous-time model building

The method of averaging has a wide applicability in modern control system theory, dynamical systems theory, nonlinear mechanics, etc. [16], [51]. The essence of the method is in separation of slow and fast components of system motion, followed by averaging out the fast motion effects. The formal analysis of the technique for continuous-time systems one can find e.g. in [51], [73] (for deterministic case) and in [73], [82] (for stochastic case).

A specific form of averaging for discrete-time stochastic systems was developed in [12], [14] and, independently in [64] and then applied to various problems in identification and adaptive control. Below the scheme of [12], [14], [64] is described.

Consider a discrete-time stochastic system

$$z_{k+1} = z_k + \gamma_k F(z_k, f_k), \qquad k = 0, 1, 2, \ldots, \tag{1}$$

where $z_k \in \mathcal{R}^n$ — state vector, $f_k \in \mathcal{R}^m$ — random disturbance vector, γ_k — gain parameter. Create the averaged continuous system (continuous model)

$$\frac{dz}{dt} = A(z), \tag{2}$$

where $A(z) = \lim_{k\to\infty} EF(z, f_k)$ (the existence of the limit is assumed). Typical relationships between the discrete-time system and its continuous model are as follows.

1. If the gains γ_k are sufficiently small ($\gamma_k \leq \gamma$) then the trajectories $\{z_k\}$ of (1) are close to the trajectories of (2) $\{z(t_k)\}$, where $t_k = \gamma_0 + \cdots + \gamma_{k-1}$.

2. If the gains γ_k tend to zero as $k \to \infty$ then some asymptotic properties of the solutions of (1) (e.g. stability, ultimate boundedness, etc.) may be similar to those of the solutions of the continuous model (2).

In the case of similarity between (1) and (2) in the above sense one can use simplified model (2) instead of (1) for the purposes of system analysis and design. Such an approach was called *the method of continuous models* [12], [14], *the ODE approach* [64] or *the Derevitskii-Fradkov-Ljung (DFL) scheme* [44]. Below the term 'method of continuous models' will be used since it takes into account two aspects:

- — averaging is not the only way of the model generating (in some cases there is a similarity between (1) and (2) even for nonstochastic disturbances f_k investigated in [14]);
- — one can use different types of models (e.g., stochastic differential equations).

III. Continuous-time model justifying

A number of rigorous results are known justifying applicability of continuous models for sufficiently small gains γ_k. Small value of the gains is prerequisite of separation of motions in system. It implies that the disturbance f_k changes faster than the system state z_k. The standard condition of averaging is weak dependence of f_k and f_s for large $|k-s|$ (e.g. independence of f_k and f_s when $k \neq s$).

Probably the first results on justifying the averaging for discrete stochastic systems in control theory belong to Meerkov [68], who used discrete averaged model

$$z_{k+1} = z_k + \gamma_k A(z_k) \tag{3}$$

(replacing (3) by (2) creates no extra mathematical problems). The proofs in [68] are based on Krylov–Bogoliubov averaging method [73]. Similarly to the 1st and 2nd Bogoliubov theorems the convergence in probability of solutions of (1) and (3) on finite time interval and, under assumption of asymptotic stability of the model (2), the closeness of the trajectories on infinite interval were established for independent f_k.

Significant progress of the method was made by Ljung [63]–[67] who also used Krylov–Bogoliubov approach. In [64] the dependent f_k were treated generated by controlled Markov chain. Moreover, the case $\gamma_k \to 0$ was examined. It was demonstrated that in this case model (2) is responsible for the stability or instability of system (1).

Further development was made by Kul'chitsky [54]–[56] who studied the averaging for some functional of of the state vector rather then for the state vector as a whole. It allowed to weaken the restrictive boundedness condition of [64].

if the gain parameter goes to zero at a suitable rate similar in spirit results were obtained [19], [20] without requirements on the dynamics of the model employing a certain set-valued deterministic model.

Another series of results [12]–[35] is based on the machinery developed by S.N. Bernstein who introduced the concept of stochastic differential equation (SDE) as early as in 1934 [24] and established the conditions of the convergence in distribution (weak convergence) of trajectories of (1) either to ODE (2) or to some SDE [25]. In [12] the mean square bounds of the model [16] accuracy were obtained both for finite and for infinite time interval. E.g. it was shown (in [12] for independent f_k and in [33] for f_k satisfying strong mixing conditions) that under Lipschitz and growth conditions

$$||A(z) - A(z')|| \leq L_1||z - z'||, b(z) \leq L_2(1 + ||z||^2), \tag{4}$$

where $b(z) = E||F(z, f_k) - A(z)||^2$ the following inequality holds:

$$E \max_{0\leq t_k\leq T} \|z_k - z(t_k)\|^2 \leq C_1 e^{C_2 T}\gamma, \tag{5}$$

where $\gamma = \max_{1\leq k\leq N} \gamma_k$, $t_N \leq T$, $C_1 > 0$, $C_2 > 0$.

In the case when the continuous model (2) is exponentially stable it was additionally shown in [12], [14] that the accuracy of approximation over infinite time interval is of order γ^α for some $0 < \alpha < 1$. Namely, there exist $\bar{\gamma} > 0$, such that for $\leq \gamma_k \leq \gamma < \bar{\gamma}$ the following inequalities hold

$$E||z_k - z(t_k)||^2 \leq C_3\gamma^\alpha, k = 1, 2, ..., \tag{6}$$

where numbers $C_3 > 0$, $\alpha > 0$ do not depend on γ.

Though the averaging scheme of [12]–[35] is similar to that of Ljung [64], the analytical results are different in that they allow to analyze dynamics of the systems over finite or infinite time intervals rather than convergence as $t \to \infty$. Moreover the results of [12]–[35] are applicable to the cases when the gain γ_k does not tend to zero which is important in many applications.

Finally an elegant approach was developed by Kushner [57]–[59] who used weak convergence theory for random functions. This framework however is convenient for the studying of asymptotics when $\gamma \to 0$ ($\gamma_k \equiv \gamma$) rather than for evaluating mean distance between the trajectories for finite values of γ.

IV. Stochastic continuous model

The inequality (5) shows that the distance between trajectories of (1) and (2) is of order $(\gamma_k)^{1/2}$. This error arises in part due to random fluctuations. Therefore the model taking in account stochasticity potentially may have higher accuracy. Employing the framework of averaging for SDE [42], [82] yields the stochastic continuous model [21], [22], [50]

$$dy = A(z(t))dt + (\gamma(t)B(z(t))^{1/2}dw, \tag{7}$$

where $\gamma(t) \equiv \gamma_k$ for $t_k \leq t \leq t_{k+1}$, $B(z) = \lim\limits_{k\to\infty} Eh(z,f_k)h(z,f_k)^{\mathrm{T}}$, $h(z,f_k) = F(z,f_k) - A(z)$, $w(t)$ — standard Wiener stochastic process, $z(t)$ — solution of deterministic model (2). In [12] the following stochastic model

$$dy = A(y(t))dt + (\gamma(t)B(y(t)))^{1/2}dw \tag{8}$$

was suggested. Model (8) does not use the solutions of deterministic model (2). It was shown that the conditional incremental covariances of solutions of both (7) and (8) coincide with corresponding characteristics of (1) with the accuracy of order γ_k^2. In [33], [14] the family of stochastic models having higher accuracy was introduced. E.g. the accuracy of model

$$dy = \left[I_n - \frac{1}{2}\gamma(t)\frac{\partial A(y)}{\partial y}\right]A(y)dt + \\ (\gamma(t)B(y(t)))^{1/2}\,dw \tag{9}$$

in terms of the conditional incremental covariances is of order γ_k^3. Note that the model (9) is nothing but the Stratonovich version of the SDE (8).

V. Extensions

In the tutorial some extensions of the previous results are presented. First the case of the systems described by more general equations than (1) are examined:

$$z_{k+1} = \Phi(z_k, f_k, \gamma_k), \tag{10}$$

Let $E\Phi(z, f_k\nu) = z + \gamma A(z) = \nu^2 a(z,\nu)$. The statement of Theorem 1 holds under additional condition $\|a(z,\gamma)\| \leq L_3(1+\|z\|)$. Note that the rigth hand side of system (10) can be defined as follows $A(z) = \lim\limits_{\gamma\to 0}\gamma^{-1}[M\Phi(z,f_k\gamma) - z]$.

The above results are extended to the case when the right hand side of (1) or (10) do not satisfy global Lipschitz condition (e.g. for some hybrid or adaptive systems). In this case the upper bound for γ_k depends on initial conditions.

VI. Continuous-time model for networks

New problems such as synchronization and control of networks have become popular during last decade. They demand for new approximation results. One of new demands is to study accuracy of continuous modes over infinite time interval under partial stability assumption for (2) instead of asymptotic stability. For such cases the following theorem can be useful [86].

Definition 1: Let $\Omega, \Omega_0,\ \Omega \subseteq \Omega_0$ be closed subsets of $\mathcal{R}^n$ and Ω consists of equilibria of (2). The set Ω is called Ω_0-pointwise stable if it is Lyapunov stable and any solution starting from Ω_0 tends to a point from Ω when $t \to \infty$.

Theorem 1: Let Lipschitz and growth conditions (4) hold. Let there exist a smooth mapping $h : \mathcal{R}^n \to \mathcal{R}^l$ and a bounded set $\Omega_0 \subseteq \mathcal{R}^n$ such that $\mathrm{rank}\partial y/partial z = l$ for $z \in \Omega = \{z \in \Omega_0 : h(z) = 0\}$ and the set Ω is Ω_0-pointwise stable. Let there exist a twice continuously differentiable function $V(z)$ and positive numbers $\varkappa_1, \varkappa_2, \varkappa_3$ such that

$$\dot{V}(z) \leq -\varkappa_1 V(z), \tag{11}$$

$$|\frac{\partial^2 V(z)}{\partial z^{(i)}\partial z^{(j)}}| \leq \varkappa_3, V(z) \geq \varkappa_2||h(z)||^2. \tag{12}$$

Then there exist numbers $\bar{\gamma} > 0$, $K_2 > 0$, $0 < \alpha < 1$ such that for $0 \leq \gamma_k \leq \gamma < \bar{\gamma}$ the following inequalities hold

$$E||y_k - y(t_k)||^2 \leq K_2\gamma^\alpha, k = 1, 2, ..., \tag{13}$$

where $y_k = h(z_k),\ y(t_k) = h(z(t_k))$.

The theorem provides an upper mean square bound for the distance between the current state and the limit manifold $\Omega = \{z \in \Omega_0 : h(z) = 0\}$. An open problem is relaxation of the pointwise stability condition. Another problem is extending the results to discontinuous models important for economic games, and pattern recognition (some special cases are considered in [46], [62]).

VII. Applications of Continuous Models

There are three stages of continuous model using: a) model building; b) model justifying; c) model analyzing (either analytic or numerical). The stage b) including checking the conditions of appropriate theorems sometimes happens to be rather involved. In many cases the theorems serve as "moral support" [64] of the designer's intuition.

Continuous models were used for the analyzing of algorithms of identification [21], [22], [23], [12], [14], [37], [55], [57], [58], [59], [65], [67], [79], optimization [28], [14], [35], [39], [50], [59], [31], filtering [21], [22], [56], [60], [70] and adaptive control [17], [13], [33], [14], [34], [36], [40], [54], [66], [74], [75], [81]; stochastic eigenvalue seeking [77], [85]; games solving [68], [80]; pattern recognition [12], [62], [65]; learning of neural networks [53]. A number of recent works open new networks related application areas: analyzing convergence of learning algorithms for coverage control of mobile sensing agents [29], distributed learning and cooperative control for multi-agent systems [30], [47], distributed topology control of wireless networks [27], etc.

VIII. CONCLUSION

Using the continuous models one can simplify the stability and performance analysis of adaptive systems and facilitate discrete-time system design by means of continuous-time design methods. Continuous models provide more detailed information about system behavior than, e.g., Lyapunov function. The main approaches to justifying averaging method for discrete-time stochastic systems are Krylov–Bogoliubov's approach [64], [68], Bernstein's approach [12]–[14] and weak convergence approach [57]. However the procedures of building the averaged (approximate) models are essentially the same. Basic conditions for applicability of averaging are stability of the system and mixing properties of disturbances.

A number of researches are devoted to analysis of the systems with constant or not tending to zero gain (learning rate). Perhaps, the first result of such kind was published in [12]. Unlike many other papers, in [12] approximation bounds were established for nonconstant not tending to zero gain. It was shown in [86] that the stability restriction of [12] can be relaxed to partial stability. Further relaxation is an avenue of further research.

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