A non-quadratic criterion for stability of forced oscillations and its application to flight control

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***Abstract*— A new test for stability of forced oscillations in nonlinear systems is applied to a marginally stable plant of the second order with a saturated PID-control and a harmonic reference signal. The range of its frequencies and amplitudes that guarantee stability of the forced oscillations is found. This result is used for stability analysis of an aircraft roll angle control system.**

***Index Terms*— nonlinear systems, time-varying systems, Lyapunov stability, incremental stability, convergent systems, flight control**

I. INTRODUCTION

The presence of actuator saturation in an otherwise linear closed-loop system can dramatically degrade the system performance. Since the feedback loop is broken when the actuator saturates the unstable modes of the regulator may then drift to undesirble values. The consequences are that undesired nonlinear oscillations appear and that the settling time may inadmissibly increase. This phenomenon is known as "integrator windup" [1], [2].

A terrifying illustration of this detrimental effect is given by the pilot-induced oscillations that entailed the *YF-22* crash in April 1992 and *Gripen* crash in August 1993 [3], [4].

Avoiding, as much as possible, occurrence of saturation during operation is an attractive task for many engineering applications since this may lead to reduction of validation costs of control laws, better use of actuators/sensors capacity, and reducing their size, mass and power consumption [5].

Various "anti-windup" techniques were proposed for preventing performance degradation due to input signal saturation, see, e.g. a survey [6] and monorgaphs [2], [5], [7]. Particularly, there is a plenty of works where the aeronautical applications of anti-windup strategies are addressed, see e.g. [8]–[12]. Typically, the anti-windup compensators are designed based on a linearized model of aircraft dynamics with taking into account only the nonlinear effects of the magnitude and rate saturation in the actuator. However, the aircraft dynamics are in fact essentially nonlinear due to nonlinear kinematics of the Euler angles, strong coupling of rotations around various axes, and nonlinear dependence of the airflow direction on the aircraft orientation. The coefficients of aerodynamic forces and moments are also highly nonlinear functions of the angle of attack, sideslip angle, airspeed, angular rates, and control surface deflections. Small perturbations in airspeed, steering angle or airflow direction may lead to significant differences in the aircraft performance. Under these circumstances, the anti-windup augmentation may fail in the real-world conditions. Traditionally the stability of forced oscillations is analyzed by the incremental version of the circle criterion, see, e.g. [13], [14], which is based on quadratic Lyapunov functions. In the literature, there are many examples of systems which do not satisfy the circle criterion and, despite that their free motion is asymtotically stable, they demonstrate the unstable steady state behavior being affected by a periodic reference signal [12], [15]–[18]. Moreover, the overall potential of incremental Lyapunov functions is strongly challenged by a recent counter example proposed by D. Angeli [19]. He showed that the Lyapunov function that proves incremental stability of a nonlinear system is not necessarily a function the difference of two trajectories of the system. So to advance to solution of the problem, an appropriate class of Lyapunov functions should be found that on the one hand, is general enough to capture the real phenomena and on the other hand, is simple enough to make the calculations possible.

The paper is organized as follows. A novel non-quadratic criterion for stability of forced oscillations is presented in Sec. II. Application of this criterion to stability analysis of the second order marginally stable plant driven by a PID-controller is given in Sec. III. Numerical example of the aircraft roll angle control is presented in Sec. IV. Concluding remarks and the future work intensions are given in Sec. V.

II. A NON-QUADRATIC CRITERION FOR STABILITY OF FORCED OSCILLATIONS

Consider the system of ordinary differential equations

$$\dot{x} = f(x,t), \qquad x \in \mathbb{R}^n \tag{1}$$

whose right-hand-side satisfies the following.

Assumption 1: The function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipshitz continuous in (x,t) and there exists a closed (maybe

Partially supported by the Ministry of Education and Science of Russian Federation, Saint Petersburg State University, Academy of Finland, and RFBR, Proj. No 12-08-01183, 12-01-00808.

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empty) set $S \subset \mathbb{R}^n \times \mathbb{R}$ such that the following two claims hold:

i) The function f is continuously differentiable everywhere on the complement $C := \mathbb{R}^n \times \mathbb{R} \setminus S$ to S;

ii) For any solution $x(t)$ of (1), the set of times t when $[x(t), t] \in S$ has the zero Lebesgue measure.

This assumption necessarily holds if the function f is continuously differentiable everywhere on $\mathbb{R}^n \times \mathbb{R}$ (then $S := \emptyset$). Claim ii) holds whenever S is a surface (i.e., n-dimensional embedded manifold) and the vector $[f(x,t); 1]$ is not tangential to S for all $(x,t) \in S$. A more general and sophisticated sufficient condition is offered by Lemma 2 in [16].

Due to the Lipshitz condition, the solution of the Cauchy problem with initial data x_0, t_0 is unique; this solution defined on the maximal interval is denoted by $x(t, t_0, x_0)$. For the sake of brevity if no confusion occurs, this solution will also be referred to as $x(t)$. Here the dependence on the initial time or state may be omitted if they are apparent from the context.

We recall that a continuous strictly increasing function $\kappa : [0,a) \to [0,\infty)$ such that $\kappa(0) = 0$ is called a *$\mathscr{K}$-function*, and a *$\mathscr{K}_\infty$-function* if in addition $a = \infty$ and $\kappa(r) \to \infty$ as $r \to \infty$ [20]. A solution $x_*(t)$ of (1) is said to be *globally $\mathscr{K}$-exponentially stable* [21] if it is defined on a time interval Δ that extends to $+\infty$ and there exists a $\mathscr{K}$-function κ defined on $[0,\infty)$ and a real $\beta > 0$ such that for arbitrary $t_0 \in \Delta$ and $x_0 \in \mathbb{R}^n$,

$$||x(t, t_0, x_0) - x_*(t)|| \leq \kappa(\|x_0 - x_*(t_0)\|) e^{-\beta(t-t_0)} \qquad (2)$$

for all $t \geq t_0$ from the domain of definition of $x(t, t_0, x_0)$. It should be noted that then the solution $x(t, t_0, x_0)$ does not blow up and is thus defined for all $t \geq t_0$ [22]. Accordingly, (2) in fact holds for all $t \geq t_0$. It is worth noting that estimate (2) implies that *all* solutions are globally $\mathscr{K}$-exponentially stable, however, as it will be shown, there is a unique solution that is bounded for positive and negative time.

The following theorem is derived in [16].

Theorem 1: Suppose that there exist a symmetric positive definite $n \times n$-matrix P, a continuous on $\mathbb{R}^n \times \mathbb{R}$ and continuously differentiable on C function $w : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$, a symmetric matrix valued function $Q : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^{n \times n}$, and a positive number $\varepsilon > 0$ such that the following statements hold:

i) The function $V(x,t) := \exp[w(x,t)] f(x,t)^{\mathrm{T}} P f(x,t)$ goes to ∞ uniformly over $t \in (-\infty, \infty)$ as $\|x\| \to \infty$: there exist $\mathscr{K}_\infty$-functions $\gamma_1, \gamma_2 : \mathbb{R}^n \to \mathbb{R}$ and a constant $c \geq 0$ such that

$$\gamma_1(||x||) - c \leq V(x,t) \leq \gamma_2(||x||) + c \qquad \forall x, t;$$

ii) There exist a continuous positive definite function $\gamma_3 : \mathbb{R}^n \to \mathbb{R}$ and a constant $c' > 0$ such that $(x,t) \in C$ and $V(x,t) \geq c'$ imply the following inequality

$$-\varepsilon V(x,t) + 2e^{w(x,t)} f(x,t)^{\mathrm{T}} P \frac{\partial f}{\partial t}(x,t) \leq -\gamma_3(x); \qquad (3)$$

iii) For any $(x,t) \in C$ the following matrix inequality is true:

$$PJ(x,t) + J^{\mathrm{T}}(x,t)P \leq Q(x,t), \quad \text{where } J(x,t) = \frac{\partial f}{\partial x}(x,t);$$

iv) For any $(x,t) \in C$, the largest solution $\Lambda(x,t)$ of the algebraic equation

$$\det[Q(x,t) - \lambda P] = 0$$

satisfies the following inequality

$$\Lambda(x,t) + \dot{w}(x,t) \leq -\varepsilon; \qquad (4)$$

v) The function $w(x,t)$ remains bounded as $t \to \pm\infty$ and x ranges over a bounded set: $W(a) := \sup_{t \in \mathbb{R}, \|x\| \leq a} |w(x,t)| < \infty$ for any $a \in [0, \infty)$.

Then the system (1) possesses a unique solution $\bar{x}(t)$ that is defined and bounded on $(-\infty, +\infty)$, and this solution is globally $\mathscr{K}$-exponentially stable. If the right-hand side of (1) is periodic in t, this solution is also periodic with the same period.

In the particular case where $w(x,t) = \text{const}$, the theorem is tantamount to the classical quadratic convergence test, see, e.g. [13], [23]. To clarify the assumptions of Theorem 1, we note that iii) and iv) mean nothing but existence of a symmetric positive definite matrix P such that

$$PJ(x,t) + J^{\mathrm{T}}(x,t)P + \dot{w}P < 0$$

uniformly in x, t. The role of the (averaging) function w is to allow the sum of the first two terms in this inequality to be positive for some $(x;t)$: the overall inequality can be kept true thanks to negative and large enough $\dot{w}$. Averaging functions first appeared in the studies of dimensions of attractors of dynamical systems, see, e.g. [24], [25] and references therein, and in estimation of the topological entropy of dynamical systems [26], [27]. The role of Assumptions i) and ii) is to ensure that the solutions of (1) are bounded. By and large, i) and ii) can be replaced with any other conditions that ensure the same.

III. Stability of forced oscillations for PID-controlled marginally stable plant of the second order

In this section, we apply Theorem 1 to the system depicted on Fig. 1 and described by the following equations

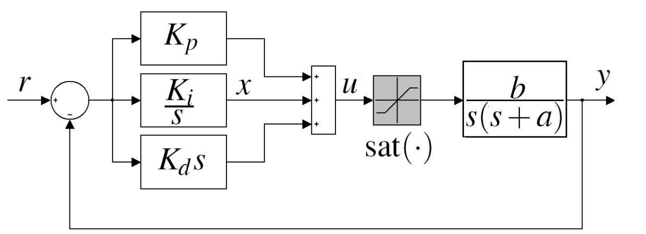

Fig. 1. Block diagram of the closed-loop system with the second order marginally stable plant and PID-controller ($s \in \mathbb{C}$ – Laplace transform variable).

$$\begin{cases} \dot{x} = -y + r(t), \\ \dot{z} = -az + b \,\mathrm{sat}\, u, \\ \dot{y} = z, \end{cases} \qquad (5)$$

$$u = K_i x - K_d z - K_p y + K_p r(t) + K_d \dot{r}(t),$$

where K_p, K_i, K_d, a, b are positive parameters and $r(t) = r_m \sin \omega t$ is the external reference action.

Let us consider the following scalar function

$$W = \gamma \frac{z^2}{2} + \frac{K_i}{b} zy + \frac{K_i a}{2b} y^2 + \int_0^u \operatorname{sat} \xi \, d\xi$$

with the parameter γ to be specified later on. It is easy to see that the time derivative of W is given by

$$\dot{W} = W_1 + (K_d \ddot{r} + K_p \dot{r} + K_i r) \operatorname{sat} u$$

where

$$W_1 = \left(-\gamma a + \frac{K_i}{b} \right) z^2 + (\gamma b - K_p + K_d a) z \operatorname{sat} u - K_d b (\operatorname{sat} u)^2.$$

Given L, the expression

$$W_1 + L(\operatorname{sat} u)^2 \tag{6}$$

is clearly nonpositive whenever $ab\gamma > K_i$ and

$$L \le K_d b - \frac{b}{4} \frac{(\gamma b - K_p + K_d a)^2}{\gamma ab - K_i}.$$

By picking γ as the maximizer of the right-hand side, we conclude that (6) is nonpositive if

$$L = K_d b, \gamma = \frac{K_p - K_d a}{b}$$

in the case where $K_p a - K_i - K_d a^2 \ge 0$, and

$$L = \frac{b}{a} \left(K_p - \frac{K_i}{a} \right), \gamma = \frac{2K_i - K_p a + K_d a^2}{ab}$$

in the case where $K_p a - K_i - K_d a^2 < 0$. Then the following dissipation inequality holds

$$\dot{W} \le -L(\operatorname{sat} u)^2 + (K_d \ddot{r} + K_p \dot{r} + K_i r) \operatorname{sat} u$$

and $L > 0$ if

$$K_p - \frac{K_i}{a} > 0. \tag{7}$$

which is assumed from now on. Since the signal r is harmonic, we have

$$K_d \ddot{r} + K_p \dot{r} + K_i r = B \sin(\omega t + \phi)$$

for some ϕ and

$$B = r_m \sqrt{(K_i - K_d \omega^2)^2 + K_p^2 \omega^2}. \tag{8}$$

Differentiating the scalar function

$$w_0(x, y, z, t) = W + \frac{B^2}{16 \omega L} \sin 2(\omega t + \phi).$$

with respect to time yields

$$\begin{aligned} \dot{w}_0 &\le L(\operatorname{sat} u)^2 + B \sin(\omega t + \phi) \operatorname{sat} u - \frac{B^2}{4L} \sin^2(\omega t + \phi) + \frac{B^2}{8L} \\ &= \frac{B^2}{8L} - \left(\sqrt{L} \operatorname{sat} u - \frac{B}{2\sqrt{L}} \sin(\omega t + \phi) \right)^2 \\ &\le \frac{B^2}{8L} =: B_L. \end{aligned} \tag{9}$$

If $B < 2L$ this estimation can be differently completed in the saturated mode where $\operatorname{sat} u = \pm 1$:

$$\dot{w}_0 \le \frac{B^2}{8L} - \left(\sqrt{L} - \frac{B}{2\sqrt{L}} \right)^2 = -L + B - \frac{B^2}{8L} =: B_S. \tag{10}$$

Now let us verify the crucial assumptions iii) and iv) of Theorem 1 for the function $w = 2\mu w_0$ where μ is a positive parameter. The other assumptions are technical and can be checked similar to [16]. The Jacobi matrices of f in the linear J_L and saturated J_S modes are

$$J_L = \begin{bmatrix} 0 & 0 & -1 \\ bK_i & -bK_d - a & -bK_p \\ 0 & 1 & 0 \end{bmatrix}, \quad J_S = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -a & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{11}$$

For a positive definite matrix P, we denote by Λ_L^P and Λ_S^P the largest solution to the algebraic equation $\det(PJ + J^{\mathrm{T}} P - \lambda P) = 0$ with $J := J_L$ and $J := J_S$, respectively. So for iii) and iv) from Theorem 1 to be true, it suffices that

$$\Lambda_L^P + 2\mu B_L < 0, \quad \Lambda_S^P + 2\mu B_S < 0, \quad B < 2L. \tag{12}$$

Now we note that the largest solution λ to the equation $\det(PJ + J^{\mathrm{T}} P - \lambda P) = 0$ is the least λ such that $PJ + J^{\mathrm{T}} P \le \lambda P$. So to establish (12) it suffices to ensure that

$$PJ_L + J_L^{\mathrm{T}} P + 2\mu B_L P < 0, \quad PJ_S + J_S^{\mathrm{T}} P + 2\mu B_S P < 0. \tag{13}$$

This system of inequalities has a positive definite solution P if and only if the following two matrices are not only both Hurwitz

$$J_L + \mu B_L I, \qquad J_S + \mu B_S I \tag{14}$$

but also have a common quadratic Lyapunov function. We emphasize that this is possible provided $B_s < 0$ since J_L is marginally unstable.

By summarizing and applying Theorem 1, we arrive at the following.

Proposition 1: Let for the system (5) with the harmonic input $r(t) = r_m \sin \omega t$ and positive parameters K_p, K_i, K_d, a, b inequality (7) be true and $B < 2L$ where B is given by (8) and

$$L = \begin{cases} K_d b, & \text{if } K_p a - K_i - K_d a^2 \ge 0 \\ \frac{b}{a} \left(K_p - \frac{K_i}{a} \right), & \text{otherwise} \end{cases} \tag{15}$$

Suppose also that for some $\mu > 0$ there exists a positive definite matrix P satisfying (13) where B_L, B_S are defined in (9), (10) respectively. Then the system possesses a unique solution that is defined and bounded on the entire $(-\infty, +\infty)$, and this solution is $2\pi/\omega$-periodic and globally $\mathscr{K}$-exponentially stable.

IV. Application example. Stability of forced oscillations of aircraft attitude control system

A. Modeling the aircraft lateral attitude motion

Let us use the following model of the aircraft lateral motion, presented in [28]:

$$\begin{bmatrix}\dot\beta\\ \dot p\\ \dot r\\ \dot\Phi\end{bmatrix} = \begin{bmatrix} Y_\beta & \sin\alpha^* & \cos\alpha^* & Y_\Phi\\ L_\beta & L_p & L_r & 0\\ N_\beta & N_p & N_r & 0\\ 0 & 1 & -\tan\theta^* & 0\end{bmatrix}\begin{bmatrix}\beta\\ p\\ r\\ \Phi\end{bmatrix} + \begin{bmatrix} Y_{\delta a} & Y_{\delta r}\\ L_{\delta a} & L_{\delta r}\\ N_{\delta a} & N_{\delta r}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\delta_a\\ \delta_r\end{bmatrix}, \tag{16}$$

where $\beta(t)$, $\psi(t)$, $\Phi(t)$ are the sideslip, yaw, and roll angles, respectively, $r(t)$ is the yaw rate, $p(t)$ is the roll rate; α^*, θ^* are the trimming attack and pitch angles. Control inputs are aileron, δ_a, and rudder, δ_r deflections.

We are interesting in the roll motion in isolation. Extraction of the pure roll dynamics model from (16) is possible provided that the yaw rate is sufficiently small (a "sluggish" yaw maneuvering) and the sideslip angle is close to zero, $\beta \approx 0$. Neglecting the corresponding terms in (16) leads to the following simplified roll model:

$$\ddot\Phi(t) + a\dot\Phi(t) = b\delta_a(t), \tag{17}$$

where the model parameters a, b are calculated based on the numerical values given in [28]: $a = 0.65\ \mathrm{s}^{-1}$, $b = 2.44\ \mathrm{s}^{-2}$. We assume that the aileron deflection δ_a is bounded by $\bar\delta_a = 0.35$ rad (20^o) and neglect the aileron servo dynamics since they are usually fast. To simplify the exposition, the servo gain is referred to as a common open-loop gain and is taken to be one. This leads to the following aileron servo model:

$$\delta_a = \bar\delta_a \,\mathrm{sat}\left(\frac{u}{\bar\delta_a}\right), \tag{18}$$

where u denotes the controller output signal.

B. Nominal controller design

For the linear system (17) with the output Φ, we consider the tracking problem for a differentiable reference signal $\Phi^*(t)$. To start with, we design a nominal controller by neglecting the constraint (18). The controller has to eliminate the steady-state error even in case of a linearly growing in time signal $\Phi^*(t)$. To comply with this demand, simultaneously with achieving the desired transient performance, the following PID control law is employed

$$\begin{aligned} &u(t) = K_p\Delta\Phi(t) + K_i\sigma(t) + K_d\dot{\Delta}\Phi(t)\\ &\dot\sigma(t) = \Delta\Phi(t),\\ &\Delta\Phi(t) = \Phi^*(t) - \Phi(t), \end{aligned} \tag{19}$$

where $\Phi^*(t)$ is the roll reference signal, K_p, K_i, K_d are the controller gains. To satisfy the specifications on the transient behavior, these gains were chosen so that the characteristic polynomial of the closed-loop system $D(s)$ is the third order Butterworth polynomial $D(s) = s^3 + 2\Omega s^2 + 2\Omega^2 s + \Omega^3$, where the parameter Ω determines the closed-loop system transient time. Specifically, the standard pole-placement technique gives the following gains:

$$K_d = (2\Omega - a)b^{-1}, \quad K_p = 2\Omega^2 b_0^{-1}, \quad K_i = \Omega^3 b_0^{-1}. \tag{20}$$

Here Ω is not arbitrary: as follows from (7), (20)), $\Omega < 2a$. This inequality restricts an agility of the closed-loop system for a given plant. For the considered example, $\Omega = 1.75a = 4.4\ \mathrm{s}^{-1}$ is chosen, which results in the following values of the controller gains: $K_i = 0.60\ \mathrm{s}^{-1}$, $K_p = 1.1$, $K_d = 0.66$ s. The step responce of the nominal (non-saturated) system (17)–(19) for $\Phi^* = 15^o$ is depicted in Fig. 2.

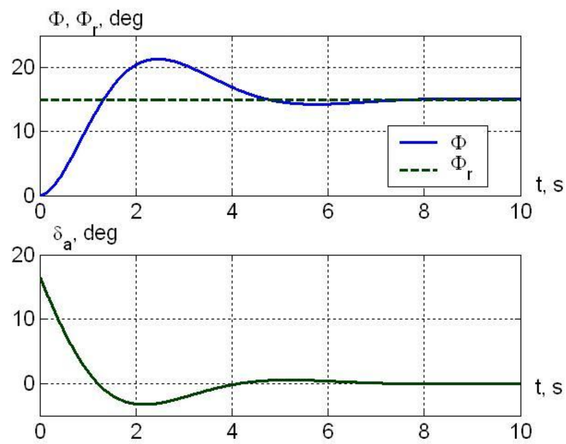

Fig. 2. Transients in the system (17)–(19) in non-saturated mode.

C. Analysis of the periodic motion stability

To apply the results of Sec. III to the aircraft control system, Eqs. (17)–(19) should be represented in the form (5), where the control input is saturated at the level one (instead of $\bar\delta_a$, as in (18)). To this end, the control gains should be divided by $\bar\delta_a$, whereas the plant parameter b should be multiplied by $\bar\delta_a$ (in the considered case, $\bar\delta_a \approx 0.35$). Overall, the following parameter substitution is performed: $K_i := K_i\bar\delta_a^{-1} = 1.72$, $K_p := K_p\bar\delta_a^{-1} = 3.15$, $K_d := K_d\bar\delta_a^{-1} = 1.90$, $b := b\bar\delta_a = 0.85$.

For $K_i = 1.72$, $K_p = 3.15$, $K_d = 1.90$, $b = 0.85$, $a = 0.65$ it follows that $L = \dfrac{b}{a}\left(K_p - \dfrac{K_i}{a}\right) = 0.66$. Let us pick up $B = 0.6$ to meet the requirement $B < 2L = 1.33$. The problem of finding μ such that the matrices (14) have a common quadratic Lyapunov matrix P can be solved numerically, e.g. with YALMIP toolbox [29]. With this matrix at hand, it is easy to see that the conditions imposed in Proposition 1 are fulfilled and so its conclusion is true for system (5). For the above numerical values of system parameters and $B = 0.6$, $\mu = 9$ the LMI problem is feasible and have the following matrix P found by YALMIP (release 2012.09.26) and SDPT3 (version 4.0) [30] packages:

$$P = \begin{bmatrix} 20.4 & -10.8 & -24.1 \\ -10.8 & 12.1 & 18.2 \\ -24.1 & 18.2 & 36.5 \end{bmatrix}.$$

By invoking (8) and summarizing, we see that the stability of the forced oscillations is guaranteed whenever the reference signal parameters (r_m, ω) satisfy the inequality:

$$r_m \leq \frac{B}{\sqrt{(K_i - K_d\omega^2)^2 + K_p^2\omega^2}}, \tag{21}$$

Its solutions fill the grey area depicted in Fig. 3.

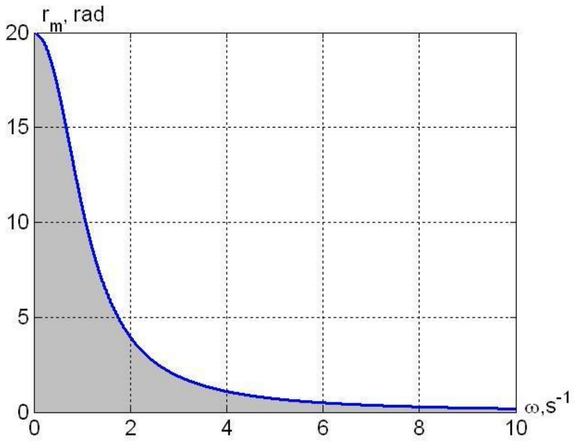

Fig. 3. Permissible region on the plane (r_m, ω) for $B = 0.6$ (shaded).

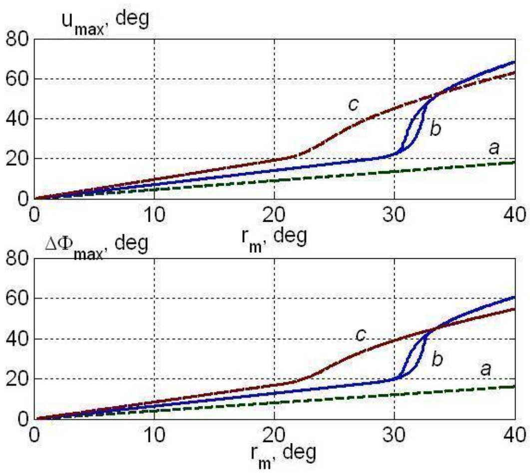

Fig. 4. Amplitudes of the PID-controller output $u(t)$ and the tracking error $\Delta\Phi(t)$ versus the amplitude r_m of the reference signal, $K_i = 0.60$ s^{-1}, $K_p = 1.1$, $K_d = 0.66$ s; *a)* $\omega = 0.8$ s^{-1}, *b)* $\omega = 1.0$ s^{-1}, *c)* $\omega = 1.2$ s^{-1}.

The system considered in the example fails to satisfy the incremental circle criterion. This criterion allows to investigate stability of forced oscillations independently of the properties of the system input, whereas for the example considered, stability depends on the amplitude and frequency of the harmonic excitation. Therefore it would be interesting to compare the obtained result with other techniques that give input-dependent criteria for stability of forced oscillations. The method proposed in [31], see also [32] is not applicable to tthe above example since the matrix J_S (Jacobian in the saturated mode) is not Hurwitz. The integral quadratic constraint (IQC) based stability criterion proposed in [33, Sec. 4] allows to find a set of frequencies of the external excitation that that give rise to stable oscillations. The method of [33] is based on Zames–Falb [34] multiplier technique applied to a memoryless monotone nonlinearity, like the saturation nonlinearity from the example. Being applied to this example, this method fails to find a set of inputs that generate stable oscillations.

D. Simulation results

Simulation tests have been performed to demonstrate the closed-loop system properties in the case of harmonic reference signal $\Phi^*(t) = r_m \sin\omega t$. In course of the simulations, the amplitude r_m was increased from zero so slowly that the transient behavior in the system was negligible. After r_m reached the prescribed maximal value $r_{m,\max}$ it was slowly decreased to zero (c.f. [15]–[17]). The results are depicted in Fig. 4, where the amplitudes of the PID-controller output signal $u(t)$ and the tracking error $\Delta\Phi(t)$ versus the amplitude r_m of the reference signal for $\omega \in \{0.8, 1.0, 1.2\}$ s^{-1} are plotted. The curve b, corresponding to $\omega = 1.0$ s^{-1}, demonstrates the *hysteresis effect*: as r_m increases, the amplitudes of the controller output u and that of the tracking error $\Delta\Phi(t)$ increase gradually until a jump occurs. Then decrease of the amplitude r_m causes monotone decrease of the amplitudes of u and $\Delta\Phi(t)$ until a second jump takes place. The hysteresis effect is caused by the lack of stability of the forced oscillations, which may depend on the reference input.

Let us compare the theoretical results with the simulation ones. As follows from Sec. III, if the reference signal parameters r_m, ω belong to the permissible area, plotted in Fig. 3 (i.e. if inequality (21) is satisfied), then the forced oscillations are stable and, consequently, no hysteresis appears at the corresponding curve in Fig. 4. The curves in Fig. 4 are calculated for the following frequencies of the reference signal: $\omega = 0.8$, 1.0, 1.2. Condition (21) is satisfied for pairs (r_m, ω) as: $(13^o, 0.8)$, $(11^o, 1.0)$, $(8.8^o, 1.2)$. The hysteresis and hence, co-existence of multiple periodic solutions, is observed for the pair $(31^o, 1.0)$, which does not belong to the permissible area. Thus the results of the simulation tests are in the conformity with the theoretical results.

V. Conclusions

A new test for stability of forced oscillations in nonlinear systems is described and applied to a marginally stable plant of the second order with a saturated PID-control and a harmonic reference signal. As a result, the range of frequencies and amplitudes that guarantee stability of the forced oscillations is found.

The theoretical results are used for stability analysis of the aircraft roll angle control system to demonstrate efficiency of the proposed method.

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