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Swinging Up of Non-affine in Control Pendulum

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Abstract

The problems of global stabilization of any prescribed energy level of the pendulum and the upright unstable equilibrium point of the pendulum are solved. The proposed stabilizing algorithm for the upright equilibrium position is based on a VSS-like modification of the energy-speed-gradient method. The qualitative description of the transient behaviour of the closed loop system is obtained.

1 Introduction

The pendulum seems to be the most popular nonlinear mechanical system which first of all serves as a benchmark example in the development of the nonlinear control theory. One of the nontrivial control problems connected with the pendulum is global stabilization of a given periodic motion and global stabilization of the upright unstable equilibrium point. Different approaches to the solution of these problems are proposed, see [2, 5, 3, 12, 8] and others.

In [5] the problem of swinging up the pendulum served as an example for the proposed method of stabilizing a specified energy level $H(q,p) = H_*$ for general Hamiltonian systems. It is worth to point out that for the pendulum, the stabilization of a specified energy level exactly means the stabilization of a special trajectories of the unforced pendulum. The method was based on the so called speed-gradient approach [4]. It was shown that the proposed speed-gradient-energy algorithm stabilizes any specified energy level if the initial energy layer $H_0 \leq H(q,p) \leq H_*$ (or $H_0 \geq H(q,p) \geq H_*$), where $H_0 = H(q(0), p(0))$ does not contain equilibria. In [5] the conditions for almost global attractivity (i. e. attraction from all initial conditions except some set of zero Lebesgue measure) were given, however, the proof was incomplete. The results in [5] were extended to the more general problems of stabilization of the invariant sets for nonlinear systems [6, 9, 10, 11].

In the papers [1, 2] the speed-gradient-energy algorithms were used for driving the pendulum to the vicinity of the upper equilibrium. Near the upright point it was proposed to switch to another (locally stabilizing) algorithm (linear in [2] and VSS-like in [1]) to achieve global stabilization of the upper equilibrium.

In paper [10] a modification of the speed-gradient-energy algorithm was proposed to stabilize exactly the upright position of the affine in control pendulum with the pivot moving along an inclined straight line.

In this paper we extend the results of [5, 10] in two directions: first, the *non-affine in control pendulum* is considered; second, the *global stabilization* problem of any specified energy level of the unforced pendulum is attacked with special attention again to the upright position stabilization problem. The main ideas being utilized here are: the construction of, in some subtle way, the set of new state feedback stabilizing regulators, which provide dissipativity of the closed loop system; the detailed analysis of a stable manifold of the upright equilibrium point in the closed loop system; VSS-modification of the proposed regulators with the objective to stabilize not only the energy level corresponding to the upright position, but exactly the upright position.

The paper is organized as follows. Section 2 contains the assumptions and the preliminary results concerning the non-affine in control pendulum. In section 3 the main results of the paper, theorem 3.1 and theorem 3.5, are presented. The example and the results of the computer simulations are discussed in section 4. Some conclusions are made in section 5.

2 Preliminaries

The motions of a non-affine in control pendulum are described by the equations

$$\begin{cases} \dot{p}(t) &= -mgl \cdot \sin q(t) + R(q,p,u), \\ \dot{q}(t) &= \dfrac{1}{ml^2} p(t), \end{cases} \tag{2.1}$$

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where q, p are generalized coordinate and moment, u is a control action; m, l, g are mass, length of pendulum and acceleration of gravity respectively; R is a smooth function, $R(q,p,0) = 0 \ \forall q,p$. It is reasonable to take as a state space of pendulum motions a cylindrical phase space with a l-radius circle at the base, i. e. $-\pi < q \leq \pi$. Therefore the points (q_1, p) and (q_2, p) if $q_1 - q_2 = 2k\pi$ for some integer k will be identified identical and in the further analysis we will consider only the 2π-periodic in q control laws $u = \mathcal{U}(q,p)$.

The full energy (or Hamiltonian function) $H_0(q,p)$ of the unforced pendulum (i.e. with $u = 0$) has the form

$$H_0(q,p) = \frac{1}{2ml^2}p^2 + mgl \cdot (1 - \cos q) \qquad (2.2)$$

and is a conserved quantity of the unforced pendulum. Let H_* be any nonnegative constant and

$$R_1(q,p) = \left. \frac{\partial R(q,p,u)}{\partial u} \right|_{u=0},$$

introduce the scalar functions $V(q,p)$ and $y(q,p)$ as follows

$$V(q,p) = \frac{1}{2}[H_0(q,p) - H_*]^2, \qquad (2.3)$$

$$y(q,p) = \frac{1}{ml^2} \cdot p \cdot R_1(q,p) \cdot [H_0(q,p) - H_*] \qquad (2.4)$$

The functions V, y possess an important property. Namely, the system (2.1) with the output function (2.4) is *weakly passive* with the storage function V, see definition 3.2 [9].

Proposition 2.1 *Suppose that the set $\{[q,p] : R_1(q,p) = 0\}$ does not contain any whole trajectory of the unforced pendulum (2.1). Let $[q(t), p(t)]$ be any motion of the unforced pendulum (2.1) subjected to the constraint: $y(t) = 0$ for all $t \geq 0$. Then there exists two possible cases:*

1.) $H_0(q(t), p(t)) = H_*$ for all $t \geq 0$;

2.) $[q(t), p(t)] \equiv [0,0]$ or $[q(t), p(t)] \equiv [\pi, 0]$.

3 Main Results

3.1 Stabilization of a Given Energy Level

Denote the function $R_2(q,p,u)u^2 = R(q,p,u) - R_1(q,p)u$. By virtue of the smoothness of the function R and the property $R(q,p,0) = 0$ for all q, p (see lemma 3.1 p. 97 [7]) we deduce that R_2 is a smooth function. Let $\rho(q,p)$ be any smooth function satisfying the inequality

$$\max_{u \leq 1} |R_2(q,p,u)| \leq \rho(q,p). \qquad (3.1)$$

Theorem 3.1 *Suppose $H^* > 0$, the set $\{[q,p] : R_1(q,p) = 0\}$ does not contain any whole trajectory of the unforced pendulum (2.1) and $\phi(z)$ is any smooth scalar function such that*

$$z\phi(z) \geq \psi(z)|\phi(z)|^2 > 0, \quad |\phi(z)| \leq 1, \quad \forall z \neq 0, \qquad (3.2)$$

where $\psi(z)$ is some scalar function with $\psi(z) \geq \psi_0 > 0$ $\forall z$. Let $\alpha(q,p)$ be any scalar smooth function such that $\hat{\alpha}(q,p) \geq \alpha(q,p) > 0$, where

$$\hat{\alpha}(q,p) = \frac{\beta}{2 \cdot \left(1 + 2 \cdot V(q,p) \cdot \rho(q,p)^2 \cdot \frac{p^2}{m^2 l^4}\right)} \qquad (3.3)$$

and β is a positive constant, $\min\{\psi_0, 1\} > \beta > 0$. Take the regulator

$$u = \begin{cases} u^*, & \text{if } q = 0,\ p = 0 \\ -\alpha(q,p)\phi(y), & \text{otherwise} \end{cases} \qquad (3.4)$$

where u^* is any constant, such that $R(0,0,u^*) \neq 0$, and the function y is defined by (2.4). Then

1). for any solution $[q(t), p(t)]$ of the closed loop system (2.1), (2.4), (3.4) the following alternatives hold:

a). $\lim_{t \to +\infty} V(q(t), p(t)) = 0$;

b). the trajectory $[q(t), p(t)]$ tends to the upright equilibrium point $[\pi, 0]$;

2). if $H_* \neq 2mgl$ then the set of initial conditions $[q_0, p_0]$, for which the case 1b) takes place, is only one smooth curve γ^0 which is invariant with respect to the solutions of the closed loop system (2.1), (2.4), (3.4) and has no intersections. The set of ω-limit points of γ^0 is only the upright position. The set of α-limit points of γ^0 is either the equilibrium $[0,0]$ or empty.

Remark 3.2 Essentially theorem 3.1 means that for any prescribed value $H_* > 0$ the H_*-energy level of the unforced pendulum is a globally asymptotically stable set with the exception of the initial conditions which lie on some smooth curve γ^0 being invariant with respect to the closed loop system (2.1), (2.4), (3.4) vector field.

Remark 3.3 Due to theorem 3.1 the stabilizing regulator is determined by the collection

$$\{\phi(z),\ \psi(z),\ \psi_0,\ \alpha(q,p),\ \hat{\alpha}(q,p),\ \rho(q,p),\ \beta\}.$$

One of the possible choice of the triple $\phi(z)$, $\psi(z)$, ψ_0 is the following

$$\phi(z) = \frac{z}{1+z^2}, \quad \psi(z) = 1 + z^2, \quad \psi_0 = 1.$$

It is also worth to mention that for any upper bound $\varepsilon > 0$ among the suggested regulators (3.2)–(3.4) there exists a regulator with $|u(q,p)| \leq \varepsilon$, $\forall q$, p.

3.2 Stabilization of the Unstable Equilibrium Point of the Pendulum

For the case $H^* = 2mgl$ theorem 3.1 provides the stabilizability of the $2mgl$-energy level of the pendulum which consists of two homoclinic curves and the upright equilibrium point. It means that the point $[\pi, 0]$ is a unique ω-limit point of any motion of the unforced pendulum belonging to the $2mgl$-energy level. In particular, this implies that the upright position will be an ω-limit point of any solution of the closed loop system (2.1), (2.4), (3.4). But theorem 3.1 does not guarantee that the point $[\pi, 0]$ will be a unique ω-limit point and that any trajectory of (2.1), (2.4), (3.4) remains in some neighbourhood of the upright position for all sufficiently large moments of time. Indeed, as it is stated in the next proposition, the solution of (2.1), (2.4), (3.4) will also pass into rotations with the infinitely damping to zero function $V = 1/2[H_0(q,p) - 2mgl]^2$. See also the results of the computer simulations.

Proposition 3.4 *Suppose $H^* = 2mgl$, the set $\{[q,p] : R_1(q,p) = 0\}$ does not contain any trajectory of the unforced pendulum (2.1) and $\alpha(q,p)$, $\phi(z)$ are any functions satisfying all the assumptions of theorem 3.1. Let $[q_0, p_0]$ be any point of the cylindrical phase space with $H_0(q_0, p_0) \neq 2mgl$. Consider the solution $[q(t), p(t)] = [q(t, q_0), p(t, p_0)]$ of the closed loop system (2.1), (2.4), (3.4). Then the set Ω of all ω-limit points of $[q(t), p(t)]$ coincides with either $[\pi, 0] \cup \Gamma_1$, or $[\pi, 0] \cup \Gamma_2$, or $[\pi, 0] \cup \Gamma_1 \cup \Gamma_2$, where Γ_1, Γ_2 are homoclinic curves of the unforced pendulum (2.1).*

The next statement contains a simple modification of theorem 3.1 which leads to the stabilization of the upright equilibrium point in a sense that this point will be a *unique* ω-limit point of any closed loop system trajectory.

Theorem 3.5 *Suppose that ε is a positive constant, $0 < \varepsilon < 2mgl$, $\phi(z)$ is any function satisfying all the assumptions of theorem 3.1 and that the set $\{[q,p] : R_1(q,p) = 0\}$ does not contain any whole trajectory of the unforced pendulum (2.1). Let $\alpha_{+\varepsilon}(q,p)$, $\alpha_{-\varepsilon}$ be any scalar smooth functions such that*

$$\hat{\alpha}_{+\varepsilon}(q,p) \geq \alpha_{+\varepsilon}(q,p) > 0, \quad \hat{\alpha}_{-\varepsilon}(q,p) \geq \alpha_{-\varepsilon}(q,p) > 0,$$

where

$$\hat{\alpha}_{+\varepsilon}(q,p) = \frac{\beta/2}{1 + [H_0(q,p) - 2mgl + \varepsilon]^2 \rho(q,p)^2 \frac{p^2}{m^2 l^4}}$$

$$\hat{\alpha}_{-\varepsilon}(q,p) = \frac{\beta/2}{1 + [H_0(q,p) - 2mgl - \varepsilon]^2 \rho(q,p)^2 \frac{p^2}{m^2 l^4}}$$

and β is a positive constant, $\min\{\psi_0, 1\} > \beta > 0$. Then the regulator

$$u = \begin{cases} u^*, & \text{if } q = 0,\ p = 0 \\ 0, & \text{if } H_0(q,p) = 2mgl \\ -\alpha_{+\varepsilon}(q,p)\phi\left(\dfrac{pR_1(q,p)}{ml^2}[H_0(q,p) - 2mgl + \varepsilon]\right), & \\ \qquad \text{if } H_0(q,p) > 2mgl & \\ -\alpha_{-\varepsilon}(q,p)\phi\left(\dfrac{pR_1(q,p)}{ml^2}[H_0(q,p) - 2mgl - \varepsilon]\right), & \\ \qquad \text{if } H_0(q,p) < 2mgl & \end{cases} \tag{3.5}$$

where u^* is any constant such that $R(0,0,u^*) \neq 0$, globally stabilizes the unstable equilibrium point $[q,p] = [\pi, 0]$ of the unforced pendulum (2.1).

Remark 3.6 The closed loop system (2.1), (3.5) is a nonlinear system with discontinuous right side in the points of switching of the control function. Thus the question of the definiteness of the solution of the system can arise. It follows from the proof of theorem 3.5 that for any initial condition the solution of the system (2.1), (3.5) is an absolutely continuous function, which is well defined on $[0, +\infty)$, and the derivative of this function is a piece-wise smooth function, which can have no more than one point of discontinuity.

4 Example and Computer Simulations

Consider as an example the following non-affine in control model of pendulum

$$\begin{cases} \dot{p} &= -mgl \cdot \sin q + ml \cdot \sin u \\ \dot{q} &= \dfrac{1}{ml^2} \cdot p \end{cases} \tag{4.6}$$

For this case

$$R_1(q,p) = \left.\frac{\partial (ml \cdot \sin u)}{\partial u}\right|_{u=0} = ml.$$

Let H^* be any positive constant, then the functions V, y defined by (2.3), (2.4) take the form

$$\begin{aligned} V(q,p) &= \frac{1}{2}[H_0(q,p) - H^*]^2, \\ y(q,p) &= \frac{1}{l} \cdot p \cdot [H_0(q,p) - H^*]. \end{aligned} \tag{4.7}$$

By theorem 3.1 the H^*-energy level of the unforced pendulum is globally (except some smooth curve of initial conditions) stabilized by any regulator (3.4) satisfying the relations (3.2), (3.3) with the function $\rho(q,p)$ being identically equal to 1.

Indeed, let us show that $|R_2(u)| \leq 1$ if $|u| \leq 1$, where the function $R_2(u)$ is defined by the equation $u^2 R_2(u) = \sin u - u$. $R_2(u)$ is an odd function so we can assume that $0 \le u \le 1$. First we show that $u = 0$ is a removable singularity of $R_2(u) = (\sin u - u)/u^2$. Taking the second derivative of the numerator and the denominator at the zero point we have

$$\left.\frac{(\sin u - u)''}{(u^2)''}\right|_{u=0} = \left.\frac{-\sin u}{2}\right|_{u=0} = 0.$$

Thus the function $|R_2(u)|$ is bounded by 1 on some neighbourhood of the origin, and can be defined as zero in $u = 0$. Consider the auxiliary function $f(u) = \sin u - u - u^2$. Taking the derivative of this function, $f'(u) = \cos u - 1 - 2u$, we obtain that the points of extremum of f on the interval $[0, 1]$ are only 0 and 1, and $f(0) = 0$, $f(1) < 0$. Hence, $f(u) \le 0$ with $0 \le u \le 1$ and, therefore, $R_2(u) = (\sin u - u)/u^2 \le 1$ with $0 \le u \le 1$.

In particular, the regulator (3.4) with the coefficients $\phi(z) = \frac{1}{\pi}\arctan(z)$, $\psi(z) = \psi_0 = 1$,

$$\alpha = \hat{\alpha} = \frac{\beta}{2\left(1 + \frac{2V(q,p)p^2}{m^2 l^4}\right)},$$

where β is any constant, $0 < \beta < 1$, and with the function y defined by (4.7) stabilizes the H^*-energy level of the unforced pendulum. An example is shown in figure 1. The parameters in the simulation were: $m = 1\,\mathrm{kg}$, $l = 4\,\mathrm{m}$, $g = 9.8\,\mathrm{m/s}^2$, $\beta = 0.99$ and $H^* = 60\,\mathrm{Nm}$, and the initial conditions $(q(0), p(0)) = (2.1, 0.7)$

Using the same control law and stabilizing the $H^* = 2mgl$ energy level, we find that for the initial conditions $(q(0), p(0)) = (2.4, 1.0)$, the closed loop system is rotating and along the solution the value of the Hamiltonian function infinitely tends to the energy level corresponding to the upright position as shown in figure 2

The modified control law (3.5) with $\epsilon = 1$ is shown in figure 3. The pendulum now converges to the upright position for the same system parameters.

5 Conclusions

In this paper the energy level stabilization problem and the upright position stabilization problem of the non-affine in control pendulum are solved. Compared to the results in [5] the presented algorithms ensure global (except some smooth curve γ^0) stabilization, while in [5] energy stabilization problem was solved only for the initial conditions from some neighbourhood of the desired attractor with the additional assumptions that this neighbourhood is invariant with respect to the motions of the unforced pendulum and that it does not contain any equilibriums of the unforced pendulum.

The stabilization of the upright unstable equilibrium point of the unforced pendulum is derived by a VSS-like modification of the regulators being used for the stabilization of any specified energy level. It is shown that this modification provides that the upright position becomes the *unique* ω-limit point of any closed loop system solution. A particular interest in the paper is drawn to the qualitative analysis of the transient behaviour of the closed loop system.

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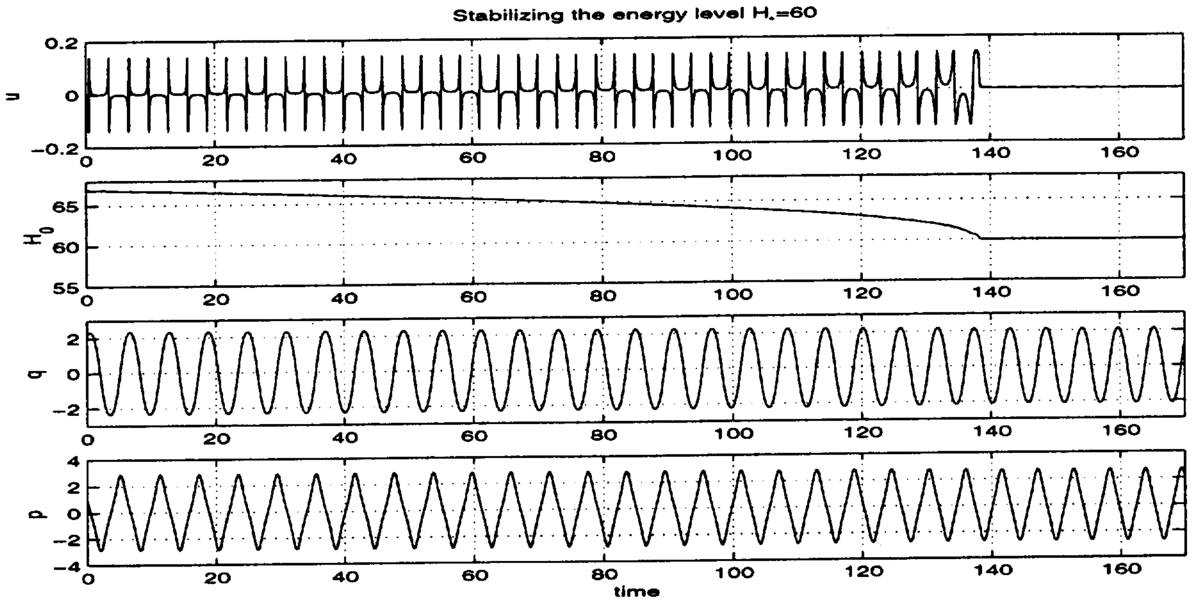

Figure 1: Convergence to a prescribed energy level

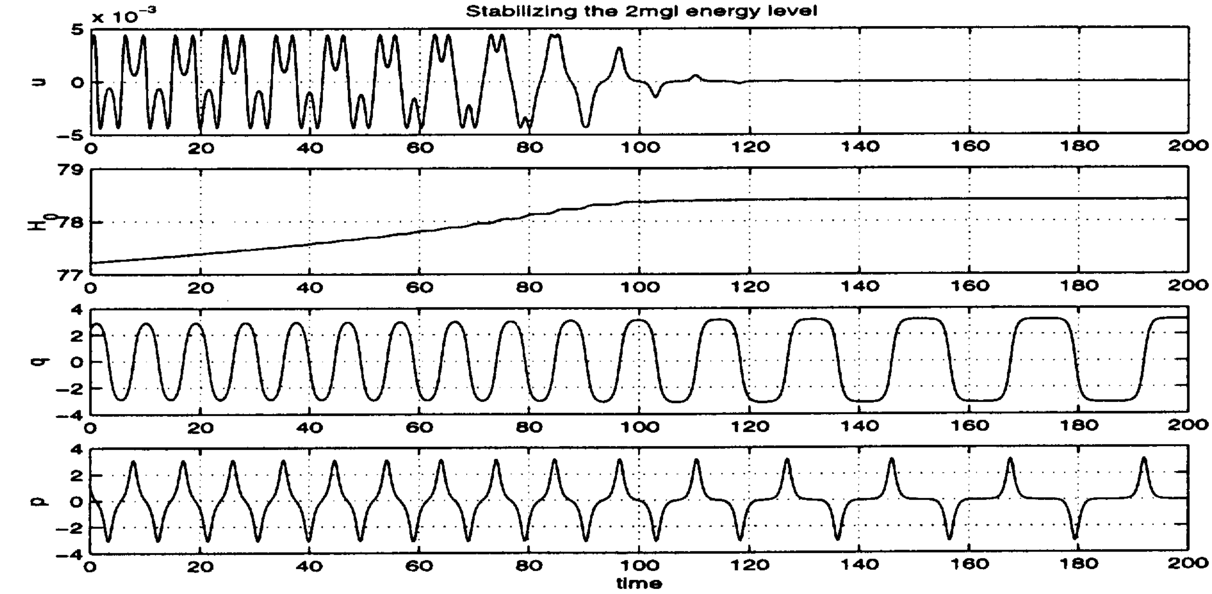

Figure 2: Infinite rotations of the pendulum

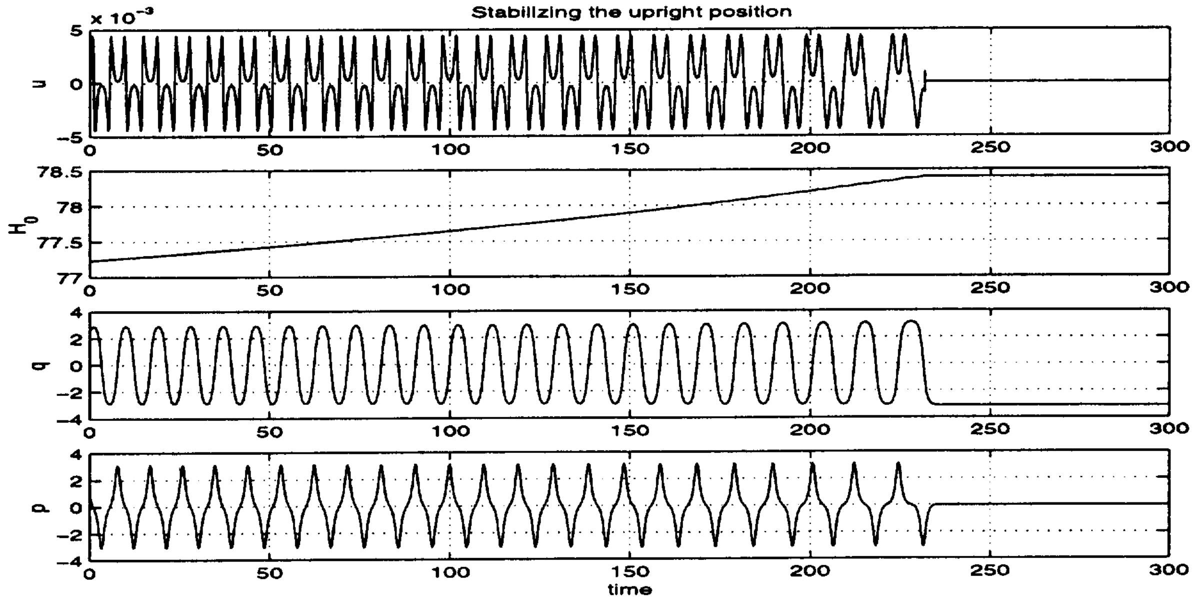

Figure 3: Convergence to the upright position