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Control of chaos: methods and applications in mechanics

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A survey of the field related to control of chaotic systems is presented. Several major branches of research that are discussed are feed-forward ('non-feedback') control (based on periodic excitation of the system), the 'Ott–Grebogi–Yorke method' (based on the linearization of the Poincaré map), the 'Pyragas method' (based on a time-delayed feedback), traditional for control-engineering methods including linear, nonlinear and adaptive control. Other areas of research such as control of distributed (spatio-temporal and delayed) systems, chaotic mixing are outlined. Applications to control of chaotic mechanical systems are discussed.

Keywords: nonlinear control; chaotic systems; mechanical systems

1. Introduction

The idea of controlling chaos in dynamical systems has come under detailed investigation during the last decade (see Fradkov & Pogromsky 1998). Starting with a few papers in 1990, the number of publications in peer-reviewed journals exceeded to 2700 in 2000, with more than half published in 1997–2000. According to the Science Citation Index, in 1997–2002, about 400 papers per year related to control of chaos were published in peer-reviewed journals, see surveys (Fradkov & Evans 2002, 2005; Andrievskii & Fradkov 2003, 2004), while in 2005 this amount exceeded 500. Authors of numerous papers have developed new methods for control of nonlinear systems and demonstrated advantages of their usage for both analysis of system dynamics and significant change of system behaviour by small forcing.

The state-of-the-art of the field related to control of chaotic systems is surveyed in this paper. Several major branches of research are discussed: the feed-forward ('non-feedback') control (based on periodic excitation of the system), the 'Ott–Grebogi–Yorke (OGY) method' (based on linearization of

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the Poincaré map), the 'Pyragas method' (based on a time-delayed feedback), traditional for control-engineering methods of nonlinear and adaptive control. Most attention is paid to the control of continuous-time chaotic systems.

A number of application examples in mechanics and mechanical engineering are discussed, including control of pendulum systems, beams, plates, control of stick–slip friction motion, control of vibroformers and control of microcantilevers.

2. Methods of chaos control

Consider a continuous-time system with lumped parameters described in state space by differential equations

$$\dot{x} = F(x, u), \tag{2.1}$$

where x is an n-dimensional vector of state variables; $\dot{x} = \mathrm{d}x/\mathrm{d}t$; and u is an m-dimensional vector of inputs (control variables). A typical goal of controlling a chaotic system is full or partial stabilization of an unstable trajectory (orbit) $x_*(t)$ of the unforced ($u=0$) system. The trajectory $x_*(t)$ may be either periodic or chaotic (non-periodic). An important requirement is the restriction of the control intensity; only small controls are of interest.

A specific feature of this problem is the possibility of achieving the goal by means of an arbitrarily small control action. Other control goals like synchronization (concordance or concurrent change of the states of two or more systems) and chaotization (generation of a chaotic motion by means of control) can also be achieved by small control in many cases.

More subtle objectives can also be specified and achieved by control, for example, to modify a chaotic attractor of the free system in the sense of changing some of its characteristics (Lyapunov exponents, entropy, fractal dimension), or delay its occurrence, or change its locations, etc.

(a) *Feed-forward control by a periodic signal*

The idea of *feed-forward* control (also called *non-feedback* or *open-loop* control) is to change the behaviour of a nonlinear system by applying a properly chosen input function or external excitation $u(t)$. The excitation can reflect influence of some physical action, e.g. external force/field, or it can be some parameter perturbation (modulation). Such an approach is attractive because of its simplicity; no measurements or extra sensors are needed. It is especially advantageous for ultrafast processes, e.g. at the molecular or atomic level where no possibility of measuring system variables exists.

The possibility of transformation of periodic motion into chaotic motion and vice versa by an external harmonic excitation was first studied in the 1980s in Moscow State University by Dudnik *et al.* (1983) and Kuznetsov *et al.* (1985) for the Lorenz system and by Alekseev & Loskutov (1985, 1987) for a fourth-order system describing dynamics of two interacting populations. Matsumoto & Tsuda (1983) demonstrated the possibility of suppressing chaos in a Belousov–Zhabotinsky reaction by adding a white noise disturbance. These results were based on computer simulations. A first account of theoretical understanding of the phenomenon was given in Pettini (1988) and Lima & Pettini (1990), where the so-called Duffing–Holmes oscillator,

$$\ddot{\varphi} - c\varphi + b\varphi^3 = -a\dot{\varphi} + d\cos(\omega t), \tag{2.2}$$

was studied by Melnikov's method. The right-hand side of equation (2.2) was considered as a small perturbation of the unperturbed Hamiltonian system. The Melnikov function, related to the rate of change of the distance between stable and unstable manifolds for small perturbations, was calculated analytically and parameter values producing chaotic behaviour of the system were chosen. Then additional excitation was introduced into the parameter of nonlinearity $b \to b(1+\eta\cos \Omega t)$ and the new Melnikov function was computed and studied numerically. It was shown that if Ω is close to the frequency of the initial excitation ω, then chaos may be destroyed. Experimental confirmation was made by a magneto-elastic device with two permanent magnets, an electromagnetic shaker and an optical sensor. The results are surveyed in Lima & Pettini (1998) where some open problems were also posed.

Recent investigations were aimed at better suppression of chaos with smaller values of excitation amplitude and providing convergence of the system trajectories to the desired periodic orbit (limit cycle). Belhaq & Houssni (1999) considered the case of quasi-periodic excitation by reducing it to the periodic case (see also Zhalnin 1999). Basios *et al.* (1999) studied the case of parametric noise excitation by Melnikov analysis. Mirus & Sprott (1999) attempted to achieve resonance of excitation with the frequency of the desired periodic excitation. Since a chaotic attractor contains trajectories close to periodic orbits with different periods, a proper choice should be made to minimize the amplitude of excitation. A numerical illustration of the approach was given for a Lorenz system and for a high-dimensional system of 32 diffusively coupled Lorenz systems. Harmonic excitation was introduced via modulation of parameter r. In Pisarchik & Corbalan (1999), stabilization of unstable periodic orbits (UPOs) by means of periodic action with frequency much lower than the characteristic frequency of the system has been studied.

In a number of papers, the choice of excitation function is based on tailoring it to the system nonlinearity. Let the controlled system be described by equations

$$\dot{x} = f(x) + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m. \tag{2.3}$$

Now let $m=n$ and $\det B \neq 0$. If the desired solution of the controlled system is $x_*(t)$, then an intuitively reasonable choice of excitation is

$$u_*(t) = B^{-1}(\dot{x}_*(t)) - f(x_*(t)), \tag{2.4}$$

because $x_*(t)$ will satisfy the equations of the excited system (see Hübler & Lusher 1989). The equation for the error $e = x - x_*(t)$ is then $\dot{e} = f(e + x_*(t)) - f(x_*(t))$. If the linearized system with matrix $A(t) = \partial f(x_*(t))/\partial x$ is uniformly stable in the sense that $A(t) + A(t)^{\mathrm{T}} \leq -\lambda I$ for some λ and for all $t \geq 0$ then all solutions of (2.3) and (2.4) will converge to $x_*(t)$. More general convergence conditions can be found in Fradkov & Pogromsky (1998). In case $m < n$ and B is singular, the same result is valid under matching conditions: vector $\dot{x}_*(t) - f(x_*(t))$ is in the span of the columns of B. Then the control can be chosen to be $u_*(t) = B^{+}(\dot{x}_*(t) - f(x_*(t)))$, where B^{+} is the pseudo-inverse matrix. Despite the fact that the uniform stability condition rules out chaotic (i.e. unstable) trajectories, it is claimed in a number of papers that some local convergence to chaotic trajectories is observed, if the instability regions are not dominating. Rajasekar *et al.* (1997) compared this approach with other methods through a second-order system, e.g. describing the so-called Murali–Lakshmanan–Chua electronic circuit and FitzHugh–Nagumo equations describing propagation of nerve pulses in a neuronal membrane. Ramesh & Narayanan (1999) investigated (numerically) different schemes of non-feedback excitation in the presence of noise. In Hsu *et al.* (1997) and Mettin (1998), similar results for the discrete-time case were obtained.

The Melnikov method was applied by Chaćon (2001, 2002) to a general model of a one-degree-of-freedom nonlinear oscillator with damping excited with biharmonic forcing. The relation between damping strength and forcing amplitudes guaranteeing either chaotic or periodic behaviour of the given trajectory of the excited system was obtained. Since the Melnikov method leads to intractable calculations for state dimensions greater than two, analytical results are known only for systems with one degree of freedom. For higher dimensions, computer simulations are used. The general problem of finding analytic conditions for creation or suppression of chaos by feed-forward periodic excitation of small or medium level still remains open.

In summary, a variety of open-loop methods have been proposed. Most of these have been evaluated by simulation for special cases and model examples. However, the general problem of finding sufficient conditions for creation or suppression of chaos by feed-forward excitation still remains open.

(*b*) *Linearization of the Poincaré map (OGY method)*

The explosion of interest in the control of chaotic systems was caused by Ott *et al.* (1990). The two key ideas introduced in this paper were:

- (i) To use a discrete system model based on linearization of the Poincaré map for controller design.
- (ii) To use the recurrent property of chaotic motions and apply control action only at time instants when the motion returns to the neighbourhood of the desired state or orbit.

The original version of the algorithm was described for discrete-time systems (iterated maps) of dimension two and for continuous-time systems of dimension three and required online computation of the eigenvectors and eigenvalues for the Jacobian of the Poincaré map. Numerous extensions and interpretations have been proposed by different authors in subsequent years and the method is commonly referred to as the 'OGY method'. The idea of the OGY method is as follows.

Let the controlled system be described by the state space equations (2.1). Usually the variable u represents a changeable parameter of the system rather than a standard 'input' control variable, but it makes no difference from a control theory point of view. The task is to obtain the desired (goal) trajectory $x_*(t)$ which is a solution of (2.1) with $u=0$. The goal trajectory may be either periodic or chaotic: in both cases it is recurrent. Draw a surface (Poincaré section)

$$S = \{x : s(x) = 0\} \tag{2.5}$$

through the given point $x_0 = x_*(0)$ transverse to the solution $x_*(t)$ and consider the map $x \mapsto P(x, u)$ where $P(x, u)$ is the point of first return to S of the solution to (2.1) with constant input u starting from x. The map $x \mapsto P(x, u)$ is called *the controlled Poincaré map*. It is well defined at least in some vicinity of the point x_0 owing to the recurrence property of $x_*(t)$. The precise definition of the controlled Poincaré map requires some technicalities (see Fradkov & Pogromsky 1998). Iterating the map, we may define a discrete-time system

$$x_{k+1} = P(x_k, u_k), \tag{2.6}$$

where $x_k = x(t_k)$, t_k is the time of the kth crossing and u_k is the value of $u(t)$ between t_k and t_{k+1}.

The next step of the control law design is to replace the initial system (2.1) by the linearized discrete system

$$\tilde{x}_{k+1} = A\tilde{x}_k + Bu_k, \tag{2.7}$$

where $\tilde{x}_k = x_k - x_0$ and find a stabilizing controller, e.g. $u_k = Cx_k$ for (2.6). Finally, the proposed control law is as follows:

$$u_k = \begin{cases} C\tilde{x}_k, & \text{if } |\tilde{x}_k| \le \Delta, \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

A key point of the method is to apply control only in some vicinity of the goal trajectory by introducing an 'outer' deadzone. This has the effect of bounding control action. Richter & Reinschke (1998) suggested determining the size and shape of the region of control action by using a Lyapunov function, which makes the mechanism of control more transparent.

Numerous simulations performed by different authors have confirmed the efficiency of such an approach. Slow convergence is often reported, but this is the price of achieving non-local stabilization of a nonlinear system by small control.

There are two important problems to solve for implementation of the OGY method: lack of information about the system model and incomplete measurements of the system state. The second difficulty can be overcome by replacing the initial state vector x by the so-called *delay coordinate vector* $X(t) = [y(t), y(t-\tau), \dots, y(t-(N-1)\tau)]^{\mathrm{T}} \in \mathbb{R}^n$, where $y = h(x)$ is the output (e.g. one of the system coordinates) available for measurement and $\tau > 0$ is the delay time. Then the control law has the form

$$u_k = \begin{cases} U(y_k, y_{k,1}, \dots, y_{k,N-1}), & \text{if } |y_{k,i} - y_*| \le \Delta, i = 1, \dots, N-1, \\ 0, & \text{otherwise,} \end{cases} \tag{2.9}$$

where $y_{k,i} = y(t_k - i\tau)$.

A special case of algorithm (2.9) introduced by Hunt (1991) is termed *occasional proportional feedback* (*OPF*). The OPF algorithm is used for stabilization of the amplitude of a limit cycle and is based on measuring local maxima (or minima) of the output $y(t)$ The Poincaré section is given by (2.5) where $s(x) = (\partial h/\partial x)F(x,0)$, which corresponds to $\dot{y} = 0$. If y_k is the value of the kth local maximum, then the OPF method employs a simple control law

$$u_k = \begin{cases} K\tilde{y}_k, & \text{if } |\tilde{y}_k| \le \Delta, \\ 0, & \text{otherwise,} \end{cases} \tag{2.10}$$

where $\tilde{y}_k = y_k - y_*$ and $y_* = h(x_0)$ is the desired upper level of oscillation.

However, only partial justification of the proposed algorithms (2.9) and (2.10) is available. The main problem is estimation of the accuracy of the linearized Poincaré map in the delayed coordinates:

$$y_k + \cdots + a_{N-1}y_{k,N-1} = b_1 u_k + \cdots + b_{N-1}u_{k-N-1}. \tag{2.11}$$

To overcome the first problem—uncertainty of the linearized plant model—Ott *et al.* (1990) and their followers (see survey papers by Grebogi *et al.* (1997), Arecchi *et al.* (1998) and Boccaletti *et al.* (2000)) suggested estimation of parameters in state space form (2.7). However, detailed methods for extracting the parameters of the model (2.7) from the measured time-series are yet to be presented. The problem is, of course, well known in identification theory and is not straightforward, because identification in closed-loop under 'good' control may prevent 'good' estimation.

In Fradkov & Guzenko (1997) and Fradkov *et al.* (2000), a justification of the above method was given for the special case when $y_{k,i} = y_{k-i}$, $i = 1, \ldots, n$. In this case, the outputs are measured and control action is changed only at the instants of crossing the surface (see also Fradkov & Pogromsky 1998). For controller design, an input–output model (2.11) was used containing fewer coefficients than (2.7). For estimation, the method of recursive goal inequalities due to Yakubovich was used, introducing an additional inner deadzone to resolve the problem of estimation in closed-loop. An inner deadzone combined with the outer deadzone of the OGY method, provides robustness of the identification-based control with respect to both model errors and measurement errors. Obradovic & Lenz (1997) proved that the OGY control approach to equilibrium point stabilization in the presence of persistent, magnitude bounded process noise is actually optimal when the system performance is measured by the l_∞-norm of the control signal.

Further modifications and extensions to the OGY method have been recently proposed. Epureanu & Dowell (1997) use only data collected over a single period of oscillation. An extension to a class of systems evolving in manifolds has been given by Aston (1998). A multi-step version was studied by Holzhuter & Klinker (1998). Epureanu & Dowell (1998) suggested a time-varying control function $u(t) = c(t)\bar{u}$ instead of a constant between crossings and $c(t)$ is chosen to minimize control energy. Iterative refinement extending the basin of attraction and reducing the transient time was proposed by Aston & Bird (2000). Basins of attraction for the initial state and parameter estimates were evaluated by Chanfreau & Lyyjynen (1999), while transient behaviour was also investigated by Holzhuter & Klinker (2000). New demonstrations of efficiency of the OGY method were obtained by computer simulations for the Copel map, the Bloch wall, magnetic domain-wall system and by physical experiments with bronze ribbon, glow discharge, non-autonomous RL-diode circuit (see references in Fradkov & Evans (2002) and

Andrievskii & Fradkov (2003)). The OPF method has been used for stabilization of the frequency emission from a tunable lead–salt stripe geometry infrared diode laser and implemented in an electronic chaos controller.

The OPF method has been applied and experimentally studied for a mechanical oscillator with dry friction nonlinearity in Moon *et al.* (2003). In this work, the control is affected by changing the normal force of the dry friction element using a magnetic actuator. It is shown to exhibit both control of chaos, i.e. the stabilization of UPOs in a strange attractor, and 'anticontrol of chaos' (use of feedback to drive a nonlinear system into a chaotic state near a periodic motion). The addition of noise or dither onto periodic oscillations can often be useful in engineering devices.

(*c*) *Delayed feedback*

During recent years, there has been increasing interest in the method of time-delayed feedback (Pyragas 1992). Pyragas, a Lithuanian physicist considered stabilization of a τ-periodic orbit of the nonlinear system (2.1) using a simple control law

$$u(t) = K[x(t) - x(t-\tau)], \tag{2.12}$$

where K is the feedback gain and τ is the time-delay. If τ is equal to the period of an open-loop periodic solution $\bar{x}(t)$ of (2.1) (for $u=0$) and the solution $x(t)$ to the closed-loop system (2.1), (2.12) starts from $\Gamma = \{\bar{x}(t)\}$, then it will remain in Γ for all $t \geq 0$. A puzzling observation was made, however, that $x(t)$ may converge to Γ even if $x(0) \bar{\in} \Gamma$.

The law (2.12) also applies to stabilization of forced periodic motions in system (2.1) with a T-periodic right-hand side. In this case, τ should be chosen equal to T. The formulation of the method for stabilization of fixed points and periodic solutions of discrete-time systems is straightforward.

An extended version of the Pyragas method has also been proposed with

$$u(t) = K \sum_{k=0}^{M} r_k [y(t - k\tau) - y(t - (k+1)\tau)], \tag{2.13}$$

where $y(t) = h(x(t)) \in \mathbb{R}^1$ is the observed output and r_k, $k = 1, \ldots, M$ are tuning parameters. For $r_k = r^k$, $|r| < 1$, and $M \to \infty$ the control law (2.13) becomes

$$u(t) = K[y(t) - y(t-\tau)] + Kru(t-\tau). \tag{2.14}$$

Although algorithms (2.12)–(2.14) look simple, analytical study of the closed-loop behaviour seems difficult. Until recently only numerical and experimental results concerning the performance and limitations of the Pyragas method have been available.

In Basso *et al.* (1997), the stability of a forced T-periodic solution of a Lurie system (system represented as a feedback connection of a linear dynamical system and a static nonlinearity) with a generalized Pyragas controller

$$u(t) = G(p)[y(t) - y(t-\tau)], \tag{2.15}$$

where $G(p)$ is a filter transfer function, was investigated. Using absolute stability theory (Leonov *et al.* 1996), sufficient conditions on the transfer function of the linear part of the controlled system and on the slope of the nonlinearity were obtained for a stabilizing $G(p)$. Extension to systems with a nonlinear nominal part and a general framework based on classical frequency-domain tools are presented in Basso *et al.* (1999).

Ushio (1996) established (for a class of discrete-time systems) a simple necessary condition for stabilizability with a Pyragas controller (2.12): the number of real eigenvalues of matrix A greater than one should not be odd, where A is the matrix of the system model linearized near the desired fixed point. Proofs for more general and continuous-time cases were given independently by Just *et al.* (1997) and Nakajima (1997). The corresponding results for an extended control law (2.13) were presented in Nakajima & Ueda (1998) and Konishi *et al.* (1999), who applied Floquet theory to the system linearized near the desired periodic solution. Using a similar approach, Just *et al.* (1999, 2000) gave a more detailed analysis and established approximate bounds for a stabilizing gain K.

Schuster & Stemmler (1997) showed that for a scalar discrete-time system $y_{k+1}=f(y_k, u_k)$ a necessary condition for existence of a discrete version of the stabilizing feedback (2.13) is $\lambda<1$, where $\lambda=\partial f/\partial y(0,0)$, following from the theorem of Giona (1991). They showed that the restriction $\lambda<1$ can be overcome by means of a periodic modulation of the gain K.

Recently, Pyragas (2001) suggested using the controller (2.14) with $|K|>1$. In that case, the controller itself becomes unstable while stability of the overall closed-loop system can still be preserved. In this case the previous limitations can be significantly relaxed and, in particular, the 'odd number' limitation can be removed. Necessary and sufficient conditions for stabilizability of discrete-time systems via delayed feedback control are obtained by Zhu & Tian (2005).

The Pyragas method was extended to coupled (open flow) systems and modified for systems with symmetries. It was also extended to include an observer estimating the difference between the system state and the desired unstable trajectory (fixed point; Konishi & Kokami 1998).

Reported applications include stabilization of coherent modes of lasers, magneto-elastic systems (Hai *et al.* 1997; Hikihara *et al.* 1997) traffic models, pulse-width modulation controlled buck converters and a paced excitable oscillator described by the FitzHugh–Nagumo model widely used in physiology.

A drawback of the control law (2.12) is its sensitivity to parameter choice, especially to the choice of the delay τ. Apparently, if the system is T-periodic and the goal is to stabilize a forced T-periodic solution, then the choice $\tau=T$ is mandatory. Alternatively, an heuristic trick is to simulate the unforced system with initial condition $x(0)$ until the current state $x(t)$ approaches $x(s)$ for some $s<t$, i.e. until $|x(t)-x(s)|<\varepsilon$. Then the choice $\tau=t-s$ will give a reasonable estimate of a period and the vector $x(t)$ will be an initial condition to start control. However, such an approach often gives overly large values of the period. Since chaotic attractors contain periodic solutions of different periods, an important problem is to find and stabilize (with small control) the solution with the smallest period. This problem remains open. Other attempts to estimate T and develop adaptive versions of the Pyragas algorithm were made by Kittel *et al.* (1995) and Chen & Yu (1999).

In Galvanetto (2002), some numerical techniques to control UPOs embedded in chaotic attractors of a particular discontinuous mechanical system are described. The control method is a continuous time-delayed feedback that modifies the stability of the orbit but does not affect the orbit itself.

(*d*) *Linear and nonlinear control*

Many papers are devoted to the demonstration of the applicability of standard control-engineering notions and techniques for the control of chaos. Indeed, in many cases even simple proportional feedback can achieve the desired control goal. For example, the so-called open-plus-closed-loop method (Jackson & Grosu 1995) applicable to models of the form $\dot{x}(t) = f(x(t)) + Bu$ with $\dim x = \dim u$ employs a combination of proportional feedback with the so called 'Hübler feed-forward action'

$$u(t) = B^{-1}[\dot{x}_*(t) - f(x_*(t)) - K(x - x_*(t))], \tag{2.16}$$

which in some cases allows stabilization of the desired trajectory $x_*(t)$. Recent results are summarized in Aquirre & Torres (2000).

Control by proportional pulses (impacts) was studied by Casas & Grebogi (1997) and Chau (1997). Proportional feedback in the extended space (x, u) (i.e. dynamic feedback) was examined by Magnitskii & Sidorov (1998) and Zhao *et al.* (1998).

The potential of dynamic feedback can be better exploited using an observer-based framework that allows for systematic use of output feedback. A survey of nonlinear observer techniques can be found in Nijmeijer & Mareels (1997) and for certain particular designs (see Grassi & Mascolo 1997; Morgul & Solak 1997). Linear high-gain observer-based control for global Lipschitz nonlinearities was studied by Liao (1998).

Note that models of chaotic systems often do not satisfy a global Lipschitz condition owing to the presence of polynomial nonlinearities x_1x_2, x^2, etc. Although trajectories of chaotic systems are bounded, this is not necessarily the case when the system is influenced by control. Therefore, special attention should be paid to provide boundedness of the solutions by appropriate choice of controls. Otherwise, the solution may escape in finite time and it does not make sense to discuss stability and convergence issues. The possibility of escape in nonlinear controlled systems is often overlooked in application papers.

A number of methods are based on continual reduction of some goal (objective) function $Q(x(t), t)$. The current value $Q(x(t), t)$ may reflect the distance between the current state $x(t)$ and the current point of the goal trajectory $x_*(t)$, such as $Q(x, t) = |x - x_*(t)|^2$, or the distance between the current state and the goal surface $h(x) = 0$, such as $Q(x) = |h(x)|^2$. For continuous-time systems, the value $Q(x)$ does not depend directly on the control u and decreasing the value of the speed $\dot{Q}(x) = (\partial Q/\partial x)F(x, u)$ can be posed as an immediate control goal instead of decreasing $Q(x)$. This is the basic idea of the *speed-gradient* (SG) method (Fradkov 1979), where a change in the control u occurs along the gradient in u of the speed $\dot{Q}(x)$. This approach was first used for control of chaotic systems in Fradkov (1994). Systematic exposition and further references can be found in Fradkov & Pogromsky (1998). The general SG algorithm has the form

$$u = -\Psi[\nabla_u \dot{Q}(x, u)], \tag{2.17}$$

where $\Psi(z)$ is vector-function forming an acute angle with its argument z. For affine controlled systems $\dot{x} = f(x) + g(x)u$, the algorithm (2.17) simplifies to

$$u = -\Psi[g(x)^{\mathrm{T}} \nabla Q(x)]. \tag{2.18}$$

For adaptive systems, SG algorithms in differential form are used where

$$\dot{u} = -\Gamma \nabla_u \dot{Q}(x, u). \tag{2.19}$$

The SG method is based on a Lyapunov function V decreasing along trajectories of the closed-loop system. The Lyapunov function is constructed from the goal function $V(x) = Q(x)$ for finite form algorithms and $V(x, u) = Q(x) + 0.5(u - u_*)^{\mathrm{T}} \Gamma^{-1} (u - u_*)$ for differential form algorithms, where u_* is the desired (ideal) value of the control variables (for differential form algorithms).

Tian (1999) used the method of macrovariables (earlier proposed by A. Kolesnikov, see Krstic *et al.* 1995) for stabilization of an invariant goal manifold $h(x) = 0$ with small control. The method provides stabilization of equilibrium, if the dynamics on the goal manifold (i.e. zero dynamics) are asymptotically stable. Khovanov *et al.* (2000) proposed an SG-like energy-optimal control algorithm for a periodically driven oscillator, moving it from a chaotic attractor to a coexisting stable limit cycle.

For stabilization of a goal point or manifold, other methods of modern nonlinear control theory have been used, e.g. feedback linearization, centre manifold theory, backstepping iterative design, passivity-based design, variable structure systems (VSS) design, absolute stability theory, H_∞ control, combination of Lyapunov and feedback linearization methods (see references in Fradkov & Evans (2002) and Andrievskii & Fradkov (2003)). Note that VSS algorithms for a switching surface $h(x) = 0$ coincide with the SG algorithm for a goal function $Q(x) = |h(x)|$.

In Ahmad & Harb (2003) and Ahmad *et al.* (2004), the problem of chaos control of three types of fractional order systems using simple state feedback gains is addressed. Electronic chaotic oscillators, mechanical 'jerk' systems and the Chen system are investigated by assuming generalized fractional orders. The static gains to place the eigenvalues of the system Jacobian matrices in a stable region whose boundaries are determined by the orders of the fractional derivatives are found. The effectiveness of the proposed controller in eliminating the chaotic behaviour from the state trajectories is numerically demonstrated. Some numerical investigation results are also presented and discussed in Ahmad & Sprott (2003).

In Lenci & Rega (2003*a*), control method of the homoclinic bifurcation is developed and applied to the nonlinear dynamics of the Helmholtz oscillator. The method consists of choosing the shape of external and/or parametric periodic excitations, which permits us to avoid, in an optimal manner, the transverse intersection of the stable and unstable manifolds of the hilltop saddle. The mathematical problem of optimization is investigated. This problem consists of determining the theoretical optimal excitation that maximizes the distance between stable and unstable manifolds for fixed excitation amplitude or equivalently, the critical amplitude for homoclinic bifurcation. The effectiveness of the proposed control is numerically studied with respect to the basin erosion and escape phenomena—the most important and dangerous practical aspects of the Helmholtz oscillator.

A fruitful direction is the use of frequency-domain methods for nonlinear control (see Basso *et al.* 1999 and references therein). In particular, approximate methods of harmonic balance for evaluation and prediction of chaotic modes are used together with rigorous absolute stability theory. An interesting method within this framework employs a selective ('washout') filter, which damps all signals with frequencies beyond some narrow range (see also Meucci *et al.* 1997). If such a filter is included in the feedback loop of a chaotic system and the base frequency of the filter coincides with the frequency of one of the existing unstable periodic solutions, then it is plausible that the system will be in a periodic motion rather than chaotic. This approach was applied to control of lasers.

In summary, the majority of nonlinear control approaches can be grouped into two large classes: Lyapunov approaches (SG, passivity-based methods) and compensation approaches (feedback linearization, geometric methods, etc.). The relationship between these classes can be illustrated as follows. Let the control goal be stabilization (to zero) of some output variable $y = h(x)$ of the affine system $\dot{x} = f(x) + g(x)u$ zero level. Lyapunov (or SG) methods introduce a goal function $Q(x) = |h(x)|^2$ and gradually decrease its derivative $\dot{Q}$ according to the condition $h^{\mathrm{T}}(\partial h/\partial x)(f + gu) < 0$, e.g. moving along the SG (antigradient of $\dot{Q}$) using

$$u = -\gamma g^{\mathrm{T}}(\nabla h)h.$$

With respect to the 'small control' requirement, it is necessary to choose the gain $\gamma > 0$ sufficiently small.

On the other hand, the compensation approaches introduce an auxiliary macrovariable $\alpha(x) = \dot{y} + \varrho y$ with some $\varrho > 0$ and immediately force it to zero with the control:

$$u = -\frac{f^{\mathrm{T}}(\nabla h) + \varrho h}{g^{\mathrm{T}}(\nabla h)}.$$

Note that $\alpha = 0$ if and only if $\dot{Q} = -2\varrho Q$, i.e. compensation is equivalent to specify a certain rate decrease of $Q(x)$. As a result, any desired 'instantaneous' transient rate can be achieved at the cost of loss of flexibility and the 'small control' property.

It is important to observe that generically, control based on change of some system parameter (as in the OGY scheme) is equivalent to the control by change of an additive force. Chen & Liu (2002*a*,*b*) have shown for a number of chaotic systems that this equivalence between linear coordinate feedback and parametric feedback can be established by an appropriate nonlinear coordinate transformation.

Analysis of published papers shows that those using the well-developed machinery of modern linear and nonlinear control theory often do not take full account of the special aspects of chaotic motions. This usually means that the 'small control' requirement is violated. On the other hand, the power of existing control theory is not fully utilized in many papers with respect to the 'small control' requirement. Frequently, only low-dimensional example models are considered and unrealistic assumptions are imposed (e.g. it is assumed in some papers that the number of controls is equal to the state dimension). Proper use of modern control theory to handle realistic problems in the control of chaos is yet to be undertaken.

(*e*) *Control of chaos in distributed systems*

Among infinite-dimensional (distributed) systems, the main classes are spatially extended (spatio-temporal) systems and retarded or delayed systems. Methods for oscillation and chaos control in infinite-dimensional systems are mainly based upon ideas developed for finite-dimensional (lumped) systems. Moreover, very often finite-dimensional models are used for control system analysis and design.

(i) *Spatio-temporal systems*

Finite-dimensional models of spatially extended controlled systems are obtained by spatial discretization of distributed models described by a partial differential equation. Such simplified models consist of ordinary differential equations describing separate space elements called cells, particles, compartments, etc. In both cases, elements (cells) interact by means of links reflecting spatial structure of the overall system (called array, lattice, etc.).

Numerous publications are devoted to study controlled *reaction–diffusion* equations

$$\frac{\partial x}{\partial t} = \varepsilon \Delta x + F(x, u), \tag{2.20}$$

where $x = x(r, t)$ is a function of space variables $r \in \boldsymbol{R} \subset \mathbb{R}^n$ and time t (possibly, vector-valued), determining the state of a physical system, $\Delta = \sum_{i=1}^{n} (\partial^2/\partial r_i)$ is the Laplace operator, specifying diffusion type of space element interactions. Boundary conditions can be specified as periodic (e.g. $x(a, t) = x(b, t)$ for $\boldsymbol{R} = [a,b] \subset \mathbb{R}^1$) or 'no flow across the boundary' conditions

$$\left(\frac{\partial x}{\partial r}\right)\Big|_{r=a} = \left(\frac{\partial x}{\partial r}\right)\Big|_{r=b} = 0.$$

The standard approach to study equation (2.20) is discretized over the space $\boldsymbol{R}$ by replacing the continuum $\boldsymbol{R}$ by a finite number of points (nodes) r_i, $i = 1, 2, \ldots, N$. The dynamics of each state x_i depends on both its internal (local) dynamics $F(x_i, u_i)$ and interactions with neighbour nodes. For example, if the space variable is one-dimensional ($r \in [a,b]$) and interactions are of the diffusive type, then the space-discretized model has the form

$$\dot{x}_i = \varepsilon(x_{i-1} - 2x_i + x_{i+1}) + F(x_i, u_i), \quad i = 1, 2, \ldots, N-1. \tag{2.21}$$

Additionally, we need to specify boundary conditions, e.g. periodic ($x_0 = x_N(t)$), or 'no-flow' ($x_0(t) = x_1(t), x_{N-1}(t) = x_N(t)$). Some authors use models that are also discretized in time and called *coupled map* or *cellular automata* models

$$\begin{gathered} x_i(n+1) = x_i(n) + \varepsilon[x_{i-1}(n) - 2x_i(n) + x_{i+1}(n)] + hF(x_i(n), u_i(n)), \\ i = 1, \ldots, N-1, \quad n = 0, 1, 2, \ldots. \end{gathered} \tag{2.22}$$

In the models (2.20) and (2.22), one can see the control influencing the dynamics of each cell corresponding to space-distributed (field) control. Another class of problems (boundary control problems) arises when the right-hand sides in (2.21) and (2.22) do not depend on the control, i.e. $F(x, u) \equiv F(x)$, while control enters only the equations of the boundary cells, e.g.

$$\dot{x}_0 = \varepsilon(x_1 - x_0) + F_0(x, u) \tag{2.23}$$

(for periodic boundary conditions). One may generalize the situation further to consider space–non-homogeneous systems. For the one-dimensional case, they are described by the following model:

$$\left.\begin{aligned} \dot{x}_i &= F_i(x_i, x_{i-1}, x_{i+1}, u), \quad i = 1, 2, \dots, N-1, \\ \dot{x}_0 &= F_0(x_0, x_1, u), \\ \dot{x}_N &= F_N(x_N, x_{N-1}, u). \end{aligned}\right\} \tag{2.24}$$

Control goals can be straightforward extensions of the goals formulated for lumped systems (see §2). In addition, specific goals can be posed that formalize specific types of inter-relations between neighbour cells.

Among specific spatio-temporal control goals the following should be mentioned.

(i) Stabilization of the given uniform (homogenous) or space-periodic field (standing wave).
(ii) Stabilization of the given time-periodic motion (travelling wave).
(iii) Creation or suppression of a spiral wave (for space dimension not less than two).
(iv) Creation or suppression of the given non-homogenous field (contrast or dissipative structure, clusters, patterns).
(v) Control of self-organization or disorganization of systems.

The methods of the first works on spatio-temporal control of chaos are similar to the finite-dimensional case: OGY/OPF, delayed feedback, etc. (see the survey by Hu *et al.* 1995). In subsequent papers, other approaches were introduced and investigated (mainly numerically).

Parmananda *et al.* (1997) considered one-dimensional array of $N=100$ cells, described by the logistic map $(F(x,u)=1-\alpha x^2+u)$, where the value of the parameter α ensured chaotic behaviour of each cell for $u\equiv 0$. It was shown by numerical experiments that local feedback,

$$u_i(n) = \gamma\left[x_i(n) - \frac{1}{N+1}\sum_{j=0}^{N} x_j(n-1)\right], \quad i = 1, 2, \dots, N-1, \tag{2.25}$$

provides stability of the spatially uniform distribution $x_i \equiv x_*$, $i=0,1,2,\dots,N$ for sufficiently large gain $\gamma > \gamma_0$. For $\gamma < \gamma_0$, a non-uniform distribution is stabilized consisting of several clusters of uniformity, each cell being periodically oscillating. Similar behaviour has been observed with local feedback in error

$$u_i(n) = \gamma[x_i(n) - x_*], \tag{2.26}$$

as well as with so called *global* feedback depending on observable average values of variables:

$$u_i(n) = -\frac{\gamma}{N+1}\sum_{j=0}^{N}[x_j(n) - x_j(n-1)], \tag{2.27}$$

or

$$u_i(n) = -\gamma\left[\frac{1}{N+1}\sum_{j=0}^{N} x_j(n) - x_*\right]. \tag{2.28}$$

The above results were justified theoretically (analytically) in Gade (1998). Kocarev *et al.* (1997*a*) studied the case of pinning control for a one-dimensional lattice of Lorenz systems, when control interacts only with every pth cell. The attractivity of the spatially uniform (coherent) yet chaotic in time motion under discrete-time control (2.26) with $\gamma=1$ applied to the first equation of the Lorenz system was established. Similar results for a two-dimensional lattice of Lorenz systems were obtained by Sinha & Gupte (1998) using integral feedback (called 'adaptive' by the authors). Analogous results were established for complex Ginzburg–Landau equation (CGLE)

$$\dot{A} = A + (1 + i\mu_1)\frac{\partial^2 A}{\partial r^2} - (1 + i\mu_2)|A|^2 A, \tag{2.29}$$

see Montagne & Collet (1997) and for Swift–Hohenberg equation describing dynamics of some types of semiconductor lasers (Bleich *et al.* 1997). Boccaletti *et al.* (1999) examined CGLE with pinning control, applied only at a finite number of points. The CGLE corresponds to a number of phenomena in laser physics, hydrodynamics, chemical turbulence, etc. It can exhibit different forms of complex behaviour, including Andronov–Hopf bifurcation, chaotic turbulent modes, contrast structures, etc. Boccaletti *et al.* (1999) numerically determined the maximum distance between nodes of control ensuring achievement of the control goal. A similar result for boundary control was obtained by Xiao *et al.* (2000).

Schuster & Stemmler (1997) have shown the possibility of stabilization of the Kuramoto–Sivashinsky equation,

$$\frac{\partial \varphi}{\partial t} + \varphi\frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial r^2} + \frac{\partial^4 \varphi}{\partial r^4} = u, \tag{2.30}$$

by periodic delayed velocity feedback,

$$u = \varepsilon^t \frac{\partial \varphi}{\partial t}(t-\tau), \tag{2.31}$$

where τ is time delay.

Pinning controls (local injections) were applied in Hu *et al.* (2000) to the stabilization of the trivial solution ($x_i(t) \equiv 0$) of coupled oscillator systems with diffusion–gradient coupling

$$\dot{x}_i = f(x_i) + \frac{\varepsilon}{2}(x_{i-1} - 2x_i + x_{i+1}) + \frac{\rho}{2}(x_{i-1} - x_{i+1}) + u_i, \tag{2.32}$$

as well as to the CGLE evolving initially in a chaotic mode. Linear high-gain feedback in each lth oscillator was employed. Stability analysis was performed based on linearized models near the goal solution.

Minimal density of the local control nodes and the optimal allocation were determined by Grigoriev *et al.* (1997) for a one-dimensional array of coupled logistic systems: $f(x)=ax(1-x)$ in (2.32) using linear feedback. A method of stabilization of the space-homogenous solution of the reaction–diffusion equation was proposed by Magnitskii & Sidorov (1999) for the case of the complex Kuramoto–Suzuki equation. A method of chaos and spiral wave suppression by a weak distributed perturbation for the Maxwell–Bloch equation with diffraction coupling was proposed by Wang & Xie (2000). Controlled synchronization in spatio-temporal system was studied in Hu *et al.* (1997), Kocarev *et al.* (1997*b*), Boccaletti *et al.* (1999, 2002), Fanceschini *et al.* (1999) and Blekhman (2000).

(*f*) *Other problems*

We give a brief account of other directions of research related to control of chaos.

(i) *Controllability*

Although controllability of nonlinear systems is well studied, only a few results are available on reachability of typical control goals by small control (see Chen 1997; Alleyne 1998; Van de Vorst *et al.* 1998; Bollt 2000; Fradkov *et al.* 2000). A very general idea that the more a system is 'unstable' (chaotic, turbulent) the 'simpler' or the 'cheaper', it is to achieve exact or approximate controllability was illustrated by Lions (1997). Quantitative estimates were obtained recently by Khryashchev (2004) who had shown that the time of transportation between two points from a chaotic attractor depends logarithmically on the inverse control intensity (power) and, therefore, control with arbitrarily small energy is possible.

(ii) *Other control goals*

Among the other control goals achieving the desired period (Fouladi & Valdivia 1997), desired process dimension (Ravindra & Hagedorn 1998), desired invariant measure (Gora & Boyarsky 1998; Antoniou & Bosco 2000; Bollt 2000), desired Kolmogorov entropy (Park *et al.* 1999), and targeting (Paskota & Lee 1997) should be mentioned. A method for the so-called *tracking chaos* problem (following a time-varying unstable orbit) proposed by Schwartz & Triandaf (1992) was justified by the continuation method for solving equations (Schwartz *et al.* 1997). Recent results are summarized in Schwartz & Triandaf (2000).

(iii) *Identification*

A number of papers are devoted to identification of chaotic systems. In most of these, conventional identification schemes are used. It has been demonstrated that the presence of chaos facilitates and improves parameter convergence (Epureanu & Dowell 1997; Petrick & Wigdorowitz 1997; Tian & Gao 1998; Poznyak *et al.* 1999; Huijberts *et al.* 2000; Maybhate & Amritkar 2000).

(iv) *Chaos in control systems*

Control of chaos should not be mixed up with *chaos in control systems*. Papers in the latter field appear since the late 1970s and study conditions for chaotic behaviour in conventional feedback control systems (Mackey & Glass 1977; Baillieul *et al.* 1980; Mareels & Bitmead 1986). Some recent results of such kind were reported for second-order systems (Alvarez *et al.* 1997), for high-order systems with hysteresis (Postnikov 1998) and for mechanical control systems (Enikov & Stepan 1998; Gray *et al.* 1998; Goodwine & Stepan 2000), to mention a few. A fruitful observation was made that the presence of chaos may facilitate control (Vincent & Yu 1991; Vincent 1997, see also Khryashchev 2004).

3. Examples of controlling chaos in mechanics and mechanical engineering

Chaos occurs widely in applied mechanical systems as mentioned in Kapitaniak *et al.* (2000). A few recent examples are mentioned below.

(*a*) *Control of pendulums, beams, plates*

A number of studies have been devoted to the control of chaos in systems of one or more pendulums. Owing to the interesting and the readily observable behaviour, pendulum systems have been used for numerical and experimental demonstration of most existing methods of chaos control (see Lenci 1998; Kaart *et al.* 1999; Thomas & Ambika 1999; Lenci & Rega 2000). Chaos suppression and creation has been studied in standard mechanical structures like beams (Bishop & Xu 1997; Heertjes & Van de Molengraft 2001), plates (Chen & Cheng 1999), impact systems (Lenci & Rega 2000; Vincent & Mees 2000), externally forced array of oscillators with nearest-neighbour visco-elastic coupling (Barratt 1997). Vincent & Mees (2000) demonstrated that driving a bouncing ball system into a chaotic mode might speed up its controlled transition to a prescribed periodic orbit.

In Pereira-Pinto *et al.* (2004), the OGY method is applied to chaos control of a simulated nonlinear pendulum based on an experimental apparatus. The pendulum consists of an aluminium disc with a lumped mass that is connected to a rotary motion sensor. A magnetic device provides an adjustable dissipation of energy. A string–spring device provides torsional stiffness to the pendulum and an electric motor excites the pendulum via the string–spring device changing the string length. In the first stage of the control process, the close-return method is employed to identify UPOs embedded in the chaotic attractor. After that, the proposed semi-continuous control method is applied to stabilize desirable orbits. Least-square fit methods are employed to estimate Jacobian matrices and sensitivity vectors. These techniques are employed to stabilize some of the identified UPOs, confirming the possibility of using such approaches to control chaotic behaviour in mechanical systems using state space reconstruction. Analysis related to the effect of noise in controlling chaos is of concern. The stabilization of orbits related to noisy time-series is more complex and an increased number of control stations tends to increase the robustness of the control procedure. Results show situations where these techniques may be used to control chaos in mechanical systems.

(*b*) *Control of friction*

It is known that a low-velocity regime of mechanical systems may be characterized by chaotic stick–slip motion caused by the interplay between static and kinetic friction forces. From a practical point of view, one may wish to control the system in such a way that the overall friction is reduced or enhanced. The chaotic mode is eliminated and smooth sliding is achieved. Such a control is of high technological importance for micromechanical devices, e.g. in computer disk drives, where the early stages of motion and the stopping process, which exhibit chaotic stick–slip, pose a real problem. Controlling frictional forces has been traditionally approached by chemical means, namely, using lubricating liquids. A different approach, proposed by Elmer (1998) and Rozman *et al.* (1998), is based on controlling the system mechanically. The goal is twofold: (i) to achieve smooth sliding at low driving velocities, which otherwise correspond to the stick–slip regime and (ii) to decrease the frictional forces. Rozman *et al.* (1998) have tested their method for a model where in addition to the macroscopic degree of freedom, i.e. the position of the sliding block, an internal degree of freedom appears which describes the state of a lubricant. Assuming that the four-dimensional state of the system can be measured, they used the normal force as the controlling parameter and linearization of Poincaré map with pole placement as the control method. The disadvantage of the method of Rozman *et al.* (1998) is the necessity of reconstructing the dynamics. This may be more or less difficult depending on the details of the dynamics of the internal degrees of freedom at the friction interface. Elmer (1998) proposed two different controlling methods to stabilize unstable continuous-sliding states of a dry-friction oscillator. Both methods rely on macroscopic equations of motion, and use the delayed-feedback mechanism and elastic deformation as the feedback (output) variable. The control parameter (input variable) is either the sliding velocity or the normal force. It is shown that both methods are able to turn stick–slip motion into continuous sliding. Velocity control is less robust than load control.

Three methods to avoid stick–slip motion in mechanical systems with friction are proposed in Popp & Rudolph (2004). They are: (i) appropriate increase of internal damping that compensates the negative damping induced by a friction characteristic, which decreases with increasing sliding speed, (ii) external excitation that breaks up the limit cycle, and (iii) passive vibration control by fluctuating normal forces. These methods can also be used for control of chaotic stick–slip motion.

Besides achieving the goals mentioned in the beginning, controlling friction provides a better understanding of friction by measuring velocity-weakening friction forces.

(*c*) *Control of chaos in the systems with impacts*

The paper by de Souza & Caldas (2004) is devoted to control of chaotic orbits in mechanical systems with impacts. By applying a small and precise perturbation on an available control parameter the desired UPOs, embedded in the chaotic invariant sets of mechanical systems with impacts, are stabilized. To obtain such perturbation numerically, the authors introduce a transcendental map (impact map) for the dynamical variables computed just after the impacts. Application of the suggested method to an impact oscillator and to an impact-pair system is demonstrated.

A model for rattling in single-stage gearbox systems with some backlash consisting of two wheels with a sinusoidal driving is considered in de Souza *et al.* (2004). A rich dynamical behaviour in such system for various control parameters is observed. It is shown that an approach based only on increase of the friction may not lead to the desired result of suppressing rattling, and a more profound analysis is necessary.

(*d*) *Control of spacecraft*

In a number of applications, irregular vibrations of mechanical units arise from rotation of unbalanced rotors, vibrations in appendages, etc. The control goal is then the suppression of these undesirable vibrations. Problems of this kind are often solved by the methods of linear control (see §2*d*). In some cases, nonlinear approaches are reported to be successful. For example, Meehan & Asokanthan (2002*a*,*b*) designed the angular velocity stabilization algorithm for spinning spacecraft using system energy as a Lyapunov (goal) function. A modification of design based on the SG method enabling reduction of the required control torque is presented in Fradkov & Andrievsky (2003) and Fradkov *et al.* (2004).

The plausibility of chaotic motion in gyrostats and methods of its control are studied in Lanchares *et al.* (1998), Iñarrea & Lanchares (2000) and Ge & Lin (2003). The gyrostat is a body with three rotational degrees of freedom and one or more internal wheels. Studies of gyrostat dynamics and control are of practical importance, because these models describe satellites performing angular spin motion and dual-spin spacecraft such as satellites with wheel motors or spinning satellites with stabilized platform.

Analysis, control and synchronization of the chaotic processes in a gyrostat subjected to external perturbations were carried out by Ge & Lin (2003). Consideration is given to the dynamics of a gyrostat having three wheels with mutually orthogonal axes of rotation. The wheels are driven by electric motors. It is assumed that a small sinusoidal ripple is superposed on the rotation moment of one of the rotors. The current in the motor of one of the wheels can be varied, thus creating a control action. The state vector of the system at hand consists of the satellite angular spin rates in the axes of the vehicle state coordinates and current in the control motor. The authors of Ge & Lin (2003) believe that studies of chaotic motions in the gyrostat are of practical value, in particular, because it can be used as a missile model. Anticontrol of the missile angular chaotic motion at attack hinders intercept, because in this case, the trajectory of motion is hardly predictable. Like Ge & Shiue (2002), this paper presents the results of using various methods for analysis of uncontrollable motion of the plant. Analysis demonstrated that angular motion of the gyrostat could become chaotic with reduction of the frequency of external perturbation. The work proposed and considered algorithms of adaptive and time-delayed feedback control to change the nature of system oscillations, i.e. make motion periodic and not chaotic. Additionally, consideration was given to the possibility of anticontrol of chaos by an arbitrarily small control. With that end in view, it is proposed to use a small constant or periodic control action. The paper then studies the synchronization of chaotic processes in two aforementioned systems. Consideration was given to synchronization with linear, sinusoidal, exponential and adaptive feedbacks. We note that the paper presents no particular treatment of the problem of synchronization for the systems under study.

The possibility of chaotic angular oscillations of the satellite and their suppression were also studied in Ge *et al.* (1998), Tsui & Jones (2000) and Chen & Liu (2000, 2002*a*,*b*). For example in Chen & Liu (2000, 2002*a*,*b*) consideration is given to the motion of a satellite having constant magnetic eigenfield under simultaneous action of the terrestrial gravity and magnetic fields. For the satellite libration angle $\varphi(t)$ in the orbit plane, the following mathematical model was established under some assumptions:

$$C\ddot{\varphi} + c\dot{\varphi} + 3\omega_c^2(B - A)\sin\varphi\cos\varphi + \mu_m iIr^{-3}(2\sin\varphi\sin\omega_c t + \cos\varphi\cos\omega_c t) = M_c(t), \tag{3.1}$$

where c is the coefficient of proper satellite damping, ω_c is the value of satellite angular speed on the orbit, A and B are its main moments of inertia ($B > A$), μ_m is the magnetic constant, I is the value of the satellite magnetic moment, r and i are the orbit radius and inclination and $M_c(t)$ is the value of the control moment. On the basis of the Melnikov method and numerical analysis, the paper proved that within some parameter domain the satellite angular motion is chaotic in the absence of control ($M_c \equiv 0$). To obtain the desired process $\varphi(t)$, the method of feedback linearization was used to construct the law of generation of the control moment M_c by output and the derivative. The feedback system was shown not only to suppress chaotic oscillations, but also to provide the desired form of $\varphi(t)$ (numerical examples of stabilization of the angle φ and harmonic oscillations with the given frequency were given). We note that solution seems rather trivial from the point of view of the system theory: the control moment is chosen so as to compensate the nonlinear—in the control error—term in the right-hand side of (3.1) and introduce proportional and differential terms.

In Kuang *et al.* (2004), the multi-body dynamics of a satellite, modelled as a central body with two hinge-connected deployable solar panel arrays is studied. The possible chaotic behaviour of the deployed satellite under the action of conservative forces is analytically investigated by the application of the Melnikov integral. The numerical simulations show that the attitude chaotic motions of the deployed satellite disturbed by the gravity-gradient torques are similar to random motions or bounded non-periodic motions, and that the chaotic dynamics are very sensitive to the initial conditions from the time-evolution history of the variables of the attitude motions.

(*e*) *Control of vibroformers*

It is known that the vibration-compaction of heated mixed paste in the manufacturing of anodes for reduction cells is much more effective than monotone compression. The vibration-compaction helps a good mixing of the material, produces well-compacted anode blocks and, above all, tends to eliminate air bubbles decreasing strength of the anodes. In Paskota (1998), a method for vibroformer control is proposed. The vibroformer is considered as an impact oscillator and is described by a version of the bouncing ball model. The frequency of the vibroformer exciter rotation is used as a control variable. To speed up changing the regime of the unit, targeting methods of Paskota & Lee (1997) are applied.

(*f*) *Control of microcantilevers*

Ashhab *et al.* (1999) studied the dynamics and control of a microcantilever system that forms the basis for the operation of atomic force microscopes. The cantilever is vibrated by a sinusoidal input, and its deflection is detected optically. The forced dynamics analysed using the Melnikov method, which reveals the region in the space of physical parameters where chaotic motion is possible. Then the Melnikov function in terms of the parameters of the proportional and derivative controller was computed and parameters eliminating chaos were designed.

(g) Stabilization of ship oscillations

Roll motion of a flooded ship was considered in Mitsubori & Aihara (2002). Large amount of water inside the hull gives rise to complicated coupled oscillations of the ship and liquid in it, that are similar to the oscillations of coupled oscillators. The picture becomes even more complicated because of the quasi-periodic external perturbations. The model of fourth-order was used to describe dynamics of roll in waves. In their preceding works, the authors relied on the numerical and laboratory studies to demonstrate the feasibility of complex chaotic oscillations of great amplitude. In Mitsubori & Aihara (2002), the problem of reducing the system to regular small-amplitude oscillations is posed and solved by the Pyragas method of time-delayed feedback. To this end, terms proportional to the differences between current and delayed values of the angular velocities of ship roll and inclination of water in it are introduced in the right-hand sides of system equations. It was shown that the chaotic process can be reduced to a small-amplitude periodic one by an appropriate choice of the delay time and the feedback coefficients.

(h) Suppression of chaotic oscillations of tachometer

Behaviour of a mechanical tachometer subjected to additional vibrations along the rotation axis was studied in Ge & Shiue (2002). Vibrations of the base obey the harmonic oscillations, $A \sin \omega t$. Characteristics of the mathematical model of the system were studied by various analytical and numerical methods. Bifurcation diagrams demonstrating that the oscillations from periodic become chaotic with growth in the vibration amplitude were constructed. To improve system quality and eliminate chaotic phenomena, various methods of control were considered such as introduction of an additional constant or periodic moment, time-delayed feedback control, adaptive control, bang-bang control, optimal control and introduction of additional pulse action. The paper presented numerous graphs depicting the results of modelling the original and controlled systems and demonstrating applicability of the proposed methods.

(i) Chaotic dynamics of rate gyro in the linear feedback control loop

An analysis of stability and chaotic dynamics for a single-axis rate gyro subjected to linear feedback control loops is given in Chen (2004). This rate gyro is supposed to be mounted on a space vehicle, which undergoes an uncertain angular velocity around its spin axis. The simultaneous acceleration occurs with respect to the output axis. The stability of the nonlinear non-autonomous system is investigated by Lyapunov stability and instability theorems. The stable regions of the autonomous system are obtained in parametric diagrams. For the non-autonomous case in which angular velocity oscillates near boundary of stability, periodic, quasi-periodic and chaotic motions can appear.

(*j*) *Chaos suppression in Duffing oscillator*

The optimal numerical control of nonlinear dynamics and chaos is investigated in Lenci & Rega (2003*b*) by means of a technique based on removal of the relevant homo/heteroclinic bifurcations, to be obtained by modifying the shape of the excitation. To highlight how the procedure works, the analysis is accomplished by referring to the Duffing equation, although the method is general and holds, at least in principle, for whatever nonlinear system. It is shown that it is possible to eliminate this bifurcation simply by adding a single superharmonic correction to the basic harmonic excitation. The optimal solutions are determined in the two cases of symmetric (odd) and asymmetric (even) excitations, and it is shown how they entail practical, though variable, effectiveness of control in terms of confinement and regularization of system dynamics.

(*k*) *Control of chaos in robot-manipulator arm*

The problems of suppressing or inducing chaotic dynamics in a model of robot arms and mechanical manipulators are studied by Nakamura *et al.* (1997), Cao *et al.* (2004) and others. For example, in Cao *et al.* (2004) it is assumed that the unperturbed systems possess multiple non-transverse homoclinic and/or heteroclinic orbits depending on the model parameters. Based on the Melnikov method and numerical computations for the Melnikov integrals, fixed points, and turning points, conditions for chaos suppression and generation are obtained.

(*l*) *Chaotic behaviour in optimal control in earthquake civil engineering*

In Liolios & Boglou (2003), a nonlinear optimal control problem arising in earthquake civil engineering is discussed. This problem concerns the elastoplastic softening–fracturing unilateral contact between neighbouring buildings during earthquakes when Coulomb friction is taken into account under second-order instabilizing effects. Hence, the earthquake response of the adjacent structures can appear as instabilities and chaotic behaviour. The problem formulation presented here leads to a set of equations and inequalities, which is equivalent to a dynamic hemi-variational inequality in the way introduced by Panagiotopoulos. The numerical procedure is based on an incremental problem formulation and on a double discretization, in space by the finite element method and in time by the Wilson-v method. The generally non-convex constitutive contact laws are piece-wise linearized, and in each time-step a non-convex linear complementarity problem is solved with a reduced number of unknowns.

(*m*) *Chaos suppression in the milling process*

Ball milling is considered in Ajaal *et al.* (2002). In a traditional ball mill, the energy exchange between the tumbling balls themselves and the powder particles tends to be chaotic. Chaotic ball motion and insufficient and uncontrolled grinding of the powders characterize this process. In order to obtain a homogeneous and reproducible product, a magnetic field is introduced to the ball mill. It is shown that the control of the ball motion during the milling of limestone leads to a reduction in grinding energy of 40% and a more homogeneous product.

(n) Control of whirling motion under mechanical resonance

Inoue *et al.* (2003) studied the influence of a whirling motion on the electric characteristics of a rotating electrical machine and proposed a method to control the whirling motion under mechanical resonance. A torque-based control method is proposed to suppress the whirling motion by controlling a torque. The control input is determined based on a stability condition for a time-varying system that is represented by a second-order vector differential equation with time-varying coefficient matrices. The effectiveness of the control method is examined by both simulation and experiment.

(o) Control of mechanical system with clearance

Mata-Jimenez & Brogliato (2003) deal with analysis and control of a rigid-body mechanical system with clearance. All the nonlinear non-smooth characteristics of this system are treated as a rigid-body mechanical system with unilateral constraints and impacts (dynamic backlash). The model is therefore a hybrid dynamical system, mixing discrete events and continuous states. The regulation and tracking capabilities of the proportional-derivative (PD) scheme are investigated. Existence of a limit cycle for non-collocated PD control is proved. A hybrid control is proposed, which may be used to track some desired trajectories in conjunction with a PD input.

4. Conclusions

State-of-the-art of the field related to control of chaotic systems is briefly surveyed and some examples for control of chaos in mechanical systems are presented. The authors do not insist that chaos should be used in realistic applications. The point of the paper is to show a variety of methods able either to increase chaos or to eliminate it. These methods often achieve the goal with smaller control power and were compared with traditional control-engineering approaches, as is demonstrated in the problem of suppression of chaotic behaviour of a spinning satellite.

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