FREQUENCY-DOMAIN CONDITIONS FOR GLOBAL SYNCHRONIZATION OF NONLINEAR SYSTEMS DRIVEN BY A MULTIHARMONIC EXTERNAL SIGNAL

A.L. Fradkov*, G.A. Leonov*†, H. Nijmeijer+‡

**Institute for Problems of Mechanical Engineering,*
Russian Academy of Sciences
61, Bolshoy, V.O., 199178, St. Petersburg, Russia
FAX:+7(812)321-4771, Tel:+7(812)321-4766, E-mail: alf@ccs.ipme.ru
†*Faculty of Mathematics and Mechanics, St.Petersburg University*
Bibliotechnaya pl.2, Staryi Petergof, 198904, St.Petersburg, Russia
FAX: +7(812)428-4677, Tel:+7(812)428-6944, E-mail: leonov@math.spbu.ru
‡*Faculty of Mathematical Sciences, University of Twente,*
P.O.Box 217, 7500 AE Enschede, The Netherlands
FAX:+31(53)434-0733, Tel:+31(53)489-3442, E-mail: h.nijmeijer@math.utwente.nl
+ *Faculty of Mechanical Engineering, Technical University of Eindhoven,*
P.O.Box 513, 5600 MB Eindhoven, The Netherlands

Keywords: global synchronization, nonlinear systems.

Abstract

The problem of entrainment (capture, synchronization) of the trajectories of nonlinear oscillatory system driven by an external periodic signal is reconsidered. The frequency-domain conditions of capture (global synchronization)are established extending the previous results (dealing with the harmonic excitation) to the case when external excitation is a multiharmonic signal, i.e. sum of a finite number of harmonics with incommensurate frequencies. Applications of the results in communications field are discussed[1].

1 Introduction

The problem of entrainment or capture of the trajectories of nonlinear oscillatory system driven by an external harmonic signal is a classical problem of oscillation theory [1, 2, 3]. The capture phenomenon, also called external synchronization, has various applications in mechanics, physics and engineering [4, 5, 6]. Perhaps, the first criterion for global external synchronization by a harmonic signal was proposed in [7] and then extended by different authors in [8, 9, 10, 11, 12]. Recently, the problem of synchronization by chaotic signals received significant attention [13, 14, 15, 16] and is motivated by potential applications in secure communications [17, 18]. However no rigorous criteria for global external synchronization were reported so far.

In this paper frequency-domain conditions are established extending the results of [7] to the case when the external excitation is a multiharmonic signal i.e. sum of a finite number of harmonics with incommensurate frequencies of large amplitude can capture all the solutions of the nonlinear system with an exponential rate.

The paper is organized as follows. In Section 2 the preliminary definitions and statements are given. In Section 3 the main result establishing conditions of global synchronization is formulated and proved. Applications of the results in the secure communications are discussed in Section 4.

2 Preliminaries

Below some definitions and auxiliary results are given. All the dynamical systems are considered on the positive time axis $[0, \infty)$ and limit properties are studied for $t \to +\infty$.

Definition 1 [20, 21]. The scalar function $f(t)$ defined on $[0, \infty)$ is called an *oscillatory signal in the sense of*

[1]The work was supported in part by the Dutch Organization for Pure Research (NWO), the Russian Foundations for Basic Research (RFBR, project 99-01-00672), and the Russian Federal Program "Integration" (projects 2.1-326.02, 2.1-589).

Yakubovich (Y-oscillation or OS), if it is bounded and

$$\overline{\lim_{t\to\infty}} f(t) > \underline{\lim_{t\to\infty}} f(t) \qquad (2.1)$$

Definition 2. The scalar function $f(t)$ is called a *strongly oscillatory signal* at the level a, if for any $\varepsilon > 0$, $L > 0$ there exist $\delta > 0$, $t_* > 0$ such that

$$\text{meas}\,\{\tau : \tau \in [t, t+L],\, |f(\tau) - a| < \delta\} < \varepsilon, \quad \forall t > t_*, \qquad (2.2)$$

where meas$\{S\}$ stands for Lebesgue measure of the set S. The signal is called strongly oscillatory (SOS), if it is strongly oscillatory at any level a from the interior of the range of $f(t)$.

Proposition 1. A function

$$f(t) = \sum_{i=1}^{N} f_i \sin(\omega_i t + \alpha_i) + f_0 \qquad (2.3)$$

is SOS, if it is not identically zero.

Proof (sketch). To prove Proposition 1 it is sufficient to consider case $a = 0$. The proof is based on establishing uniform boundedness of the number of zeros of $f(t)$ on any time interval $[0, L], L > 0$ with respect to the initial phases $\alpha = \{\alpha_i\}$ and on the observation that $f(t)$ and $f(t + L)$ coincide modulo shift of initial phases.

Remark. It follows from the proof that the set S_ε of measure less than ε from (2.2) can be taken as a finite number of open intervals constituting ε/n_L-neighborhoods of zeros of $f(t)$ on $[0, L]$.

Consider the linear system

$$\dot{x} = Ax + Bu, \quad y = Cx, \qquad (2.4)$$

where $x \in \mathbb{R}^n$ is the state vector, u, y are scalar input and output, respectively and matrices A, B, C are of corresponding dimensions. The rational function $W(\lambda) = C(\lambda I - A)^{-1}B$, is called the *transfer function of (2.4)*. We denote $j = \sqrt{-1}$, and $A_\lambda = \lambda I - A, \lambda \in \mathbb{C}$.

Proposition 2. If A is Hurwitz matrix and the input $u(t)$ of (2.4) is of the form (2.3) then its output $y(t)$ is SOS, provided that $W(j\omega_0)f(i) \neq 0$ at least for one $i = 1, \ldots, N$.

The proof follows from Proposition 1 and the fact that the steady state mode of the system (2.4) with matrix A being Hurwitz and input of form (2.3) is of form (2.3) too.

3 Main result

Consider the single input, single output Lur'e system

$$\dot{x} = Ax + B[\varphi(y,t) + \gamma f(t)], \quad y = Cx, \qquad (3.5)$$

where $f(t)$ has the form (2.3), the matrix A is Hurwitz, and $\gamma > 0$.

The main result of the paper is formulated as follows.

Theorem 1. *Let the following assumptions be valid.*

A1. The function $\varphi(y,t)$ is nondecreasing in y in the regions $(-\infty, y^-]$, $[y^+, +\infty)$ and satisfies a Lipschitz-like condition with constant μ, i.e.

$$0 \leq \frac{\varphi(y_1,t) - \varphi(y_2,t)}{y_1 - y_2} \leq \mu \qquad (3.6)$$

for some $y^- < 0$, $y^+ > 0$, $\mu > 0$.

A2. The function $\varphi(y,t)$ satisfies a Lipschitz condition in y with constant M uniformly in $t \in [0, \infty)$:

$$\left|\frac{\varphi(y_1,t) - \varphi(y_2,t)}{y_1 - y_2}\right| \leq M \qquad (3.7)$$

A3. The function $\varphi(y,t)$ is bounded: $|\varphi(y,t)| < l$ for all $y \in \mathbb{R}, t \in \mathbb{R}^+$.

A4. The frequency-domain condition holds

$$\operatorname{Re} W(j\omega) \leq \frac{1}{\mu'} < \frac{1}{\mu} \quad \forall\omega \in \mathbb{R}. \qquad (3.8)$$

where $W(\lambda) = c(\lambda I - A)^{-1}B$.

Then there exist $\gamma > 0$, $\varrho > 0$, $C > 0$ such that

$$\|x_1(t) - x_2(t)\| \leq C\|x_1(0) - x_2(0)\|e^{-\varrho t} \qquad (3.9)$$

for any pair of solutions $x_1(t)$, $x_2(t)$ of (3.5).

Corollary. *Under the conditions of the Theorem 1 all solutions of (3.5) converge for sufficiently large $\gamma > 0$ to the steady-state solution*

$$\overline{x}_\gamma(t) = \gamma \sum_{i=1}^{N} \operatorname{Re} W(j\omega_1) f_i e^{\omega_i t + \alpha_i} \qquad (3.10)$$

of the auxiliary linear system

$$\dot{x} = Ax + B\gamma f(t) \qquad (3.11)$$

Remark 1. When a margin $\lambda > 0$ in the frequency-domain condition (3.8) is known, i.e.

$$A4'. \qquad \operatorname{Re} W(j\omega - \lambda) \leq \frac{1}{\mu'} < \frac{1}{\mu}, \forall\mu \in \mathbb{R} \qquad (3.12)$$

is valid instead of *A4*, the conclusion of the theorem reads as: there exist $\gamma_* > 0$, $C > 0$ such that (3.9) holds for any $\gamma > \gamma_*$, any $\varrho :\ 0 < \varrho < \lambda$ and any pair of solutions $x_1(t)$, $x_2(t)$ of (3.5).

Remark 2. For the special case that $f(t) = \sin\omega t$ the problem was solved in [7] where also analytical bounds for γ_*, ϱ were given.

Remark 3. If the system (3.5) is dissipative in the sense of Levinson, i.e. its trajectories fall into some bounded set $B \subset \mathbb{R}^n$, then the condition *A3* can be replaced by:

A3'. The function $\varphi(y,t)$ is uniformly bounded in some neighborhood of B.

In this case the theorem applies to the Van der Pol and Duffing system which are Levinson dissipative according to the results of [22].

Remark 4. Theorem 1 also holds for any bounded excitation signal $f(t)$, if $f(t)$ is SOS at the zero level after passing through the linear filter (3.11).

To prove the theorem we need the following simple result.

Lemma. *Consider the two systems*

$$\dot{x} = Ax + B\varphi(y,t) + g(t), \quad y = Cx \qquad (3.13)$$

$$\dot{\tilde{x}} = A\tilde{x} + g(t) \qquad (3.14)$$

where A is Hurwitz matrix, $g(t)$ is measurable bounded function, $\varphi(\cdot)$ is Lipschitz in y and $|\varphi(y,t)| \le l$.

Then the solutions $x(t)$, $\tilde{x}(t)$ of (3.13) and (3.14), respectively, satisfy the condition

$$\overline{\lim_{t\to\infty}} |Cx(t) - C\tilde{x}(t)| \le l\nu, \qquad (3.15)$$

where $\nu = \int\limits_0^\infty Ce^{At}B\,dt$.

The proof of Lemma 1 is similar to that of Lemma 1 of [10].

Proof of Theorem 1. Consider two arbitrary solutions $x_1(t)$, $x_2(t)$ of (3.5). Their difference $z(t) = x_1(t) - x_2(t)$ satisfies the equation

$$\dot{z} = Az + B\psi(t), \qquad (3.16)$$

where $\psi(t) = \varphi(x_1(t),t) - \varphi(x_2(t),t)$. Consider the quadratic function

$$V(z) = z^T Hz, \qquad (3.17)$$

where $H = H^T$ is a symmetric $n \times n$-matrix and evaluate its derivative along trajectories of (3.16)

$$\begin{aligned} \dot{V} &= 2z^T H\dot{z} = 2z^T H(Ax + B\psi) \\ &= F(z,\psi) - 2\lambda V - \psi(\eta - \psi/\mu), \qquad (3.18) \end{aligned}$$

where $\eta = Cz$ is a scalar; $\lambda > 0$ is chosen so that the matrix $A + \lambda I$ is Hurwitz, and $F(z,\psi)$ is the quadratic form in z, ψ:

$$\begin{aligned} F(z,\psi) &= 2z^T H(Ax + B\psi) + 2\lambda z^T Hz + \psi(\eta - \psi/\mu) \\ &= 2z^T H[(A+\lambda I) + B\psi] + \psi(\eta - \psi/\mu). (3.19) \end{aligned}$$

Since $A + \lambda I$ is Hurwitz, it follows from the Kalman-Yakubovich lemma that there exists a positive definite matrix $H = H^T > 0$ such that $F(z,\psi) \le 0$ for all $z \in \mathbb{R}^n$, $\psi \in \mathbb{R}$ if and only if

$$\tilde{F}((A+\lambda I)^{-1}_{j\omega}B\psi, \psi) \le 0, \qquad (3.20)$$

where $\tilde{F}$ is the Hermitean extension of the quadratic form F. It follows from condition *A4* that (3.20) is fulfilled, since

$$\tilde{F}((A+\lambda I)^{-1}_{j\omega}B\psi, \psi) = \left[\operatorname{Re} W(j\omega - \lambda) - \frac{1}{\mu}\right]|\psi|^2.$$

Define the set $D \subset \mathbb{R}^n \times \mathbb{R}^n$ in the state space of two identical systems (3.5) as

$$\begin{aligned} D &= \{(x_1,x_2):\, y_1 \ge y^+, y_2 \ge y^+\} \\ &\bigcup \{(x_1,x_2):\, y_1 \le y^+, y_2 \le y^+\}, \end{aligned}$$

where $y_1 = Cx_1$, $y_2 = Cx_2$.

First, assume that $(x_1,x_2) \in D$. In this case *A1* yields $0 \le \psi/\eta \le \mu$ which is equivalent to the inequality $\psi(\eta - \psi/\mu) \ge 0$. Taking into account $F(z,\psi) \le 0$ we obtain from (3.16) the inequality

$$\dot{V} \le -2\lambda V. \qquad (3.21)$$

In case $(x_1,x_2) \notin D$ we employ *A2* and the inequality $\psi\eta \ge 0$ which follows from *A2*. Then (3.16) yields

$$\begin{aligned} \dot{V} &\le -2\lambda V - \psi(\eta - \psi/\mu) \le -2\lambda V + |\psi|^2/\mu \\ &\le -2\lambda V + M^2|\eta|^2/\mu. \end{aligned}$$

Since the matrix H is positive definite, it follows that $H > \delta_1 \frac{M^2}{\mu} CC^T$ or $V(z) \ge \delta_1 (M\eta)^2/\mu$ for some $\delta_1 > 0$, which gives

$$\dot{V} \le (-2\lambda + 1/\delta_1)V. \qquad (3.22)$$

Denote $V_t = V(x(t) - \overline{x}_\gamma(t))$, where $x_\gamma(t)$ is defined in (3.10). We show that for any $L > 0$ there exist $\gamma > 0$, $t_* > 0$:

$$V_{t+L} \le e^{-2\varrho L} V_t \quad \forall t \ge t_* \qquad (3.23)$$

for $0 < \varrho < \lambda$.

Indeed, choose $L > 0$ and let $\varepsilon = 2\delta_1 L(\lambda - \varrho)$. By Proposition 2, the output $C\overline{x}_\gamma(t)$ of the linear system (3.11) is SOS. Hence, there exists $\gamma > 0$ such that

$$|C\overline{x}_\gamma(t)| > \max\{y^+, y^-\} + l\nu + 1$$

for $\tau \in R_\varepsilon(t)$, where $R_\varepsilon(t) = [t, t+L]\backslash S_\varepsilon(t)$ and $S_\varepsilon(t)$ is a set of measure less than ε. (E.g. we may take $\gamma > (\max\{y^+, y^-\} + l\nu + 1)/\delta$, where $\delta > 0$ comes from the definition of SOS. Further, by Lemma 1, there exists $t_* > 0$ such that $|Cx(t) - C\overline{x}_\gamma(t)| \le l\nu + 1$ for all $t > t_*$. Therefore, for $t > t_*$ and $\tau \in R_\varepsilon(t)$ the inequality $|Cx(t)| > \max\{y^+, y^-\}$ holds, i.e. $(x(\tau), \overline{x}_\gamma(\tau)) \in D$ and (3.21) is valid. Apparently, for $\tau \in S_\varepsilon(t) = [o, L]\backslash R_\varepsilon$ the inequality (3.22) is valid.

In view of the remark after the Proposition 1, the set $S_\varepsilon(t)$ consists of a finite number of intervals (t'_k, t''_k), $k = 1, \cdots, n_L$, such that

$$t \le t'_1 < t''_1 < t'_2 < t''_2 < \cdots < t'_{n_L} < t''_{n_L} \le t + L$$

Integration of (3.22) over the intervals $(t'_k, t''_k) \subset S_\varepsilon(t)$ yields

$$V_{t''_k} \le e^{-2\lambda + 1/\delta_1)(t''_k - t'_k)} V_{t'_k}, \qquad (3.24)$$

while integration of (3.21) over the remaining intervals from $R_\varepsilon(t)$ yields

$$V_{t'_{k+1}} \le e^{-2\lambda(t'_{k+1} - t''_k)} V''_{t_k}, \qquad (3.25)$$

where $k = 1, \cdots, n_L$ and $t''_0 = t$, $t_{n_L+1} = t + L$. Iterating the inequalities (3.24), (3.25) and taking into account that meas$S_\varepsilon(t) < \varepsilon$ we arrive at the inequality $V_{t+L} \leq e^{-2\lambda L+\varepsilon/\delta_1} V_t$ which obviously coincides with (3.23) for the above chosen ε.

To end the proof of the theorem pick up some $L > 0$, e.g. $L = 1$. Then iterating (3.23) immediately yields the exponential bound (3.9) for integer $t \geq t_*$ (since γ, ε can be chosen independently of t). To establish (3.9) for non-integer t we integrate (3.22) over the interval $[[t], t]$, where $[t]$ stands for the integer part of t: $V_t \leq e^{1/\delta_1} V_{[t]}$. Therefore (3.9) for all $t > t_*$ follows by appropriately increasing the value of C.

4 Application to signal transmission

During recent years different schemes for secure communications have been studied based on the synchronization of two nonlinear systems, usually called the transmitter and the receiver. To encode the message a change of the transmitter parameters ("parameter modulation") is used, while to ensure privacy the message is hidden by some masking signal generated by the transmitter itself. However, a multiharmonic signal containing a large number of incommensurate frequencies looks quite similar to chaotic one.

Apart from considering message encoding issues we discuss only the first stage of the communication scheme design, aimed at the choice of transmitter and receiver structures allowing synchronization of transmitter and receiver under "ideal" conditions. Particularly, the goal is the reconstruction of the transmitter state vector, i.e. solving the observer problem, see [23].

In order to apply our main result assume that transmitter is described as

$$\dot{x} = Ax + B[\varphi(y) + \gamma f(t)], \quad y = Cx, \tag{4.26}$$

where $y(t)$ is the transmitted signal. The receiver is designed as an observer, [23] described by

$$\dot{\hat{x}} = A\hat{x} + B[\varphi(y) - \varphi(y - \hat{y})], \quad \hat{y} = C\hat{x}, \tag{4.27}$$

Then the error equation is

$$\dot{e} = Ae + B[\varphi(y_e) + \gamma f(t)], \quad y_e = Ce, \tag{4.28}$$

where $e = x - \hat{x}$.

It follows from Theorem 1 that $e(t) - \overline{x}_\gamma(t) \to 0$ for large $\gamma > 0$, where $\overline{x}_\gamma(t)$ is the steady-state trajectory (3.10) of (4.28).

Therefore, for reconstruction of the transmitter state the simple relations can be used

$$x = \hat{x} + e \tag{4.29}$$

or

$$x = \hat{x} + \overline{x}_\gamma \tag{4.30}$$

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