ENERGY CONTROL OF ONE-DEGREE-OF-FREEDOM OSCILLATORS IN PRESENCE OF BOUNDED FORCE DISTURBANCES

I.G. Polushin, A.L. Fradkov,
Institute for Problems of Mechanical Engineering
of Russian Academy of Sciences
61, Bolshoy, V.O., St. Petersburg, 199178, Russia
fax : +7 (812) 321-4771
e-mail: polushin@ccs.ipme.ru, alf@ccs.ipme.ru

V.V. Putov, K.A. Rogov
St.Petersburg Electrotechnical
University
5, Prof. Popova str., St. Petersburg,
197376, Russia

Keywords: Nonlinear Oscillations Control, Input Disturbances, Hamiltonian Systems.

Abstract

Energy level stabilization problem for one-degree-of-freedom Hamiltonian systems in presence of bounded input (force) disturbances is considered. It is shown that for arbitrary uniformly bounded force disturbances with sufficiently small bound speed-gradient control law ensures ultimate boundedness of energy error. As an auxiliary result the new sufficient conditions for ultimate boundedness of Lyapunov function along the trajectories of nonlinear nonstationary dynamical system are obtained.

1 Introduction

In the paper we consider a problem of energy level stabilization for one-degree-of-freedom Hamiltonian systems in presence of bounded input disturbances. This problem is of interest in the oscillations control area where the level of energy determines the mode of oscillations. For Hamiltonian systems stabilization of energy level implies stabilization of a certain invariant manifold in phase space. Special case when the desired level of energy is equal to its minimum value (stabilization of equilibrium) is well studied, see e.g. [1, 2]. In [3, 4] the speed-gradient control was used to solve the problem for arbitrary desired value of energy. It was shown that the stabilization of energy level is achieved if there is no equilibria in the initial energy layer (between the initial and the desired levels of energy). In [5, 6] further development of the result was given. However no results concerning behavior of the closed loop systems under disturbances were available so far.

In this paper we extend the results of [3] to the case of presence of the bounded input (force) disturbances. Roughly speaking, the main result of the paper is as follows. For a broad class of controlled one-degree-of-freedom Hamiltonian systems we state that under uniformly bounded input disturbances with sufficiently small bound $D > 0$ the speed-gradient control law ensures ultimate boundedness of error between actual value of energy and desired one. Moreover the upper bound of the error tends to zero as $D \to 0$.

Essential feature of the input disturbances case is that the time derivative of Lyapunov function is not negative semidefinite even for large values of V. Hence methods of papers [3, 5, 6] essentially based on assumption $\dot{V} \leq 0$ and the standard theorems on ultimate boundedness (like Yoshizawa's theorem) are not applicable. In this paper we exploit the fact that trajectories of the system "pass through" the regions of sign indefiniteness of Lyapunov function time derivative. Based on this circumstance under some further conditions it is possible to show that the ultimate boundedness of V holds along an arbitrary trajectory of system. We prove the corresponding lemma in section 2. Proof of the main result reduces to checking the lemma conditions for closed loop system "Hamiltonian system + speed-gradient control law + disturbances".

This paper is organized as follows. In section 2 we state the lemma on sufficient conditions for ultimate boundedness of Lyapunov function along the trajectories of nonlinear nonstationary dynamical system. The main result of the paper is presented in section 3. In section 4 we consider an application of the main theorem to control of oscillations of simple controlled pendulum. Appendix of the paper contains a proof of a certain auxiliary claim.

2 Lyapunov-type characterization of ultimate boundedness

By $L_F V(x)$ denote the Lee derivative of function V along vector field F. Suppose $V: X \to Y$ and $Y_1 \subset Y$; then by $V_{Y_1}^{-1}$ we denote

$$V_{Y_1}^{-1} \equiv \{x \in X \colon V(x) \in Y_1\}. \qquad (1)$$

A function $V: X \to R^+$ is called *proper* if for all $\delta \geq 0$ the set $V_{[0,\delta]}^{-1} \equiv \{x \in X \colon 0 \leq V(x) \leq \delta\}$ is compact set.

Let X be an n-dimensional smooth manifold. Suppose $F(x,t)$ is a time-dependent vector field on X that is smooth on x, piecewise continuous on t, and bounded on any compact subset of X uniformly on t. Consider the system

$$\dot{x} = F(x,t), \qquad (2)$$

where F is a nonstationary vector field on X. Suppose $V: X \to R^+$ is a smooth ($V \in C^1$) proper function. Consider the following properties of system (2).

Property 1. There exists a constant $\alpha \geq 0$ s.t. $L_F V(x,t) \leq \alpha$ for all $x \in X$, $t \geq 0$.

Property 2. There exist positive constants v_0, v_3, $0 < v_0 < v_3 < \infty$, and continuous functions $\beta, \epsilon \colon V_{[v_0,v_3]}^{-1} \to R^+ \setminus \{0\}$ s.t. for arbitrary trajectory $x_0(\cdot)$ of system (2) if

$$x_0(t) \in V_{[v_0,v_3]}^{-1} \setminus \Pi \quad \text{for all } t \in [t_1, t_2],$$

where

$$\Pi \equiv \{x \in X \colon L_F V(x,t) < -\beta(x) \text{ for all } t \geq 0\},$$

then

i) $t_2 - t_1 \leq A$ for some constant $A \geq 0$, and

ii) there exists $t_3 > t_2$ s.t.

$$V(x_0(t_3)) - V(x_0(t_1)) \leq -\epsilon(x_0(t_1)),$$

and for all $t \in (t_2, t_3)$

$$x_0(t) \in V_{[v_0,v_3]}^{-1} \quad \Longrightarrow \quad x_0(t) \in \Pi.$$

We shall say that the system (2) *satisfies the Assumption A* if it has properties 1 and 2 and the corresponding constants v_0, v_3, α, A satisfy the condition $\alpha A < (v_3 - v_0))/2$.

Now suppose the system (2) satisfies the Assumption A. Denote

$$\begin{aligned} v_1 &= v_0 + \alpha A \\ v_2 &= v_3 - \alpha A. \end{aligned} \qquad (3)$$

Obviously, $v_1 < v_2$.

Lemma 1. Suppose the system (2) satisfies the Assumption A. Let $x(\cdot)$ be arbitrary trajectory of system (2) satisfy the condition $x(t_0) \in V_{[0,v_2]}^{-1}$. Then $x(t) \in V_{[0,v_3]}^{-1}$ for all $t \geq t_0$ and there exists $T \geq 0$ such that $x(t) \in V_{[0,v_1]}^{-1}$ for all $t \geq t_0 + T$.

Proof. First we claim that under the conditions of Lemma

$$V(x(t_0)) \in [v_0, v_2)$$
$$\Downarrow$$
$$V(x(t)) \leq V(x(t_0)) + \alpha A \text{ for all } t \geq t_0.$$

Indeed, consider a set

$$\Omega \equiv \left\{t \geq t_0 \colon x(t) \in V_{[v_0,v_3]}^{-1} \setminus \Pi\right\}.$$

Since Π is open set, we see that the set $\Omega \in R^1$ is a union of finite or denumerable number of disjoint closed intervals

$$\Omega = [t_1, t_1'] \cup [t_2, t_2'] \cup \ldots,$$

Consider the interval $[t_1, t_1']$. It is clear that $V(x(t_1)) \leq V(x(t_0))$. Suppose there exists $t' \in [t_1, t_1']$ such that $V(x(t')) > V(x(t_0)) + \alpha A$; then from property 1 it follows that $t_1' - t_1 \geq t' - t_1 > A$. The last is in contradiction with the property 2. Then for all $t \in [t_1, t_1']$ we have $V(x(t)) \leq V(x(t_0)) + \alpha A < v_3$. In particular we get $V(x(t_1')) < v_3$. Then it is easily proved that $V(x(t)) < V(x(t_1'))$ for all $t \in [t_1', t_2]$. Due to property 2 we have $V(x(t_2)) \leq V(x(t_1))$. Continuing in the same way, we see that for all $t \in [t_2, t_2']$ we have $V(x(t)) \leq V(x(t_0)) + \alpha A < v_3$, and so on.

Now we claim that there exist $T_M > 0$ such that from $V(x(t_0)) \in [v_0, v_2)$ it follows that $V(x(t)) = v_0$ for some $t \in [t_0, t_0 + T_M]$. Indeed, suppose $V(x(t_0)) \in [v_0, v_2)$ and $v_0 \leq V(x(t)) \leq v_3$ for any $t \in [t_0, t_0 + T]$ where $T > 0$ is a constant. Denote

$$\begin{aligned} \beta_0 &= \min_{x \colon v_0 \leq V(x) \leq v_3} \beta(x) > 0, \\ \epsilon_0 &= \min_{x \colon v_0 \leq V(x) \leq v_3} \epsilon(x) > 0, \\ \epsilon_1 &= \max_{x \colon v_0 \leq V(x) \leq v_3} \epsilon(x) \geq \epsilon_0 > 0. \end{aligned}$$

It is easy to prove (see Appendix) that

$$V(t) \leq V_3 - \frac{\beta_0 \epsilon_0}{\beta_0 A + \alpha A + \epsilon_1}(t - t_0 - A), \qquad (4)$$

for all $t \in [t_0, t_0 + T]$. It follows that $T \leq T_M = \frac{(v_3 - v_0)(\beta_0 A + \alpha A + \epsilon_1)}{\beta_0 \epsilon_0} + A$. Since $V(x(t)) \leq v_3$ for all $t \geq t_0$, we obtain $V(x(t_*)) = v_0$ for some $t_* \in [t_0, t_0 + T_M]$. Finally from (1) it follows that $V(x(t)) \leq v_0 + \alpha A = v_1$ for all $t \geq t_0 + T_M$. This completes the proof.

3 Main result

Let X be a two-dimensional smooth manifold. Consider a controlled Hamiltonian system on X defined by equations

$$\dot{q} = \frac{\partial H}{\partial p} \qquad (5)$$

$$\dot{p} = -\frac{\partial H}{\partial q} + \Delta(t) + u \qquad (6)$$

where, q,p are the generalized coordinate and momenta respectively, $H(q,p)$ is a Hamiltonian function of the system, $u \in R^1$ is a controlled input and $\Delta(t)$ are disturbances. Suppose $\Delta(t)$ are piecewise continuous and bounded $|\Delta(\cdot)| \leq D$.

Suppose H(q,p) is a proper function on phase space X, and $H(q,p) = K(p) + P(q)$ where $K\colon R \to R^+$ is a smooth positive definite convex $\left(\frac{d^2K}{dq^2} > 0\right)$ even $(K(p) = K(-p))$ function represents kinetic energy and $P(q)$ is smooth function with strict local minimum at a point $q = 0$.

Let h be a positive number. Consider a set

$$X_h \equiv \{(q,p)\colon\, 0 \leq H(q,p) \leq h\}\,. \tag{7}$$

Suppose $q = 0$, $p = 0$ is the unique equilibrium point of free ($u \equiv 0$, $\Delta(t) \equiv 0$) system (5), (6) on the set X_h.

Consider the problem of energy level stabilization of system (5), (6) in presence of bounded input disturbances. By H_* denote the desired value of Hamiltonian function, $0 < H_* < h$. Consider the control law (speed-gradient algorithm)

$$u = -\gamma_0\,(H - H_*)\,\frac{\partial H}{\partial p}, \tag{8}$$

where $\gamma_0 > 0$ is a gain.

Suppose h_1, h_2, h_3 are positive constants satisfy the condition

$$0 < h_1 < h_2 < h_3 < \min\{H_*, h - H_*\}\,. \tag{9}$$

The main result of the paper is the following theorem.

Theorem 1. For any constants h_1, h_2, h_3 satisfy the condition (9) there exist $D > 0$ s.t. for for any given initial condition $(q,p)(0) \in H^{-1}_{[H_*-h_2, H^*+h_2]}$ trajectories of closed loop system (5), (6), (8) satisfy $(q,p)(t) \in H^{-1}_{[H_*-h_3, H_*+h_3]}$ for all $t \geq 0$ and there exists $T > 0$ s.t. $(q,p)(t) \in H^{-1}_{[H_*-h_1, H_*+h_1]}$ for all $t \geq T$.

Proof. Consider a function $V = 1/2\,(H - H_*)^2$. Let

$$v_0 = \frac{1}{8}h_1^2, \tag{10}$$

$$v_3 = \frac{1}{2}h_3^2. \tag{11}$$

Clearly

$$V^{-1}_{[0,v_0]} = H^{-1}_{\left[H_* - \frac{h_1}{2}, H_* + \frac{h_1}{2}\right]},$$
$$V^{-1}_{[0,v_3]} = H^{-1}_{[H_*-h_3, H_*+h_3]}.$$

The time derivative of function V along the trajectories of closed loop system (5), (6), (8) is

$$\dot V = \frac{\partial V}{\partial H}\dot H = \frac{\partial V}{\partial H}\left(\frac{\partial H}{\partial q}\dot q + \frac{\partial H}{\partial p}\dot p\right) = (H - H_*)\frac{\partial H}{\partial p}\left(\Delta(\cdot) - \gamma_0 (H - H_*)\frac{\partial H}{\partial p}\right).$$

Further

$$\dot V \leq -\gamma_0 (H - H_*)^2\left(\frac{\partial H}{\partial p}\right)^2 + \mu (H - H_*)^2\left(\frac{\partial H}{\partial p}\right)^2 + 1/4\mu\Delta^2(\cdot)$$

for arbitrary $\mu > 0$. By choosing $\mu < \gamma_0$ we get

$$\dot V \leq -\gamma (H - H_*)^2\left(\frac{\partial H}{\partial p}\right)^2 + \Delta^*. \tag{12}$$

where $\gamma = \gamma_0 - \mu > 0$, $\Delta^* = \frac{1}{4\mu}D^2 > 0$. It follows from (12) that the closed loop system (5), (6), (8) has property 1.

Now let us prove that the closed loop system has property 2. First, due to properties of kinetic energy $K(p)$ (recall that $K(p)$ is a smooth positive definite convex even function) we get that $\partial H/\partial p = \partial K/\partial p$ is odd strictly increasing function of p. Denote $L = \lim_{p\to\infty}|\partial K/\partial p|$. The inverse function $(\partial K/\partial p)^{-1}$ is defined on interval $(-L, L)$. Let

$$\beta = D^2/8\mu,$$
$$p_0 = (\partial K/\partial p)^{-1}\left(\sqrt{\frac{D^2}{2\gamma\mu h_1^2}}\right), \tag{13}$$

where p_0 is well defined for sufficiently small $D > 0$. Using (12) and (13), we get that for sufficiently small $D > 0$ from $V \in [v_0, v_3]$ and $\dot V \leq -\beta$ it follows that $|p| \leq p_0$.

Second, we claim that for sufficiently small $D > 0$ there exist $B > 0$ such that from $V \in [v_0, v_3]$ and $|p| \leq p_0$ it follows that $|\dot p| \geq B$. Indeed, due to properties of potential energy $P(q)$ and from the fact that $q = 0$, $p = 0$ is the unique equilibrium point of free system in X_h we get that

$$\left|\frac{\partial P}{\partial q}(q)\right| > 0 \qquad \text{for all} \quad q \in P^{-1}_{(0,v_3]}.$$

Further, for sufficiently small $D > 0$ from $V \in [v_0, v_3]$ and $|p| \leq p_0$ it follows that $P(q) \geq v_0 - K(p_0)$. Let

$$B = \frac{1}{2}\inf_{|q| \in P^{-1}_{[v_0 - K(p_0), v_3]}}\left|\frac{\partial P}{\partial q}(q)\right| > 0.$$

Obviously

$$\left|\frac{\partial P}{\partial q}(q)\right| \geq 2B. \tag{14}$$

From the other hand for sufficiently small $D > 0$ from $V \in [v_0, v_3]$ and $|p| \leq p_0$ it follows that

$$|\gamma\,(H(t) - H_*)\,\frac{\partial H}{\partial p}| + |\Delta(t)| \leq B. \tag{15}$$

Finally, combining (6), (14), (15), we obtain

$$|\dot p(t)| \geq B. \tag{16}$$

Let $A = 2P_0/B$. It is easy to see that for arbitrary trajectory of closed loop system (5), (6), (8) from $V(t) \in [v_0, v_3]$ and $|p(t)| \leq p_0$ for all $t \in [t_1, t_2]$ it follows from (16) that

$$t_2 - t_1 \leq A. \tag{17}$$

Now let us prove that there exist $t_3 > t_2$ and $\epsilon > 0$ such that $\dot{V}(t) < -\beta$ for all $t \in (t_2, t_3)$ and

$$V(x_0(t_3)) - V(x_0(t_1)) \leq -\epsilon.$$

Denote

$$q_0 = \min\left\{ |q| : \quad P(q) = \frac{H_* - h_3}{2} \right\}.$$

Suppose q_0' determined by the condition $|q_0'| = q_0, \mathrm{sign} q_0' = -\mathrm{sign} q(t_2)$. Consider a strip

$$\Upsilon = \{(q,p) : \ q \in (q(t_2), q_0')\}.$$

We claim that there exists an instant $t_3 > t_2$ such that $q(t_3) = q_0'$ and for all $t \in (t_2, t_3)$

$$\begin{array}{ll} i) & (q,p)(t) \in \Upsilon \\ ii) & |p(t)| > p_0. \end{array}$$

Put

$$K_0 = \frac{\partial K}{\partial p}(p_0) = \sqrt{\frac{D^2}{2\gamma\mu h_1^2}}.$$

Due to properties of $K(p)$ and continuity of $\dot{q}(t)$ it is sufficient to prove that

$$\begin{array}{ll} iii) & \mathrm{sign}\,\dot{q}(t_2) = -\,\mathrm{sign}\,q(t_2) \\ iv) & |\dot{q}| > K_0 \quad \text{on the strip } \Upsilon. \end{array}$$

From (5) and due to properties of $K(p)$ we have $\mathrm{sign}\,\dot{q}(t_2) = \mathrm{sign}\,p(t_2)$. Further, by definition of t_2 we get $\frac{d|p(t_2)|}{dt} > 0$; therefore, $\mathrm{sign}\,\dot{p}(t_2) = \mathrm{sign}\,p(t_2)$. Finally, from (6), (14), (15) and due to properties of $P(q)$ we get $\mathrm{sign}\,\dot{p}(t_2) = -\,\mathrm{sign}\,q(t_2)$. Combining the=20 foregoing, we obtain *iii*). Fulfillment of *iv*) follows from *iii*), (6), (14), (15), and from considerations of continuity.

Let us estimate $V(t_3) - V(t_1)$. We have

$$V(t_2) - V(t_1) \leq \alpha A \to 0 \quad \text{=D0=D2=C9} \quad D \to 0.$$

From the other hand

$$\begin{array}{c} V(t_3) - V(t_2) \leq \\ -\int\limits_{t_2}^{t_3} \gamma\,(H(t) - H_*)^2 \left(\frac{\partial H}{\partial p}\right)^2 (t)dt + \Delta^*(t_3 - t_2) \leq \\ -\int\limits_{t_2}^{t_3} \gamma \left((H(t) - H_*)^2 \left(\frac{\partial H}{\partial p}\right)^2 (t) - \frac{1}{2}h_1^2 K_0^2 \right) dt \leq \\ -\frac{1}{2}\gamma h_1^2 \int\limits_{t_2'}^{t_3} \left(\frac{\partial H}{\partial p}\right)^2 dt = -\frac{1}{2}\gamma h_1^2 \int\limits_{t_2'}^{t_3} \dot{q}^2(t)dt \leq \\ -\frac{1}{2}\gamma h_1^2 \frac{1}{t_3 - t_2'} \left(\int\limits_{t_2'}^{t_3} \dot{q}(t)dt \right)^2 = -2\gamma h_1^2 \frac{1}{t_3 - t_2'} (q_0)^2 . \end{array}$$

It is easy to see that

$$t_3 - t_2' \leq \frac{2q_0}{C},$$

where

$$C = \left(\frac{\partial H}{\partial p}\right)^{-1} \left(\frac{H_* - h_3}{2}\right).$$

Then

$$V(t_3) - V(t_2) \leq -\gamma h_1^2 q_0 C.$$

Let

$$\epsilon = \frac{1}{2}\gamma h_1^2 q_0 C > 0.$$

Then for sufficiently small $D > 0$ we get

$$V(t_3) - V(t_1) \leq -\epsilon < 0.$$

We see that all conditions of Lemma 1 are fulfilled. ¿From Lemma 1 it follows that there exist $D > 0$ s.t. for for any given initial condition $(q,p)(0) \in V^{-1}_{[0,v_2]}$ trajectories of closed loop system (5), (6), (8) satisfy $(q,p)(t) \in H^{-1}_{[0,v_3]}$ for all $t \geq 0$ and there exists $T > 0$ s.t. $(q,p)(t) \in H^{-1}_{[0,v_1]}$ for all $t \geq T$, where $v_1 = v_0 + A\Delta^*$, $v_2 = v_3 - A\Delta^*$. Since A, $\Delta^* \to 0$ as $D \to 0$, from (10), (11) we see that for sufficiently small $D > 0$

$$\begin{array}{rcl} V^{-1}_{[0,v_1]} & \subset & H^{-1}_{[h_1,h_1^*]}, \\ H^{-1}_{[h_2,h_2^*]} & \subset & V^{-1}_{[0,v_2]}. \end{array}$$

The statement of Theorem 1 follows.

4 Example: control of pendulum oscillations under disturbances

As an illustration of above results consider a problem of oscillation control of simple pendulum under bounded force disturbances. The pendulum is described by the following equation

$$J\ddot{\phi} + mgl\sin\phi = u + \Delta(t), \tag{18}$$

where ϕ is an angle of pendulum defined to be zero in the lower position, u is a controlling torque, J, m, and l are the inertia, mass, and length of pendulum respectively, g is an acceleration due to gravity, $\Delta(t)$ are input disturbances, $\Delta(t) \leq D$, where $D > 0$ is a constant.

The pendulum equation (18) can be represented in the form (5), (6) by choosing $q = \phi$, $p = J\dot{\phi}$ and

$$H(q,p) = \frac{1}{2J}p^2 + mgl(1 - \cos q). \tag{19}$$

For our purpose it is convenient to consider a cylinder with unit circle at the base $-\pi < q \leq \pi$ as a phase space of pendulum. It means that we identify the points (q_1, p) and (q_2, p) iff $q_2 - q_1 = 2\pi k$, where k is an integer number. Then it is easy to check that $H(q,p)$ given by (19) is proper function on cylindrical phase space. Further, the kinetic energy $K(p) = \frac{1}{2J}p^2$ is a smooth positive definite convex even function, and the potential energy $P(q) = mgl(1 -$

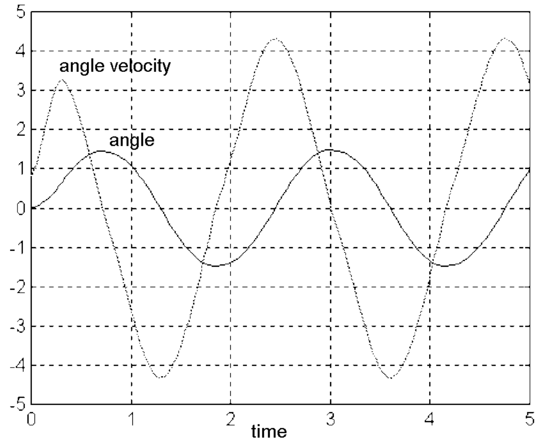

Figure 1: State variables of controlled pendulum

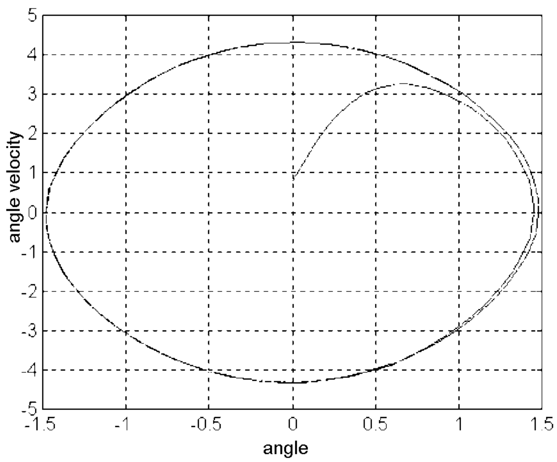

Figure 2: Phase portrait

$\cos q)$ is smooth function with strict local minimum at a point $q = 0$. Let $h = 2mgl$. Then the point $(0,0)$ is a unique equilibrium point of free system on X_h defined by (7). We see that the system (18) satisfy all necessary conditions for applying the Theorem 1. The control law (8) becomes

$$u = -\gamma_0 (H - H_*) p. \tag{20}$$

In particular, from Theorem 1 it follows that the control law (20) swing the pendulum (18) from any initial state $(q(t_0), p(t_0))$ satisfy $H(q(t_0), p(t_0)) \in (0, 2mgl)$ to an oscillation mode of given energy level $H_* \in (0, 2mgl)$ with prescribed accuracy if the disturbances are of sufficiently small level. Moreover during the swinging the pendulum does not turn into rotation mode.

The obtained results are confirmed by the extensive simulations. The example of the simulations is presented in Figures 1-3. The parameter values are $m = l = 1$, and $\gamma = 1$. The dry friction type disturbances are represented by $\Delta(t) = -\Delta_* \operatorname{sign} \dot{\phi}(t)$, $\Delta_* = 2$. The initial conditions are $\phi(0) = 0$, $\dot{\phi}(0) = 0.8$, and thus $H(0) = 0.32$. Energy level to be stabilized is $H_* = mgl = 9.81$. From the

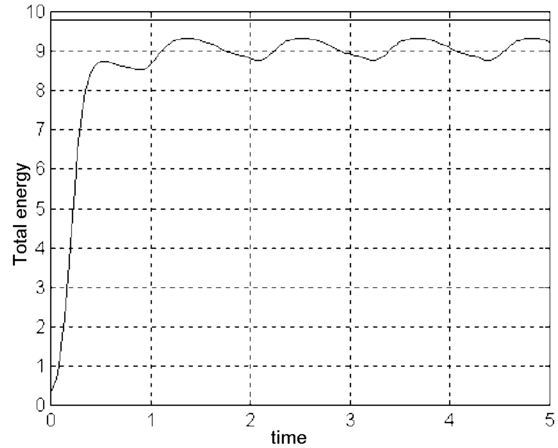

Figure 3: Total energy

Figures 1-3 it can be seen that the algorithm swings the pendulum up to oscillations with energy close to H_*.

5 Conclusions

In this paper we address to the problem of energy control of one-degree-of-freedom Hamiltonian system in presence of disturbances. Main result of the paper is Theorem 1. To prove this result we state an auxiliary Lemma 1 gives sufficient conditions for ultimate boundedness of Lyapunov function along the trajectories of nonlinear nonstationary system.

We call the reader's attention to a number of results recently obtained by D.Aeyels and J.Peuteman (see [7, 8]). The results are concerned with sufficient conditions for asymptotic as well as exponential stability without assumption of negative semidefiniteness of of Lyapunov function's time derivative. These results in some sense close in spirit to Lemma 1.

Acknowledgements

This research was partially supported by Russian Federal Program "Integration" (project 2.1-589), and by RFBR (grant 99-01-00672). The authors are grateful to Dr. Anton Shiriaev for useful discussions.

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Appendix. Proof of the formula (4)

Consider a function

$$W(t) = V_3 - \frac{\beta_0\epsilon_0}{\beta_0 A + \alpha A + \epsilon_1}(t - t_0 - A) - V(x(t)).$$

Condition (4) is equivalent to nonnegativity of $W(t)$ for all $t \in [t_0, t_0 + T]$. From $V(x(t_0)) \le v_2 = v_3 - \alpha A$ it follows that

$$W(t_0) \ge \frac{(\beta_0\epsilon_0 + \alpha)\,A}{\beta_0 A + \alpha A + \epsilon_1}.$$

Let $x(\cdot)$ be arbitrary trajectory of system ('2). If $x(t) \in \Pi$ for some $t \ge t_0$ then

$$\dot W \ge -\frac{\beta_0\epsilon_0}{\beta_0 A + \alpha A + \epsilon_1} + \beta_0 \ge 0.$$

Suppose $t_1 \ge t_0$ is the least instant of time such that $x(t_1) \notin \Pi$, and $[t_1, t_2]$ is the maximal interval such that $x(t) \notin \Pi$ for all $t \in [t_1, t_2]$. Then due to the condition of the Lemma $t_2 - t_1 \le A$ and

$$\dot W(t) \ge -\frac{\beta_0\epsilon_0}{\beta_0 A + \alpha A + \epsilon_1} - \alpha$$

for all $t \in [t_1, t_2]$. Further $t \in [t_1, t_2]$ we have

$$W(t) - W(t_1) \ge -\frac{(t - t_1)(\beta_0\epsilon_0 + \alpha)}{\beta_0 A + \alpha A + \epsilon_1} \ge -\frac{A\,(\beta_0\epsilon_0 + \alpha)}{\beta_0 A + \alpha A + \epsilon_1}.$$

Due to $W(t_1) \ge W(t_0)$ we get

$$W(t) \ge W(t_0) - \frac{A\,(\beta_0\epsilon_0 + \alpha)}{\beta_0 A + \alpha A + \epsilon_1} \ge 0$$

for all $t \in [t_1, t_2]$. Due to conditions of Lemma $W(t)$ increase at least on the interval $[t_2, t_3]$. Then $W(t) \ge W(t_2) \ge 0$ for all $t \in [t_2, t_3]$. Since

$$t_3 - t_2 \le \frac{\alpha A + \epsilon_1}{\beta_0},$$

we see that

$$W(t_3) - W(t_2) \ge \alpha A + \frac{\epsilon_0\beta_0 A}{\beta_0 A + \alpha A + \epsilon_1}.$$

It follows that

$$W(t_3) \ge W(t_1) \ge W(t_0).$$

We see that $W(t) \ge 0$ for all $t \in [t_0, t_3]$, and $W(t_3) \ge W(t_0)$. Note that $t_3 \ge t_3 - t_2 \ge \epsilon_0/M > 0$, where $M = \max\left\{|L_F V(x)|\colon x \in V^{-1}_{[v_0, v_3]}\right\}$. Continuing this line of reasoning, we see that $W(t) \ge 0$ on arbitrary finite interval of time.