FEEDBACK RESONANCE IN NONLINEAR OSCILLATORS

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Keywords: Nonlinear oscillators, Nonlinear resonance, feedback.

Abstract

The possibilities of studying nonlinear physical systems by small feedback action are discussed. Analytical bounds of possible system energy change by feedback are established. It is shown that for 1-DOF nonlinear oscillator the change of energy by feedback can reach the limit achievable for linear oscillator by harmonic (nonfeedback) action. Such a phenomenon is called *feedback resonance.* Speed-Gradient based method of creating feedback resonance in nonlinear multi-DOF oscillators is described. Example problem of escape from a potential well is studied by computer simulation.

1 Introduction

The concept of *resonance* is one of the core concepts for physics and mechanics. It has numerous applications in spectroscopy, optics, mechanical engineering, laser and communication technologies, etc. Perhaps it was first described by Galileo Galilei in "Discorsi a Dimostrazioni Matemache" (1638) [1]. Small resonant forces applied to a system lead to significant changes in its behavior. The resonance phenomenon is well understood and perfectly studied for linear systems. If, however, the dynamics of the system is nonlinear, the resonance is much more complicated because interaction of different harmonic signals in nonlinear system may create complex and even chaotic behavior [2, 3]. Nonlinear resonance caused by aperiodic force was studied in [4].

New avenue of research in physics was opened in the 90s by the works on control and synchronization of chaos. It was discovered by E.Ott, C.Grebogi, J.Yorke [5] and their numerous successors that even small feedback action can dramatically change the behavior of a nonlinear system, e.g. turn chaotic motions into periodic ones and *vise versa*. Although similar phenomenon for nonchaotic systems was previously known to control theorists, an interest in control of chaos has gained interaction between physics and control theory, see, e.g. [6, 7, 8].

In order to examine possibilities and limitations of small control, the notion of *swingability* was introduced in [12]. The system was called *swingable with respect to a characteristics (functional) G*, if the value of G can be changed in an arbitrarily broad range by means of arbitrarily small external action (control). It was shown in [12] that finite-dimensional continuous-time conservative systems are swingable with respect to energy, if the energy layer between initial and final states does not contain equilibria of unforced system. This result was extended to swingability with respect to several integrals (invariants or Casimirs) of free (unforced) system [13, 6]. Finally, the notion of *feedback resonance* was introduced in [9, 10].

In this paper further investigation of feedback resonance is performed. It is shown that for 1-DOF nonlinear oscillator the change of energy by feedback can reach the limit achievable for linear oscillator by harmonic (nonfeedback) action. Speed-Gradient based method of creating feedback resonance in nonlinear multi-DOF oscillators is described. Example problem of escape from a potential well is studied by computer simulation. Extension to a class of strictly passive nonlinear systems is given.

2 Feedback Resonance in 1-DOF Oscillator

Consider the controlled 1-DOF oscillator, modeled after appropriate rescaling by the differential equation

$$\ddot{\varphi} + \Pi(\varphi)' = u, \tag{1}$$

where φ is the phase coordinate, $\Pi(\varphi)$ is the potential energy function and u is the controlling variable. The state vector of the system (1) is $x = (\varphi, \dot{\varphi})$ and its important characteristic is the total energy $H(\varphi, \dot{\varphi}) = \frac{1}{2}\dot{\varphi}^2 + \Pi(\varphi)$. The state vector of the uncontrolled (free) system moves along the energy surface (curve) $H(\varphi, \dot{\varphi}) = H_0$. The behavior of the free system depends on the shape of $\Pi(\varphi)$ and the value of H_0. E.g., for a simple pendulum we have $\Pi(\varphi) = \omega_0^2(1 - \cos\varphi) \geq 0$. Obviously, choosing $H_0 : 0 < H_0 < 2\omega_0^2$ we obtain oscillatory motion with amplitude $\varphi_0 = \arccos(1 - H_0/\omega_0^2)$. For $H_0 = 2\omega_0^2$ the motion along the separatrix, including upper equilibrium, is observed, while for $H_0 > 2\omega_0^2$ the energy curves become infinite and the system exhibits permanent rotation with average angular velocity $< \dot{\varphi} > \approx \sqrt{2H_0}$.

Let us put the question: is it possible to significantly change the energy (i.e., behavior) of the system by means of arbitrarily small controlling action?

The answer is well known when the potential is quadratic, $\Pi(\varphi) = \frac{1}{2}\omega_0^2\varphi^2$, i.e., system dynamics are linear:

$$\ddot{\varphi} + \omega_0^2\varphi = u. \tag{2}$$

In this case we may use harmonic external action

$$u(t) = \overline{u}\sin\omega t \tag{3}$$

and for $\omega = \omega_0$ watch the unbounded resonance solution $\varphi(t) = -\frac{\overline{u}t}{2\omega_0}\cos\omega_0 t$.

However for nonlinear oscillators the resonant motions are more complicated with interchange of energy absorption and emission. It is well known that even for a simple pendulum the harmonic excitation can give birth to chaotic motions. The reason is, roughly speaking, that the natural frequency of a nonlinear system depends on the amplitude of oscillations.

Therefore the idea arises: to create resonance in a nonlinear oscillator by changing the frequency of external action as a function of oscillation amplitude. To implement this idea we need to make $u(t)$ depending on the current measurements $\varphi(t), \dot{\varphi}(t)$, which exactly means introducing a feedback

$$u(t) = U\left(\varphi(t), \dot{\varphi}(t)\right). \tag{4}$$

Now the problem is: how to find the feedback law (4) in order to achieve the energy surface $H(\varphi, \dot{\varphi}) = H_*$. This problem falls into the field of control theory. To solve it we suggest using so called Speed-Gradient(SG) method (see [6, 11, 12, 13] and Section 4 below). For the system (1) the SG-method with the choice of the goal function $Q(x) = [H(x) - H_*]^2$ produces simple feedback laws:

$$u = -\gamma\left(H - H_*\right)\dot{\varphi}, \tag{5}$$

$$u = -\gamma\,\mathrm{sign}\,\left(H - H_*\right)\cdot\mathrm{sign}\,\dot{\varphi}, \tag{6}$$

where $\gamma > 0$, $\mathrm{sign}(H) = 1$, for $H > 0$, $\mathrm{sign}(H) = -1$ for $H < 0$ and $\mathrm{sign}(0) = 0$. It can be proven (see Section 6 for general statement) that the goal $H(x(t)) \to H_*$ in the system (1), (5) (or (1), (6)) will be achieved from almost all initial conditions, provided that the potential $\Pi(\varphi)$ is smooth and its stationary points are isolated. It is worth noticing that, since the motion of the controlled system belongs to the finite energy layer between H_0 and H_*, the right hand side of (5) is bounded. Therefore, taking sufficiently small gain γ, we can achieve the given energy surface $H = H_*$ by means of *arbitrarily small* control. Of course this seemingly surprising result holds only for conservative (lossless) systems.

Let now losses be taken into account, i.e., system is modeled as

$$\ddot{\varphi} + \varrho\dot{\varphi} + \Pi(\varphi)' = u \tag{7}$$

where $\varrho > 0$ is the damping coefficient. Then it is not possible any more to reach an arbitrary level of energy. The lower bound $\overline{H}$ of the energy value reachable by a feedback of amplitude $\overline{u}$ can be calculated as (see Section 4)

$$\overline{H} = \frac{1}{2}\left(\frac{\overline{u}}{\varrho}\right)^2. \tag{8}$$

In order to achieve the energy (8) the parameters of feedback should be chosen properly. Namely, parameter values of the algorithm (5) providing energy (8) under restriction $|u(t)| \leq \overline{u}$ should be as follows: $H_* = 3\overline{H}, \gamma = \varrho/(2\overline{H})$. For the algorithm (6) any value H_* exceeding $\overline{H}$ is appropriate if $\gamma = \overline{u}$ (see Section 6).

Note that H_* does not have the meaning of the desired energy level in presence of losses. It leaves some freedom of parameter choice. Exploiting this observation we may take H_* sufficiently large in the algorithm (6) and arrive to its simplified form

$$u = -\gamma\,\mathrm{sign}\dot{\varphi}, \tag{9}$$

that looks like introducing negative Coulomb friction into the system.

It is worth comparing the bound (8) with the energy level achievable for linear oscillator

$$\ddot{\varphi} + \varrho\dot{\varphi} + \omega_0^2\varphi = u(t), \tag{10}$$

where $\varrho > 0$ is the damping coefficient, by harmonic (nonfeedback) action. The response of the model to the harmonics $u(t) = \overline{u}\sin\omega t$ is also harmonics $\varphi(t) = A\sin(\omega t + \varphi_0)$ with the amplitude

$$A = \frac{\overline{u}}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \varrho^2\omega^2}}. \tag{11}$$

Let ϱ be small, $\varrho^2 < 2\omega_0^2$. Then A reaches its maximum for resonant frequency: $\omega^2 = \omega_0^2 - \varrho^2/4$, and the system energy averaged over the period is

$$\overline{H} = \frac{1}{2}\left(\frac{\overline{u}}{\varrho}\right)^2 + \mathrm{O}(\varrho^2), \tag{12}$$

Comparison of (8) and (12) shows that, for a nonlinear oscillator affected by feedback, the change of energy can reach the limit achievable for a linear oscillator by harmonic (nonfeedback) action, at least in the case of small damping. Therefore, feedback allows a nonlinear oscillator to achieve as deep a resonance as can be achieved by harmonic excitation for the linear case.

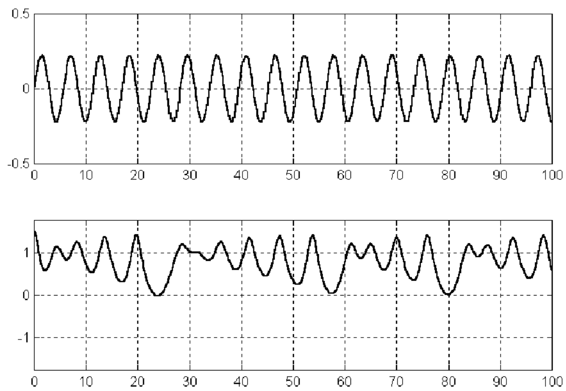

Figure 1: Behavior of Duffing system under harmonic input

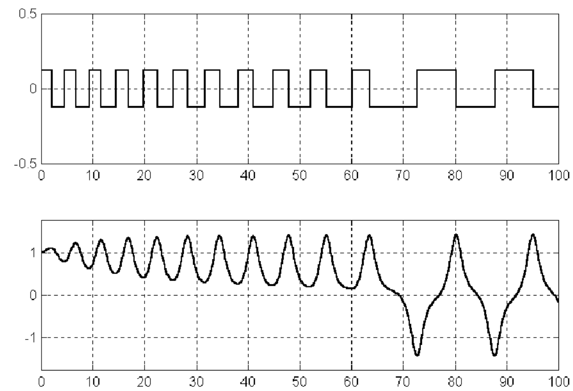

Figure 2: Behavior of Duffing system under feedback input

3 Escape from a Potential Well

The study of escape from potential wells is important in many fields of physics and mechanics [14, 15]. Sometimes escape is an undesirable event and it is important to find conditions preventing it (e.g., buckling of shells, capsize of ships, etc.) In other cases escape is useful and the conditions guaranteeing it are needed. In all cases the conditions of achieving the escape by means of as small an external force as possible are of interest.

In [15] such a possibility (optimal escape) has been studied for typical nonlinear oscillators (7) with a single-well potential $\Pi_e(\varphi) = \varphi^2/2 - \varphi^3/3$ (so called "escape equation") and a twin-well potential $\Pi_d(\varphi) = -\varphi^2/2 + \varphi^4/4$ (Duffing oscillator). The least amplitude of a harmonic external forcing $u(t) = \overline{u}\sin\omega t$ for which no stable steady state motion exists within the well was determined by intensive computer simulations. For example, for escape equation with $\varrho = 0.1$ the optimal amplitude was evaluated as $\overline{u} \approx 0.09$, while for the Duffing twin-well equation with $\varrho = 0.25$ the value of amplitude was about $\overline{u} \approx 0.212$. Our simulation results agree with [15]. The typical time histories of input and output for $\overline{u} = 0.211$ are shown in Fig.1. It is seen that escape does not occur.

Using feedback forcing we may expect reduction in the escape amplitude. In fact using the results of section 2, the amplitude of feedback (5), (6) leading to escape can be easily calculated, by just substituting the height of the potential barrier $\max_{\Omega}\Pi(\varphi) - \min_{\Omega}\Pi(\varphi)$ for $\overline{H}$ into equation (8) where Ω is the well corresponding to the initial state.

For example taking $\overline{H} = 1/6, \varrho = 0.1$ for $\Pi_e(\varphi)$ gives $\overline{u} = 0.0577$, while for the Duffing oscillator the choice $\overline{H} = 1/4, \varrho = 0.25$ yields $\overline{u} = 0.1767$. These values are substantially smaller than those evaluated in [15]. The less the damping, the bigger the difference between the amplitudes of feedback and nonfeedback signals leading to escape. Simulation exhibits still stronger results: escape for the Duffing oscillator occurs even for $\overline{u} = \gamma = 0.122$, if the law (6) is applied, see Fig.2.

4 Speed-gradient Algorithms and Energy Control

Various algorithms for control of nonlinear systems were proposed in the literature, see e.g., [16, 17]. However the overwhelming part of control theory and technology deals with stabilization of prespecified points and trajectories. For purposes of "small control" design the following "speed-gradient" procedure is convenient [6, 11, 12, 13].

Let the controlled system be modeled as

$$\dot{x} = F(x, u), \tag{13}$$

where $x \in R^n$ is the state and $u \in R^m$ is the input (controlling signal). Let the goal of control be expressed as the limit relation

$$Q(x(t)) \to 0 \text{ when } t \to \infty. \tag{14}$$

In order to achieve the goal (14) we may apply the SG-algorithm in the *finite* form

$$u = -\Psi(\nabla_u \dot{Q}(x, u)), \tag{15}$$

where $\dot{Q} = (\partial Q/\partial x)F(x, u)$ is the speed of changing $Q(x(t))$ along the trajectories of (13), vector $\Psi(z)$ forms a sharp angle with the vector z, i.e. $\Psi(z)^{\mathrm{T}} z > 0$ when $z \neq 0$ (superscript "T" stands for transpose). The first step of the speed-gradient procedure is to calculate the speed $\dot{Q}$. The second step is to evaluate the gradient $\nabla_u \dot{Q}(x, u)$ with respect to controlling input u. Finally the vector-function $\Psi(z)$ should be chosen to meet sharp angle condition. E.g. the choice $\Psi(z) = \gamma z, \gamma > 0$ yields the *proportional* (with respect to speed-gradient) feedback

$$u = -\gamma \nabla_u \dot{Q}(x, u), \tag{16}$$

while the choice $\Psi(z) = \gamma \mathrm{sign} z$, where sign is understood componentwise, yields the *relay* algorithm

$$u = -\gamma \mathrm{sign}(\nabla_u \dot{Q}(x, u)). \tag{17}$$

The *integral* form of SG-algorithm

$$\frac{du}{dt} = -\gamma \nabla_u \dot{Q}(x,u), \tag{18}$$

also can be used as well as combined, e.g. proportional-integral form.

The underlying idea of the choice (16) is that moving along the antigradient of the speed $\dot{Q}$ provides decrease of $\dot{Q}$. It may eventually lead to negativity of $\dot{Q}$ which, in turn, yields decrease of Q and, eventually, achievement of the primary goal (14). However, to prove (14) some additional assumptions are needed, see [6, 11, 12, 13].

Let us illustrate derivation of SG-algorithms for the Hamiltonian controlled system of the form

$$\begin{aligned} \dot{q} &= \nabla_p H(q,p) + \nabla_p H_1(q,p)u, && (19)\\ \dot{p} &= -\nabla_q H(q,p) - \nabla_q H_1(q,p)u, && (20) \end{aligned}$$

where $x = (q,p)$ is the $2n$-dimensional state vector, H is the Hamiltonian of the free system, H_1 is the interaction Hamiltonian. In order to control the system to the desired energy level H_*, the energy related goal function $Q(q,p) = (H(q,p) - H_*)^2$ is worth choosing. The first step of speed-gradient design yields

$$\begin{aligned} \dot{Q} &= 2(H - H_*)\dot{H} \\ &= 2(H-H_*)[(\nabla_q H)^{\mathrm{T}} \nabla_p H_1 - (\nabla_p H)^{\mathrm{T}} \nabla_q H_1]u \\ &= 2(H-H_*)\{H, H_1\}u, \end{aligned}$$

where $\{H, H_1\}$ is Poisson bracket. Since $\dot{Q}$ is linear in u, the second step yields $\nabla_u \dot{Q} = 2(H - H_*)\{H, H_1\}$. Now different forms of SG-algorithms can be produced. For example proportional form (16) is as follows

$$u = -\gamma (H - H_*)\{H, H_1\}, \tag{21}$$

where $\gamma > 0$ is the gain parameter. For a special case $n = 1$, $H_1(q,p) = q$, $q = \varphi$ it turns into the algorithm (5). Analysis of the system containing the feedback is based on the following result (proof see in [6]).

Theorem 1. *Let functions H, H_1 and their partial derivatives be smooth and bounded in the region $\Omega_0 = \{(q,p) : |H(q,p) - H_*| \leq \Delta\}$. Let the unforced system (for $u = 0$) have only isolated equilibria in Ω_0.*

Then any trajectory of the system with feedback either achieves the goal or tends to some equilibrium. If, additionally, Ω_0 does not contain stable equilibria then the goal will be achieved for almost all initial conditions from Ω_0.

Similar results are also valid for the goals expressed in terms of several integrals of motions and for the general nonlinear systems with SG-algorithms (see [13]).

Now consider the 1-DOF oscillator with losses (7) controlled by the algorithm (9) (Extension to n-DOF systems see in [10]). Let the goal be increasing the energy of the system. Evaluating the energy change and substituting $u(t)$ from (9) with $\gamma = \overline{u}$ yields

$$\dot{H} = \frac{\partial H}{\partial p}(-\varrho p + u) = -\varrho p^2 + pu = |p|(\overline{u} - \varrho|p|).$$

Therefore $\dot{H} \geq 0$ in the region defined by the inequality $|p| \leq \overline{u}/\varrho$ which is equivalent to the restriction on the kinetic energy $p^2/2 \leq (\overline{u}/\varrho)^2/2$. The latter inequality holds if the condition

$$H \leq \frac{1}{2}\left(\frac{\overline{u}}{\varrho}\right)^2 \tag{22}$$

is imposed on the total energy of the system. Hence the energy increases as long as (22) remains valid. It justifies the estimate (8).

In the case when the feedback (5) is used we obtain $\dot{H} = -\varrho p^2 - (H - H_*)\gamma p^2 = p^2(\gamma(H - H_*) - \varrho)$, and $\dot{H} \geq 0$ within the region $H \leq H_* - \varrho/\gamma$. It yields the estimate

$$\overline{H} = H_* - \frac{\varrho}{\gamma}. \tag{23}$$

However H_* cannot be taken arbitrarily large because of control amplitude constraint $|u| \leq \overline{u}$ which is equivalent to $\gamma|H - H_*|\ |p| \leq \overline{u}$. The above inequality is valid if

$$\gamma^2 (H - H_*)^2 H \leq \frac{1}{2}\overline{u}^2. \tag{24}$$

Since (24) should be valid in the whole range of energies $0 \leq H \leq H_*$, it is sufficient to require it for $H = H_*/3$ providing maximum of the left hand side of (24). Therefore the maximum γ consistent with (24) is $\gamma = \overline{u}/\left(\frac{2}{3}H_*\right)^{3/2}$. Substituting the above γ into (23) and taking maximum over H_* we obtain that the bound (8) is achieved with the choice $\gamma = \varrho/(2\overline{H})$, $H_* = 3\overline{H}$.

5 Excitation of Multi-DOF Systems. Excitability Index

The above consideration can be extended to a class of multivariable (multi-DOF) systems. Consider Hamiltonian systems with dissipation having Hamiltonian function

$$H = \frac{1}{2}p^T A^{-1}(q)p + \Pi(q) \tag{25}$$

and dissipation function (Rayleigh function) $R(p)$, where $q = (q_1, \cdots, q_n)$ is the generalized coordinate, $p = (p_1, \cdots, p_n)$ is the generalized momentum, $x = (q,p)$ is the state of the system. Assume that $\Pi(q) \geq 0$, and

$$0 < \alpha|p|^2 \leq p^T A(q)p \leq \overline{\alpha}|p|^2 \tag{26}$$

$$|\nabla R(p)| \leq \epsilon\varrho|p| \tag{27}$$

for some $\alpha > 0$, $\overline{\alpha} > 0$, $\epsilon > 0$, $\varrho > 0$. It means that the kinetic energy matrix $A(q)$ is uniformly bounded and uniformly positive definite, viscous damping is bounded. It was shown in [10] that under above conditions the lower bound of the energy level achievable by the control satisfying

$$|u(t)| \leq \epsilon\overline{u} \tag{28}$$

is as follows:

$$\overline{H} = \frac{\alpha}{2}\left(\frac{\overline{u}}{2}\right)^2. \qquad (29)$$

To achieve the level (29) the Speed-Gradient control algorithm (17) of Section 3 can be used which is locally optimal (it minimizes the energy growth rate dH/dt over all controls saisfying (28)). It follows from the results of [18] that for small $\epsilon > 0$ the locally optimal control (17) provides a suboptimal solution for the problem of the terminal energy level minimization. In addition, the achievable energy has the order of $C\ (\overline{u}/\varrho)^2$.

The factor C depends on the shape of the potential and its evaluation is not an easy task. It follows from (29) that for the Hamiltonian systems with dissipation $C \geq \alpha/2$, where $\alpha > 0$ is lower eigenvalue of inertia matrix $A(q)$. Using energy balance method (see [21]) for simple pendulum the estimate $C = 8/\pi^2$ can be obtained. Numerous simulations show that $C \leq \alpha$. We may conjecture that inequality in general case.

The suboptimally property holds for the class of strictly passive nonlinear systems, satisfying relation

$$V(x(t)) - V(x(0)) = \int_0^t [u(s)y(s) - W(x(s))]\, ds \qquad (30)$$

for some nonnegative function $V(x)$ and positive definite function $W(x)$, where $y = h(x)$ is a system output. In this case the control law

$$u(t) = \epsilon\overline{u}\,\mathrm{sign}\, y(t) \qquad (31)$$

provides suboptimal (for small $\epsilon > 0$) solution to the following optimization problem:

$$\sup_{|u(t)|\leq\epsilon\overline{u}} V(x(t)). \qquad (32)$$

The above considerations motivate introducing a new characteristic of the physical system measuring its resonance properties. Recall the relation between the oscillation amplitude of a linear 1-DOF system and its energy: $A = \sqrt{2H}$. The value $(\sqrt{2H})/\overline{u}$ has the meaning of maximum amplifucation of the input signal (exciting force) having amplitude $\overline{u}$. It characterizes the depth of the resonance achievable in the system. The same is true also for nonlinear systems if we agree to define resonance mode as one corresponding to a maximum excitation of system output by means of a bounded feedback signal.

Having in mind the above arguments we introduce the function $A_F(v)$ measuring resonance property of a nonlinear system under feedback excitation as follows:

$$A_F(v) = \frac{1}{v}\sqrt{2\overline{V}(v)}, \qquad (33)$$

where $\overline{V}(v)$ is optimal value of the problem

$$\sup_{|u(t)|\leq v} V(x(t)). \qquad (34)$$

The function $A_F(v)$ is called *excitabilty index*. It can be measured experimentally by applying a feedback signal $u(t) = \mathcal{U}(x(t))$ to the system. Note that the magnitude frequency response for linear system is measured in a similar way by means of applying a harmonic signal to the systems. The value of $A_F(v)$ characterizes the damping properties of nonlinear systems. The lower bound for $A_F(v)$ is provided by output amplitude created by the Speed-Gradient input signal (31). Moreover, if $v = \epsilon\overline{u}$, $W(x) = \epsilon\varrho|y|^2$ and $\epsilon > 0$ is small, this estimate is suboptimal.

The role of excitability index for nonlinear systems is analogous to that of maximum magnitude frequency response for linear ones. (The difference is that the maximum magnitude frequency response is measured by scanning over frequency range, while the excitability index can be calculated by scanning over a range of input amplitudes). For example, it is possible to use it for reformulating stability criteria for feedback interconnection of two nonlinear systems. To this end, we take stability criteria based on passivity theorem [19, 20] and substitute $\frac{1}{2}(vA(v))^2$ instead of storage function value.

For special case when one of the two subsystems is linear (the whole system in that case is called *Lur'e system*) we obtain classical circle criterion by this procedure.

Thus the introduced above excitability index allows to extend classical absolute stability results to the system with nonlinear nominal part. Note that the notion of excitability index is most useful for weakly damped (close to conservative) systems, where $A_F(v)$ can be well determined by measuring system output as response to small v. It follows from (8) that for 1-DOF systems with small damping $\epsilon\varrho$ the estimate $A_F(\epsilon)/\epsilon \approx 1/\varrho$, where ϱ is viscous damping coefficient holds for small $\epsilon > 0$.

Conclusions

The fundamental question of physics, mechanics and other natural sciences is: what is possible and why? In this paper we attempted to investigate what is possible to do with physical system by feedback. It was shown that if system is close to conservative, its energy can be changed in a broad range by small feedback forcing. The 1-DOF nonlinear oscillator was taken as example but similar results hold for much more general systems. The introduced concept of *excitability index* characterizes the depth of the resonance in the nonlinear system achievable by feedback forcing.

Acknowledgements

The paper was partly written when author was staying at the University of Twente in March-July 1998. The author acknowledges support of the Russian Foundation of Basic Research (grants RFBR 96-01-01151, 99-01-00672), and of the Dutch Organization for Pure Research (NWO).

Author thanks Boris Andrievsky for help with performing simulations.

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