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Energy Control of Hamiltonian Systems under Disturbances*

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Abstract

The problem of the energy level stabilization for Hamiltonian systems in presence of disturbances is considered. First, it is shown that for 1-DOF systems under sufficiently small uniformly bounded force disturbances the speed-gradient control law ensures ultimate boundedness of energy error. As an auxiliary result the new sufficient conditions for ultimate boundedness of Lyapunov function along the trajectories of nonlinear nonstationary dynamical system are obtained. Second, for n-DOF systems with dissipation-like disturbances the bounds for achievable energy level are given.

1 Introduction

In the paper we consider the problem of energy level stabilization for Hamiltonian systems in presence of disturbances. This problem is of interest in the oscillations control area where the level of energy determines the mode of oscillations. For Hamiltonian systems stabilization of energy level is equivalent to stabilization of a certain invariant manifold in phase space. The special case when the desired level of energy is equal to its minimum value (stabilization of equilibrium) is well studied, see e.g. (Nijmeijer and van der Schaft, 1990; Rumyantsev, 1970). In (Fradkov, 1996; Andrievsky *et al.*, 1996) the speed-gradient control was used to solve the problem for arbitrary desired value of energy. It was shown that the stabilization of energy level is achieved if there is no equilibria in the initial energy layer (between the initial and the desired levels of energy). In (Fradkov *et al.*, 1997; Shiriaev and Fradkov, 1998; Fradkov and Pogromsky, 1998) further development of the result was given. However no results concerning behavior of the closed loop systems under disturbances were available so far.

In this paper we extend the results of (Fradkov, 1996) in two directions. Firstly, the case of presence of the bounded input (force) disturbances is considered. For a broad class of controlled one-degree-of-freedom Hamiltonian systems we state that under uniformly bounded input disturbances with sufficiently small bound $D > 0$ the speed-gradient control law ensures ultimate boundedness of error between actual value of energy and desired one. Moreover the upper bound

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of the error tends to zero as $D \to 0$. Secondly, for n-DOF systems with disturbances satisfying the dissipation-like inequality the bounds for the energy level achievable by a bounded control are obtained.

The difficulty of the input disturbances case is that the time derivative of Lyapunov function is not negative semidefinite even for large values of V. Hence methods of papers (Fradkov, 1996; Fradkov *et al.*, 1997; Shiriaev and Fradkov, 1998) essentially based on assumption $\dot{V} \leq 0$ and the standard theorems on ultimate boundedness (like Yoshizawa's theorem) are not applicable. In this paper we exploit the fact that trajectories of the system "pass through" the regions of sign indefiniteness of Lyapunov function's time derivative. Based on this circumstance under some further conditions it is possible to show that the ultimate boundedness of V holds along an arbitrary trajectory of system. We prove the corresponding lemma in section 3. Proof of the main result reduces to checking the lemma conditions for closed loop system "Hamiltonian system + speed-gradient control law + disturbances".

The paper is organized as follows. In section 2 the results on the Speed-gradient energy control of Hamiltonian systems without disturbances are recalled from (Fradkov and Pogromsky, 1998). In section 3 we state the auxiliary lemma on sufficient conditions for ultimate boundedness of Lyapunov function along the trajectories of a nonlinear time-varying dynamical system. The result for the 1-DOF systems is presented in Section 4, while the case of n-DOF systems with dissipation is considered in Section 5. In Section 6 the example of the energy control of pendulum is considered.

2 Speed-gradient energy control of Hamiltonian systems

A convenient mathematical description for a controlled oscillatory system is the Hamiltonian form. It allows an explicit description of surfaces of constant energy which unforced oscillatory motions belong to. The Hamiltonian form of controlled plant equations is as follows:

$$\dot{q}_i = \frac{\partial H_c(q,p,u)}{\partial p_i}, \dot{p}_i = -\frac{\partial H_c(q,p,u)}{\partial q_i}, \quad i = 1, \dots, n, \tag{1}$$

where $q = \mathrm{col}(q_1, \dots, q_n)$, $p = \mathrm{col}(p_1, \dots, p_n)$ are the vectors of generalized coordinates and momenta, respectively, $H_c(q,p,u)$ is the controlled Hamiltonian function, and $u(t) \in \mathbb{R}^m$ is the control input (generalized force). The model (1) can be also rewritten as follows

$$\begin{cases} \dot{q} = \nabla_p H_c(q,p,u), \\ \dot{p} = -\nabla_q H_c(q,p,u). \end{cases} \tag{2}$$

Let $H(q,p) = H_c(q,p,0)$ be the "internal" Hamiltonian describing the unforced system

$$\begin{cases} \dot{q} = \nabla_p H(q,p), \\ \dot{p} = -\nabla_q H(q,p). \end{cases} \tag{3}$$

Consider the following control goal

$$H(q(t),p(t)) \to H_*, \quad \text{when} \quad t \to \infty. \tag{4}$$

In what follows we assume that the Hamiltonian is linear in control:

$$H_c(q,p,u) = H(q,p) + H_1(q,p)^T u,$$

where $H(q,p)$ is the internal Hamiltonian and $H_1(q,p)$ is an m-dimensional vector of interaction Hamiltonians (Nijmeijer and van der Schaft, 1990). Define the Poisson bracket for smooth functions $f(q,p)$ and $g(q,p)$ in a standard manner

$$[f,g] = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right).$$

If the functions f, g are the vector-functions then the Poisson bracket is defined componentwise. For example if the function f is scalar and g is an m-dimensional vector (column) then $[f,g]$ is an m-dimensional co-vector (row). More generally, if f and g are an l-dimensional and m-dimensional vectors, respectively, then $[f,g]$ is an $l \times m$ matrix.

To find the control algorithm providing the goal (4) according to the Speed-gradient method (Fradkov, 1996; Fradkov and Pogromsky, 1998) first the objective function is introduced

$$Q(x) = \frac{1}{2}\left(H(q,p) - H_*\right)^2, \tag{5}$$

where $x = \mathrm{col}(q,p)$. Then the control goal (4) is reformulated as follows

$$Q(x(t)) \to 0 \quad \text{when} \quad t \to \infty. \tag{6}$$

To design the SG algorithm calculate $\dot{Q}$:

$$\dot{Q} = (H - H_*)\left(\frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial q}\dot{q}\right) = (H - H_*)[H, H_1]u \tag{7}$$

and the speed gradient: $\nabla_u \dot{Q} = (H - H_*)[H, H_1]^T$. The linear and relay forms of Speed-gradient control algorithms are as follows:

$$u = -\gamma(H - H_*)[H, H_1]^T, \tag{8}$$

$$u = -\gamma \mathrm{sign}\left\{(H - H_*)[H, H_1]^T\right\}, \tag{9}$$

where $\gamma > 0$ is the gain factor. We may consider also the general speed pseudogradient algorithm

$$u = -\psi\left((H - H_1)[H, H_1]^T\right), \tag{10}$$

where ψ is a smooth vector function with values in $\mathbb{R}^m$ which satisfies the strict pseudogradient condition $\psi(z)^T z > 0$ for $z \neq 0$. To analyze the behavior of the system with algorithm (10) the following result can be used (Fradkov and Pogromsky, 1998).

Theorem 1 *Let the their first and second derivatives of the functions H, H_1 be bounded on the set $\Omega_0 = \{x \ : \ Q(x) \le Q_0\}$ for some $Q_0 > 0$.*

Then the algorithm (10) with $x(0) \in \Omega_0$ ensures $u(t) \to 0$ when $t \to \infty$ and ensures either the goal (4) or convergence $[H, H_1](x(t)) \to 0$ when $t \to \infty$.

If, additionally, the following two conditions hold:

H1. *For any $c \neq H_*$ there exists $\varepsilon > 0$ such that any nonempty connected component of the set $D_{\varepsilon,c} = \{x \ : \ |[H(x), H_1(x)]| \le \varepsilon, |H(x) - c| \le \varepsilon\} \cap \Omega_0$ is bounded.*

H2. *The largest invariant set $M \subset D_0$ of the free system (i.e. the set M of whole trajectories of (3) contained in D_0), where $D_0 = \{x \ : \ [H(x), H_1(x)] = 0\} \cap \Omega_0$, consists of finite or countable number of isolated points.*

Then any solution of the system (2), (10) either achieves the goal (4) or tends to a point of the set D_0 which is an equilibrium of the free system (3). Moreover, the set of initial conditions from which the solution of (2), (10) tends to unstable[1] equilibrium of the free system has zero Lebesgue measure.

Corollary 1 *If D_0 is empty, i.e. $[H, H_1](x) \neq 0$ for $x \in \Omega_0$, then the control goal is achieved for all $x(0) \in \Omega_0$.*

The theorem shows, loosely speaking, that algorithm (10) ensures the goal (4) almost always unless there are "false" goals: stable or neutral equilibria of the free system which are reachable from the initial point within the energy layer

$$\Omega_0 = \{(q,p) \ : \ |H(q,p) - H_*| \leq |H(q(0), p(0)) - H_*|\}.$$

In other words, the goal (4) will be achieved for almost all initial conditions from the set Ω_0 if it does not contain local potential wells. Moreover, it is clear from the proof (see (Fradkov and Pogromsky, 1998)) that the set of exceptional initial conditions is contained within a finite or countable number of manifolds, i.e., the complement of this set is open dense set in Ω_0.

The result of the Theorem 1 holds with obvious change of notations if the system (2) evolves on a smooth $2n$-dimensional manifold possessing standard Poisson structure (see (Nijmeijer and van der Schaft, 1990)).

3 Lyapunov-type characterization of ultimate boundedness

By $L_F V(x)$ denote the Lie derivative of function V along vector field F. Suppose $V: X \to Y$ and $Y_1 \subset Y$; then by $V_{Y_1}^{-1}$ we denote

$$V_{Y_1}^{-1} \equiv \{x \in X \colon V(x) \in Y_1\}. \tag{11}$$

A function $V: X \to R^+$ is called *proper* if for all $\delta \geq 0$ the set $V_{[0,\delta]}^{-1} \equiv \{x \in X \colon 0 \leq V(x) \leq \delta\}$ is compact.

Let X be an n-dimensional smooth manifold. Suppose $F(x,t)$ is a time-dependent vector field on X that is smooth on x, piecewise continuous on t, and bounded on any compact subset of X uniformly on t. Consider the system

$$\dot{x} = F(x,t), \tag{12}$$

where F is a nonstationary vector field on X. Suppose $V: X \to R^+$ is a smooth ($V \in C^1$) proper function. Consider the following properties of system (12).

Property 1. There exists a constant $\alpha \geq 0$ s.t. $L_F V(x,t) \leq \alpha$ for all $x \in X$, $t \geq 0$.

Property 2. There exist positive constants v_0, v_3, $0 < v_0 < v_3 < \infty$, and continuous functions β,ϵ: $V_{[v_0,v_3]}^{-1} \to R^+ \setminus \{0\}$ s.t. for arbitrary trajectory $x_0(\cdot)$ of system (12) if

$$x_0(t) \in V_{[v_0,v_3]}^{-1} \setminus \Pi \quad \text{for all } t \in [t_1, t_2],$$

where $\Pi \equiv \{x \in X \colon L_F V(x,t) < -\beta(x) \text{ for all } t \geq 0\}$, then

[1] Instability of an equilibrium means that the Jacobi matrix of the system calculated at the equilibrium point has at least one eigenvalue with positive real part.

i) $t_2 - t_1 \le A$ for some constant $A \ge 0$, and

ii) there exists $t_3 > t_2$ s.t.

$$V\left(x_0(t_3)\right) - V\left(x_0(t_1)\right) \le -\epsilon\left(x_0(t_1)\right),$$

and for all $t \in (t_2, t_3)$

$$x_0(t) \in V^{-1}_{[v_0,v_3]} \quad \Longrightarrow \quad x_0(t) \in \Pi.$$

We shall say that the system (12) *satisfies the Assumption A* if it has properties 1 and 2 and the corresponding constants v_0, v_3, α, A satisfy the condition $\alpha A < (v_3 - v_0))/2$.

Now suppose the system (12) satisfies the Assumption A. Denote

$$\begin{aligned} v_1 &= v_0 + \alpha A \\ v_2 &= v_3 - \alpha A. \end{aligned} \tag{13}$$

Obviously, $v_1 < v_2$.

Lemma 1. Suppose the system (12) satisfies the Assumption A. Let $x(\cdot)$ be arbitrary trajectory of system (12) satisfying the condition $x(t_0) \in V^{-1}_{[0,v_2]}$. Then $x(t) \in V^{-1}_{[0,v_3]}$ for all $t \ge t_0$ and there exists $T \ge 0$ such that $x(t) \in V^{-1}_{[0,v_1]}$ for all $t \ge t_0 + T$.

Proof. First we claim that under the conditions of Lemma

$$V\left(x(t_0)\right) \in [v_0, v_2) \quad \Longrightarrow \quad V\left(x(t)\right) \le V\left(x(t_0)\right) + \alpha A \text{ for all } t \ge t_0. \tag{14}$$

Indeed, consider a set

$$\Omega \equiv \left\{ t \ge t_0\colon x(t) \in V^{-1}_{[v_0,v_3]} \setminus \Pi \right\}.$$

Since Π is open set, we see that the set $\Omega \in R^1$ is a union of finite or denumerable number of disjoint closed intervals

$$\Omega = [t_1, t_1'] \cup [t_2, t_2'] \cup \ldots,$$

Consider the interval $[t_1, t_1']$. It is clear that $V(x(t_1)) \le V(x(t_0))$. Suppose there exists $t' \in [t_1, t_1']$ such that $V(x(t')) > V(x(t_0)) + \alpha A$; then from property 1 it follows that $t_1' - t_1 \ge t' - t_1 > A$. The last is in contradiction with the property 2. Then for all $t \in [t_1, t_1']$ we have $V(x(t)) \le V(x(t_0)) + \alpha A < v_3$. In particular we get $V(x(t_1')) < v_3$. Then it is easily proved that $V(x(t)) < V(x(t_1'))$ for all $t \in [t_1', t_2]$. Due to property 2 we have $V(x(t_2)) \le V(x(t_1))$. Continuing in the same way, we see that for all $t \in [t_2, t_2']$ we have $V(x(t)) \le V(x(t_0)) + \alpha A < v_3$, and so on.

Now we claim that there exists $T_M > 0$ such that from $V\left(x(t_0)\right) \in [v_0, v_2)$ it follows that $V(x(t)) = v_0$ for some $t \in [t_0, t_0 + T_M]$. Indeed, suppose $V\left(x(t_0)\right) \in [v_0, v_2)$ and $v_0 \le V\left(x(t)\right) \le v_3$ for any $t \in [t_0, t_0 + T]$ where $T > 0$ is a constant. Denote

$$\begin{aligned} \beta_0 &= \min_{x:\, v_0 \le V(x) \le v_3} \beta\left(x\right) > 0, \\ \epsilon_0 &= \min_{x:\, v_0 \le V(x) \le v_3} \epsilon\left(x\right) > 0, \\ \epsilon_1 &= \max_{x:\, v_0 \le V(x) \le v_3} \epsilon\left(x\right) \ge \epsilon_0 > 0. \end{aligned}$$

It is easy to prove that

$$V(t) \le V_3 - \frac{\beta_0 \epsilon_0}{\beta_0 A + \alpha A + \epsilon_1}(t - t_0 - A), \tag{15}$$

for all $t \in [t_0, t_0 + T]$. It follows that $T \le T_M = \frac{(v_3 - v_0)(\beta_0 A + \alpha A + \epsilon_1)}{\beta_0 \epsilon_0} + A$. Since $V(x(t)) \le v_3$ for all $t \ge t_0$, we obtain $V(x(t_*)) = v_0$ for some $t_* \in [t_0, t_0 + T_M]$. Finally from (14) it follows that $V(x(t)) \le v_0 + \alpha A = v_1$ for all $t \ge t_0 + T_M$. This completes the proof.

4 Case of 1-DOF systems under input disturbances

In what follows we consider the controlled Hamiltonian system on a $2n$-dimensional smooth manifold M governed by equations

$$\dot{q} = \nabla_p H(q,p), \tag{16}$$

$$\dot{p} = -\nabla_q H(q,p) + \Delta(q,p,t) + u. \tag{17}$$

where, $(q,p) \in M$, $H(q,p)$ is Hamiltonian function of the free system, $u \in R^n$ is controlling input and $\Delta(q,p,t)$ is vector-function of=20 disturbances. The form (16), (17) is more simple than (2) since $H_1 = q$ in this case. On the other hand the presence of disturbances makes the known results nonapplicable.

In this Section we consider the case when $n = 1$ and $\Delta(t)$ is piecewise continuous and bounded: $|\Delta(\cdot)| \leq D$.

Suppose $H(q,p)$ is a proper function on M, and $H(q,p) = K(p) + P(q)$ where $K\colon R \to R^+$ is a smooth positive definite convex $\left(\frac{d^2K}{dq^2} > 0\right)$ even $(K(p) = K(-p))$ function representing kinetic energy and $P(q)$ is smooth function with strict local minimum at a point $q = 0$, representing potential energy.

Let h be a positive number. Consider a set

$$X_h \equiv \{(q,p)\colon\ 0 \leq H(q,p) \leq h\}. \tag{18}$$

Suppose $q = 0$, $p = 0$ is the unique equilibrium point of free ($u \equiv 0$, $\Delta(t) \equiv 0$) system (16), (17) on the set X_h.

Consider the problem of energy level stabilization of system (16), (17) in presence of bounded input disturbances. By H_* denote the desired value of Hamiltonian function, $0 < H_* < h$. Consider the control law (speed-gradient algorithm)

$$u = -\gamma_0 \left(H - H_*\right) \frac{\partial H}{\partial p}, \tag{19}$$

where $\gamma_0 > 0$ is a gain.

Suppose h_1, h_2, h_3 are positive constants satisfying the condition

$$0 < h_1 < h_2 < h_3 < \min\left\{H_*, h - H_*\right\}. \tag{20}$$

The main result of the Section is the following theorem.

Theorem 2 *For any constants h_1, h_2, h_3 satisfying the condition (20) there exists $D > 0$ s.t. for for any given initial condition $(q,p)(0) \in H^{-1}_{[H_*-h_2, H^*+h_2]}$ trajectories of closed loop system (16), (17), (19) satisfy $(q,p)(t) \in H^{-1}_{[H_*-h_3, H_*+h_3]}$ for all $t \geq 0$ and there exists $T > 0$ s.t. $(q,p)(t) \in H^{-1}_{[H_*-h_1, H_*+h_1]}$ for all $t \geq T$.*

Proof. Consider a function $V = 1/2\left(H - H_*\right)^2$. Let

$$v_0 = \frac{1}{8} h_1^2, \tag{21}$$

$$v_3 = \frac{1}{2} h_3^2. \tag{22}$$

Clearly

$$\begin{aligned} V^{-1}_{[0,v_0]} &= H^{-1}_{\left[H_*-\frac{h_1}{2}, H_*+\frac{h_1}{2}\right]}, \\ V^{-1}_{[0,v_3]} &= H^{-1}_{[H_*-h_3, H_*+h_3]}. \end{aligned}$$

The time derivative of function V along the trajectories of closed loop system (16), (17), (19) is

$$\dot{V} = \frac{\partial V}{\partial H}\dot{H} = \frac{\partial V}{\partial H}\left(\frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p}\right) = (H - H_*)\frac{\partial H}{\partial p}\left(\Delta(\cdot) - \gamma_0(H - H_*)\frac{\partial H}{\partial p}\right).$$

Further

$$\dot{V} \le -\gamma_0(H - H_*)^2\left(\frac{\partial H}{\partial p} = 20\right)^2 + \mu(H - H_*)^2\left(\frac{\partial H}{\partial p}\right)^2 + 1/4\mu\Delta^2(\cdot)$$

for arbitrary $\mu > 0$. By choosing $\mu < \gamma_0$ we get

$$\dot{V} \le -\gamma(H - H_*)^2\left(\frac{\partial H}{\partial p}\right)^2 + \Delta^*. \tag{23}$$

where $\gamma = \gamma_0 - \mu > 0$, $\Delta^* = \frac{1}{4\mu}D^2 > 0$. It follows from (23) that the closed loop system (16), (17), (19) has property 1.

Now let us prove that the closed loop system has property 2. First, due to properties of kinetic energy $K(p)$ (recall that $K(p)$ is a smooth positive definite convex even function) we get that $\partial H/\partial p = \partial K/\partial p$ is odd strictly increasing function of p. Denote $L = \lim_{p\to\infty}|\partial K/\partial p|$. The inverse function $(\partial K/\partial p)^{-1}$ is defined on the interval $(-L, L)$. Let

$$\begin{aligned} \beta &= D^2/8\mu, \\ p_0 &= (\partial K/\partial p)^{-1}\left(\sqrt{\frac{D^2}{2\gamma\mu h_1^2}}\right), \end{aligned} \tag{24}$$

where p_0 is well defined for sufficiently small $D > 0$. Using (23) and (24), we get that for sufficiently small $D > 0$ from $V \in [v_0, v_3]$ and $\dot{V} \le -\beta$ it follows that $|p| \le p_0$.

Second, we claim that for sufficiently small $D > 0$ there exist $B > 0$ such that from $V \in [v_0, v_3]$ and $|p| \le p_0$ it follows that $|\dot{p}| \ge B$. Indeed, due to properties of potential energy $P(q)$ and from the fact that $q = 0$, $p = 0$ is the unique equilibrium point of free system in X_h we get that

$$\left|\frac{\partial P}{\partial q}(q)\right| > 0 \qquad \text{for all} \quad q \in P^{-1}_{(0,v_3]}.$$

Further, for sufficiently small $D > 0$ from $V \in [v_0, v_3]$ and $|p| \le p_0$ it follows that $P(q) \ge v_0 - K(p_0)$. Let

$$B = \frac{1}{2}\inf_{|q|\in P^{-1}_{[v_0-K(p_0), v_3]}} = \left|\frac{\partial P}{\partial q}(q)\right| > 0.$$

Obviously

$$\left|\frac{\partial P}{\partial q}(q)\right| \ge 2B. \tag{25}$$

On the other hand for sufficiently small $D > 0$ from $V \in [v_0, v_3]$ and $|p| \le p_0$ it follows that

$$|\gamma\,(H(t) - H_*)\,\frac{\partial H}{\partial p}| + |\Delta(t)| \le B. \tag{26}$$

Finally, combining (17), (25), (26), we obtain

$$|\dot{p}(t)| \geq B. \tag{27}$$

Let $A = 2P_0/B$. It is easy to see that for arbitrary trajectory of closed loop system (16), (17), (19) from $V(t) \in [v_0, v_3]$ and $|p(t)| \leq p_0$ for all $t \in [t_1, t_2]$ it follows from (27) that

$$t_2 - t_1 \leq A. \tag{28}$$

Now let us prove that there exist $t_3 > t_2$ and $\epsilon > 0$ such that $\dot{V}(t) < -\beta$ for all $t \in (t_2, t_3)$ and

$$V(x_0(t_3)) - V(x_0(t_1)) \leq -\epsilon.$$

Denote

$$q_0 = \min\left\{|q|: \quad P(q) = \frac{H_* - h_3}{2}\right\}.$$

Consider a strip

$$\Upsilon = \left\{(q,p): \ q \in \left(q(t_2), q_0'\right); \ |q_0'| = q_0, \ \mathrm{sign} q_0' = -\mathrm{sign} q(t_2)\right\}.$$

We claim that there exists an instant $t_3 > t_2$ such that $q(t_3) = q_0'$ and for all $t \in (t_2, t_3)$

$$\begin{array}{ll} i) & (q,p)(t) \in \Upsilon \\ ii) & |p(t)| > p_0. \end{array}$$

Put

$$K_0 = \frac{\partial K}{\partial p}(p_0) = \sqrt{\frac{D^2}{2\gamma\mu h_1^2}}.$$

Due to properties of $K(p)$ and continuity of $\dot{q}(t)$ it is sufficient to prove that

$$\begin{array}{ll} iii) & \mathrm{sign}\,\dot{q}(t_2) = -\,\mathrm{sign}\,q(t_2) \\ iv) & |\dot{q}| > K_0 \quad \text{on the strip } \Upsilon. \end{array}$$

From (16) and due to properties of $K(p)$ we have $\mathrm{sign}\,\dot{q}(t_2) = \mathrm{sign}\,p(t_2)$. Further, by definition of t_2 we get $\frac{d|p(t_2)|}{dt} > 0$; therefore, $\mathrm{sign}\,\dot{p}(t_2) = \mathrm{sign}\,p(t_2)$. Finally, from (17), (25), (26) and due to properties of $P(q)$ we get $\mathrm{sign}\,\dot{p}(t_2) = -\,\mathrm{sign}\,q(t_2)$.=20 Combining the above, we obtain $iii)$. Fulfillment of $iv)$ follows from $iii)$, (17), (25), (26), and from the continuity arguments.

Let us estimate $V(t_3) - V(t_1)$. We have

$$V(t_2) - V(t_1) \leq \alpha A \to 0 \quad \text{as} \quad D \to 0.$$

On the other hand

$$V(t_3) - V(t_2) \leq -\int_{t_2}^{t_3} \gamma\left(H(t) - H_*\right)^2 \left(\frac{\partial H}{\partial p}\right)^2 (t)dt + \Delta^*(t_3 - t_2) \leq$$

$$-\int_{t_2}^{t_3} \gamma\left((H(t) - H_*)^2 \left(\frac{\partial H}{\partial p}\right)^2 (t) - \frac{1}{2}h_1^2 K_0^2\right) dt \leq -\frac{1}{2}\gamma h_1^2 \int_{t_2'}^{t_3} \left(\frac{\partial H}{\partial p}\right)^2 dt =$$

$$-\frac{1}{2}\gamma h_1^2 \int_{t_2'}^{t_3} \dot{q}^2(t)dt \leq -\frac{1}{2}\gamma = h_1^2 \frac{1}{t_3 - t_2'} \left(\int_{t_2'}^{t_3} \dot{q}(t)dt \right)^2 = -2\gamma h_1^2 \frac{1}{t_3 - t_2'} (q_0)^2 .$$

It is easy to see that

$$t_3 - t_2' \leq \frac{2q_0}{C},$$

where

$$C = \left(\frac{\partial H}{\partial p} \right)^{-1} \left(\frac{H_* - h_3}{2} \right).$$

Then

$$V(t_3) - V(t_2) \leq -\gamma h_1^2 q_0 C.$$

Let

$$\epsilon = \frac{1}{2}\gamma h_1^2 q_0 C > 0.$$

Then for sufficiently small $D > 0$ we get

$$V(t_3) - V(t_1) \leq -\epsilon < 0.$$

We see that all conditions of Lemma 1 are fulfilled. From Lemma 1 it follows that there exist $D > 0$ s.t. for for any given initial condition $(q,p)(0) \in V_{[0,v_2]}^{-1}$ trajectories of closed loop system (16), (17), (19) satisfy $(q,p)(t) \in H_{[0,v_3]}^{-1}$ for all $t \geq 0$ and there exists $T > 0$ s.t. $(q,p)(t) \in H_{[0,v_1]}^{-1}$ for all $t \geq T$, where $v_1 = v_0 + A\Delta^*$, $v_2 = v_3 - A\Delta^*$. Since A, $\Delta^* \to 0$ as $D \to 0$, from (21), (22) we see that for sufficiently small $D > 0$

$$\begin{aligned} V_{[0,v_1]}^{-1} &\subset H_{[h_1,h_1^*]}^{-1}, \\ H_{[h_2,h_2^*]}^{-1} &\subset V_{[0,v_2]}^{-1}. \end{aligned}$$

The statement of Theorem 2 follows.

5 Energy control of n-DOF Hamiltonian systems

Consider again the controlled Hamiltonian system with disturbances (16), (17) and suppose that the disturbance function $\Delta(\cdot)$ satisfies the inequality

$$||\Delta(q,p,t)|| \leq \varrho ||p|| \tag{29}$$

for some $\varrho > 0$. Inequality (29) means that the disturbances vanish at the zero-momentum manifold. It holds, e.g. for disturbances caused by viscous damping.

Below the conditions guaranteeing achievement of the goal (4) are established.

Theorem 3 *Consider the system (16), (17) under condition (29) and the following assumptions.*

A1. *$H(q(0),p(0)) \leq H_*$ and the set $\Omega = \{(q,p) : H(q(0),p(0)) \leq H(q,p) \leq H_*\}$ is bounded.*

A2. *The Hamiltonian satisfies the inequalities $H(q,0) \geq 0$ and for some $\alpha > 0$*

$$H(q,p) - H(q,0) \geq \alpha ||p||^2. \tag{30}$$

A3. $\nabla_p H(q,p) \neq 0 \quad \forall (q,p) \in \Omega.$

Then the goal (4) is achieved in the system (16), (17) controlled by the algorithm (9) if

$$H_* \leq \alpha \left(\frac{\gamma}{\varrho}\right)^2. \tag{31}$$

Proof. It is sufficient to prove that $\dot{Q}(q,p) < 0$ in the set Ω, if $Q(q,p) \neq 0$. Calculation of $\dot{Q}(q,p)$ and taking into account (29) and A2 yields

$$\begin{aligned} \dot{Q} &= (H - H_*)\nabla_p H(q,p)^T \Delta - |H - H_*| \|\nabla_p H(q,p)\| \gamma \\ &\leq |H - H_*| \|\nabla_p H(q,p)\| (\varrho \|p\| - \gamma) \\ &\leq |H - H_*| \|\nabla_p H(q,p)\| \left(\varrho \sqrt{\frac{H(q,p) - H(q,0)}{\alpha}} - \gamma \right). \end{aligned} \tag{32}$$

Therefore if $H(q,p) < H_*$, then $\dot{Q}(q,p) < 0$ in the set Ω provided the inequality $\varrho\sqrt{\frac{H_*}{\alpha}} \leq \gamma$ holds which is equivalent to (31).

6 Example: control of pendulum oscillations under disturbances

As an illustration of above results consider the problem of oscillation control of simple pendulum under bounded force disturbances. The pendulum is described by the following equation

$$J\ddot{\phi} + mgl \sin\phi = u + \Delta(t), \tag{33}$$

where ϕ is an angle of pendulum defined to be zero in the lower position, u is a controlling torque, J, m, and l are the inertia, mass, = and length of pendulum respectively, g is an acceleration due to gravity, $\Delta(t)$ are input disturbances, $\Delta(t) \leq D$, where $D > 0$ is a constant.

The pendulum equation (33) can be represented in the form (16), (17) by choosing $q = \phi$, $p = J\dot{\phi}$ and

$$H(q,p) = \frac{1}{2J}p^2 + mgl(1 - \cos q). \tag{34}$$

For our purpose it is convenient to consider a cylinder with unit circle at the base $-\pi < q \leq \pi$ as a phase space of pendulum. It means that we identify the points (q_1, p) and (q_2, p) iff $q_2 - q_1 = 2\pi k$, where k is an integer number. Then it is easy to check that $H(q,p)$ given by (34) is proper function on cylindrical phase space. Further, the kinetic energy $K(p) = \frac{1}{2J}p^2$ is a smooth positive definite convex even function, and the potential energy $P(q) = mgl(1 - \cos q)$ is smooth function with strict local minimum at a point $q = 0$. Let $h = 2mgl$. Then the point $(0,0)$ is a unique equilibrium point of free system on X_h defined by (18). We see that the system (33) satisfy all the conditions of the Theorem 2. The control law (19) becomes

$$u = -\gamma_0 (H - H_*) p. \tag{35}$$

In particular, it follows from Theorem 2 that the control law (35) drives the pendulum (33) from any initial state $(q(t_0), p(t_0))$ satisfying $H(q(t_0), p(t_0)) \in (0, 2mgl)$ to the oscillatory mode of given energy level $H_* \in (0, 2mgl)$ with prescribed accuracy if the disturbances are of sufficiently small intensity. Moreover during the swinging the pendulum does not turn into rotation mode.

The conditions of the Theorem 3 are also fulfilled in this case. Therefore we may conclude that adding the viscous damping does not change qualitatively the behavior of the pendulum controlled by the algorithm

$$u = -\gamma \operatorname{sign}((H - H_*)p), \tag{36}$$

if $\gamma > \varrho\sqrt{2mgl/J}$, where ϱ is damping coefficient.

7 Conclusions

In the paper we address the problem of the Hamiltonian system energy control to the arbitrary (not necessarily equilibrium) desired level in presence of disturbances. The problem of such kind is important for control of oscillatory systems. However it was not considered in the previous publications. For 1-DOF systems it is shown (Theorem 2) that under uniformly bounded input disturbances with sufficiently small bound $D > 0$ the speed-gradient control law ensures ultimate boundedness of error between actual value of energy and desired one. Moreover the upper bound of the error tends to zero as $D \to 0$. To prove this result an auxiliary Lemma 1 is stated which gives sufficient conditions for ultimate boundedness of Lyapunov function along the trajectories of nonlinear nonstationary system. Note that related results were obtained recently in (Aeyels and Peuteman, 1997; Peuteman and Aeyels, 1998). The results of (Aeyels and Peuteman, 1997; Peuteman and Aeyels, 1998) establish conditions for asymptotic as well as for exponential stability without assumption of negative semidefiniteness for the time derivative of Lyapunov function. These results are close in spirit to Lemma 1.

The case when the disturbances vanish on zero momentum submanifold turnes out to be simpler. In this case similar Speed-gradient algorithm allows to achieve zero error in terms of energy for n-DOF systems.

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