Application of passification method to controlled synchronization of tree networks under information constraints

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***Abstract*— The synchronization problem in network having line, star or combined linestar (tree) topology via communication links with limited capacity is considered. Limit possibilities of nonlinear controlled synchronization systems with firstorder coders for multi-dimensional master-slave Lurie systems (linear part plus nonlinearity depending only on measurable outputs) are presented. It is shown that the upper and lower bounds of limited synchronization error are proportional to the maximum coupling signal rate and inversely proportional to the minimum information transmission rate of network communication links (minimum channel capacity). The results are illustrated by example of synchronization in the network of Chua systems coupled via communication links with limited capacity.**

***Index Terms*— Information constraints, Synchronization, Passification method, Communication channel**

I. INTRODUCTION

Synchronization problems for complex systems and networks are attracting a lot of interest recently. Possibilities of design of links between parts of complex systems providing their synchronous behavior were studied in [1]–[8]. Essentially those and other similar papers were studying possibilities of modification of complex dynamical system behavior by means of feedback actions. The results significantly depend on models of interconnection between nodes. In some works interconnections are modeled as delay elements. However, due to spatial distribution of the nodes modeling connections by means of communication channels with limited capacity looks more realistic. In this case study of the overall system should include both dynamical and information aspects. The problems of estimation and control over communication networks were not addressed until recently. Perhaps, the first results were published in [9], [10]. However existing results deal mainly with linear systems and stabilization problems. First results concerning limit possibilities of observer-based synchronization of nonlinear (chaotic) continuous-time systems under information constraints were proposed in [11], [12]. In [13], [14] the result of [11] is extended to *controlled synchronization* scheme for two nonlinear systems. A major difficulty with the controlled synchronization problem arises because the coupling is implemented in a restricted manner via the control signal. Key tools used to solve the problem are quadratic Lyapunov functions and *passification method* [15]–[17].

Following [18], where the observer-based synchronization over communication networks having line, star, or tree topology under information constraints was considered, in the present work we analyse *controlled* synchronization over the networks and provide practically applicable coding procedure for network communication links. We apply the results of [13], where the passification method was used for controlled synchronization of a pair (master–slave) nonlinear systems to networks. To reduce technicalities we restrict our analysis by Lurie systems (linear part plus nonlinearity depending only on measurable outputs). After explaining the basics of the information transmission through digital communication channels and coding procedures in Section II, the analyze controlled synchronization over networks with tree topology under information constraints is presented in Section III. Preliminary results for controlled synchronization of unidirectionally coupled Lur'e systems over the limited-band communication channel are given in Sections III-A, III-B. Similarly the case when the information graph of the network has the tree topology and the leader sits in the root of the tree is considered. It is shown that the upper bound of the *limit synchronization error* (LSE) is proportional to the upper bound of the transmission error. As a consequence, the upper and lower bounds of LSE are proportional to the maximum rate of the coupling signal and inversely proportional to the information transmission rate (channel capacity). In Section IV a numerical case study for a typical chaotic system, namely the Chua circuit, over a line network in presence of information constraints is presented.

II. BASICS OF CONTROLLED SYNCHRONIZATION THROUGH THE COMMUNICATION CHANNEL

In this paper the controlled synchronization through the communication channel is considered. It is supposed that the drive and response nodes of the network are connected via the link with data rate limitations.

A. Controlled synchronization of drive–response nonlinear systems over the communication channel

Consider two identical dynamical systems modeled in Lurie form (i.e. the right hand sides are split into a linear part and a nonlinear part which depends only on the measurable outputs). Let one of the systems be controlled by a scalar control function $u(t)$ whose action is restricted by a vector

The work was supported by the Russian Foundation for Basic Research (Project # 07-01-92166 (PICS), 08-01-00775, 09-08-00803), the Council for grants of the RF President to support young Russian researchers and leading scientific schools (project NSh-2387.2008.1) and the Program of basic research of OEMPPU RAS #2 "Control and safety in energy and technical systems".

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of control efficiencies B. The controlled system model is as follows:

$$\dot{x}(t) = Ax(t) + B\varphi(y_1),\ y_1(t) = Cx(t), \tag{1}$$

$$\dot{z}(t) = Az(t) + B\varphi(y_2) + Bu,\ y_2(t) = Cz(t), \tag{2}$$

where $x(t)$, $z(t)$ are n-dimensional (column) vectors of state variables; $y_1(t)$, $y_2(t)$ are scalar output variables; A is an $(n \times n)$-matrix; B is $n \times 1$ (column) matrix; C is an $1 \times n$ (row) matrix, $\varphi(y)$ is a continuous nonlinearity, acting in the span of control; vectors $\dot{x}$, $\dot{z}$ stand for time-derivatives of $x(t)$, $z(t)$ correspondingly. System (1) is called *master* (*leader*) *system*, while the controlled system (2) is *slave system* (*follower*). Our goal is to evaluate limitations imposed on the synchronization precision by limited the transmission rate between the systems. The intermediate problem is to find a control function $\mathcal{U}(\cdot)$ depending on measurable variables such that the synchronization error $e(t)$, where $e(t) = z(t) - x(t)$ becomes small as t becomes large. We are also interested in the value of output synchronization error $\varepsilon(t) = y_2(t) - y_1(t) = Ce(t)$.

A key difficulty arises because the output of the master system is not available directly but only through a communication channel with limited capacity. This means that the signal $y_1(t)$ must be coded at the transmitter side and codewords are then transmitted with only a finite number of symbols per second thus introducing some error. We assume that the observed signal $y_1(t)$ is coded with symbols from a finite alphabet at discrete sampling time instants $t_k = kT$, $k = 0, 1, 2, \ldots$, where T is the sampling time. Let the coded symbol $\bar{y}_1[k] = \bar{y}_1(t_k)$ be transmitted over a digital communication channel with a finite capacity. To simplify the analysis, we assume that the observations are not corrupted by observation noise; transmission delay and transmission channel distortions may be neglected. Therefore, the discrete communication channel with sampling period T is considered, but it is assumed that the coded symbols are available at the receiver side at the same sampling instant $t_k = k$, as they are generated by the coder. Assume that *zero-order extrapolation* is used to convert the digital sequence $\bar{y}_1[k]$ to the continuous-time input of the response system $\bar{y}_1(t)$, namely, that $\bar{y}_1(t) = \bar{y}_1[k]$ as $kT \le t < (k+1)T$. Then the *transmission error* is defined as follows:

$$\delta_y(t) = y_1(t) - \bar{y}_1(t). \tag{3}$$

On the receiver side the signal is decoded introducing additional error and the controller can use only the signal $\bar{y}_1(t) = y_1(t) + \delta_y(t)$ instead of $y(t)$.

We restrict consideration to simple control functions in the form of static linear feedback

$$u(t) = -K\varepsilon(t), \tag{4}$$

where $\varepsilon(t) = y_2(t) - y_1(t)$ denotes an output synchronization error, K is a scalar controller gain. The problem of finding static output feedback even for linear systems is one of the classical problems of control theory. Although substantial effort has been devoted to its solution and various necessary and sufficient conditions for stabilizability by static output feedback have been obtained, most existing conditions are not testable practically [19], [20]. Since we are dealing with a nonlinear problem further complicated by information constraints, we restrict our attention to sufficient conditions for solvability of the problem and evaluate upper bounds for synchronization error. To this end we introduce an upper bound on the limit synchronization error $Q = \sup \varlimsup_{t\to\infty} \|e(t)\|$, where the supremum is taken over all admissible transmission errors. In the next two sections the coding and decoding procedures are described and a bound on admissible transmission errors $\delta_y(t)$ is evaluated.

Under the assumption that a sampling period may be properly chosen, in [11] optimality of the binary coding in the sense of demanded transmission rate is established, and the relation between synchronization accuracy and an optimal sampling period is found. On the basis of these results, the present paper deals with a binary coding procedure. Let us describe this procedure in more detail.

B. Static binary coder

At first, introduce the memoryless (static) binary coder to be a discretized map $q_M : \mathbb{R} \to \mathbb{R}$ as

$$q_M(y) = M \operatorname{sign}(y), \tag{5}$$

where $\operatorname{sign}(\cdot)$ is the *signum* function: $\operatorname{sign}(y) = 1$, if $y \ge 0$, $\operatorname{sign}(y) = -1$, if $y < 0$; parameter M may be referred to as a *coder range* or as a *saturation value*. Notice that for binary coder each codeword symbol contains $R = 1$ bit. The discretized output of the considered coder is found as $\bar{y} = q_M(y)$ and we assume that the coder and decoder make decisions based on the same information.

C. Binary coder with memory

The static coder (5) is a part of the time-varying coders with memory, see e.g. [11], [21]–[24]. Two underlying ideas are used for this kind of coders:

– reducing the coder range M to cover the some area around the predicted value for the $(k+1)$th observation $y[k+1]$, $y[k+1] \in \mathcal{Y}[k+1]$. This means that the quantizer range M is updated during the time and a time-varying quantizer (with different values of M for each instant, $M = M[k]$) is used. Using such a "zooming" strategy it is possible to increase coder accuracy in the steady-state mode, and, at the same time, to prevent coder saturation at the beginning of the process;

– introducing memory into the coder, which makes possible to predict the $(k+1)$th observation $y[k+1]$ with some accuracy and, therefore, to transmit over the channel only encrypted innovation signal.

To introduce memory into coder, use the sequence of *central numbers* $c[k]$, $k = 0, 1, 2, \ldots$ with initial condition $c[0] = 0$. At step k the coder compares the current measured output $y[k]$ with the number $c[k]$, forming the deviation signal $\partial y[k] = y[k] - c[k]$. Then this signal is discretized with a given $M = M[k]$ according to (5). The output signal

$$\bar{\partial} y[k] = q_{M[k]}(\partial y[k]) \tag{6}$$

is represented as an R-bit information symbol from the coding alphabet and transmitted over the communication channel to the decoder. Then the central number $c[k+1]$ and the range parameter $M[k]$ are renewed based on the available information about the master system dynamics. Assuming that the master system output y changes at a slow rate, i.e. that $y[k+1] \approx y[k]$. we use the following update algorithms:

$$c[k+1] = c[k] + \bar{\partial}y[k], \quad c[0] = 0, \quad k = 0, 1, \dots, \tag{7}$$

$$M[k] = (M_0 - M_\infty)\rho^k + M_\infty, \quad k = 0, 1, \dots, \tag{8}$$

where $0 < \rho \leq 1$ is the decay parameter, M_∞ stands for the limit value of $M[k]$. The initial value M_0 should be large enough to capture all the region of possible initial values of y_0.

The equations (5), (6), (8) describe the coder algorithm. The similar algorithm is realized by the decoder. Namely: the sequence of $M[k]$ is reproduced at the receiver node utilizing (8); the values of $\bar{\partial}y[k]$ are restored with given $M[k]$ from the received codeword; the central numbers $c[k]$ are found in the decoder in accordance with (7). Then $\bar{y}[k]$ is found as a sum $c[k] + \bar{\partial}y[k]$.

D. Adaptive coder

If parameter M_∞ in (8) is mistakenly chosen too small, the data transmission process by means of the described procedure may failed due to the coder saturation. In the present work, the time-based zooming procedure with the event-based correction is proposed: if the coder is not saturated, the quantizer range M is exponentially decreased; if the coder saturation appears, the quantizer range M is increased. Such an *adaptive* coding makes it possible to maintain the minimal discretization interval δ and, therefore, minimize transmission errors. At the same time, this prevents fail in tracking the signal $y[k]$ due to the saturation.

The proposed method for tuning the quantizer range M is described by the following recurrent algorithm:

$$\begin{aligned} &\lambda[k] = (\bar{\partial}y[k] + \bar{\partial}y[k-1])/2, \\ &M[k] = m + \begin{cases} \rho M[k-1], & \text{if } \ |\lambda[k]| \leq 0.5 \\ M[k-1]/\rho, & \text{otherwise,} \end{cases} \end{aligned} \tag{9}$$

where $0 < \rho \leq 1$ is the *decay parameter*; $m = (1-\rho)M_\infty$, M_∞ assigns the minimal possible value for $M[k]$. The procedure (9) leads to time-based decreasing of $M[k]$ while signs of the successive values of $\bar{\partial}y[k]$ alternate. The sort of discrete-time sliding-mode tracking the transmitted signal appears in that case, and $M[k]$ recursively tends to the limiting value M_∞. When the moving average of the transmission error exceeds the threshold, the second alternative of algorithm (9) is realized, and the quantizer range $M[k]$ increases. It should be noticed that the adaptive quantizer with a memoryless (static) coder was considered in [25]. For a special case of a first-order system the adaptive coding was analyzed in [26], where for zooming-in and zooming-out phases the different values of ρ, $0 < \rho_1 < 1$ and $\rho_2 > 1$ were proposed.

III. Controlled synchronization of networks

A growing number of publications are devoted to control and estimation of networks under communication constraints. Limit possibilities for linear dynamical networks are studied in the book [10]. However, in the most papers the coding procedures are rather complex.

To analyze the network synchronization error, let us recall the existing results for synchronization of a pair master–slave nonlinear systems over the limited-band communication channel.

A. Evaluation of synchronization error in master–slave configuration

Following [11], [13], find a relation between the transmission rate and the achievable accuracy of the coder–decoder pair, assuming that the growth rate of $y_1(t)$ is uniformly bounded. Obviously, the exact bound L_y for the rate of $y(t)$ is $L_y = \sup\limits_{x\in\Omega} |C\dot{x}|$, where $\dot{x}$ is from (2). To analyze the coder–decoder accuracy, evaluate the upper bound $\Delta = \sup\limits_t |\delta_y(t)|$ of the transmission error $\delta_y(t) = y_1(t) - \bar{y}_1(t)$. The total transmission error for each interval $[t_k, t_{k+1}]$ satisfies the inequality:

$$|\delta_y(t)| \leq M + L_y T \tag{10}$$

Inequality (10) shows that in order to meet the inequality $|\delta_y(t)| \leq \Delta = 2M$ for all t, the sampling interval T should satisfy condition

$$T < \Delta / L_y. \tag{11}$$

To evaluate the limit synchronization error analitically assume that the transfer function $W(\lambda) = C(\lambda I - A)^{-1}B$ of the linear part of the system (2) is Hyper-Minimum Phase (HMP). Recall that the HMP property for a rational function $W(\lambda) = b(\lambda)/a(\lambda)$ where $a(\lambda)$ is a polynomial of degree n, $b(\lambda)$ is a polynomial of degree not greater than $n-1$ means that $b(\lambda)$ is a Hurwitz polynomial of degree $n-1$ with positive coefficients [16].

Subtracting (1) from (2) and taking into account the control law (4) we derive an equation for the synchronization error in the form

$$\dot{e}(t) = A_K e(t) + B\zeta(t) - BK\delta_y(t), \tag{12}$$

where $A_K = A - BKC$, $\zeta(t) = \varphi(y_2(t)) - \varphi(y_1(t))$.

Evaluate the total guaranteed synchronization error $Q = \sup \overline{\lim\limits_{t\to\infty}} \|e(t)\|$, where $\|\cdot\|$ denotes the Euclidian norm of a vector, and the supremum is taken over all admissible transmission errors $\delta_y(t)$ not exceeding the level Δ in absolute value. The ratio $C_e = Q/\Delta$ (the relative error) can be interpreted as the norm of the transformation from the input function $\delta_y(\cdot)$ to the output function $e(\cdot)$ generated by the system (12). Owing to the nonlinearity of the equation (12) evaluation of the norm C_e is nontrivial and it even may be infinite for rapidly growing nonlinearities $\varphi(y)$. To obtain a reasonable upper bound for C_e we assume that the nonlinearity is Lipschitz continuous along all the trajectories of the drive system (2). More precisely, we assume existence of a positive number $L_\varphi > 0$ such that

$$|\varphi(y) - \varphi(y+\delta)| \leq L_\varphi |\delta|$$

for all $y = Cx$, $x \in \Omega$, where Ω is a set containing all the trajectories of the drive system (1), starting from the set of initial conditions Ω_0, $|\delta| \leq \Delta$. For Lipschitz nonlinearities $\zeta(t)$ satisfies inequality $|\zeta(t)| \leq L_\varphi |\varepsilon(t)|$. After the change $K \to K + L_\varphi$, the error equation (12) can be represented as

$$\dot{e}(t) = A_K e(t) + B\xi(t) - B(K + L_\varphi)\delta_y(t), \qquad (13)$$

where the variable $\xi(t) = L_\varphi \varepsilon(t) + \zeta$, apparently, satisfies sector inequality $\xi(t)\varepsilon(t) \geq 0$ for all $t \geq 0$.

The problem is reduced to quantifying the stability properties of (13) for bounded input $\delta_y(t)$. We first analyze behavior of the system (13) for $\delta_y(t) = 0$. To this end, we find conditions for the existence of a quadratic Lyapunov function $V(e) = e^{\mathrm{T}} P e$ and controller gain K satisfying inequality $\dot{V}(e) \leq -\mu V(e)$ for some $\mu > 0$ for $\delta_y(t) = 0$ and for all ξ satisfying the quadratic inequality $\xi\varepsilon \geq 0$. Such conditions are derived from the Passification theorem [15]–[17]. Namely, such V and K exist if and only if the transfer function of the linear part of (2) $W(\lambda) = C(\lambda I - A)^{-1}B$ is *Hyper-Minimum Phase* (HMP). Now consider the case $\delta_y(t) \neq 0$, assuming that the HMP condition holds and matrix P and gain K are chosen properly and the modified Lyapunov inequality $PA_K + A_K^{\mathrm{T}} P \leq -\mu P$ is valid for some $\mu > 0$. Evaluating the time derivative of function $V(e)$ along trajectories of (2), (1) with initial conditions in Ω_0, using standard quadratic inequality $|e^{\mathrm{T}} PB| \leq \sqrt{V(e)}\sqrt{V(B)}$ after simple algebra we get

$$\dot{V} \leq -\mu V + |e^{\mathrm{T}} PB(K + L_\varphi)\delta_y| \leq -\mu V + \sqrt{V}\nu,$$

where $\nu = \sqrt{V(B)}(|K| + L_\varphi)\Delta$. Since $\dot{V} < 0$ within the set $\sqrt{V} > \mu^{-1}\nu$, the value of $\overline{\lim\limits_{t\to\infty}} \sup V(t)$ cannot exceed $\Delta^2\big(L_\varphi + |K|\big)^2 \lambda_{\max}(P)/\mu^2$. In view of positivity of P, $\lambda_{\min}(P)\|e(t)\|^2 \leq V(t)$, where $\lambda_{\min}(P)$, $\lambda_{\max}(P)$ are minimum and maximum eigenvalues of P, respectively. Hence

$$\overline{\lim_{t\to\infty}} \|e(t)\| \leq C_e^+ \Delta, \qquad (14)$$

where

$$C_e^+ = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \frac{L_\varphi + |K|}{\mu}. \qquad (15)$$

The inequality (14) shows that the total synchronization error is proportional to the upper bound on the transmission error Δ.

As was shown in [11], a binary coder is optimal in the sense of bit-per-second rate, and the optimal sampling time T for this coder is

$$T = \Delta/(\beta L_y),, \qquad (16)$$

where $\beta \approx 1.688$. Then the channel bit-rate $R = 1/T$ is as follows:

$$R = \beta L_y/\Delta, \qquad (17)$$

and this bound is tight for the considered class of coders. It worth to mention that the HMP property is essential for validity of the proposed solution to the controlled synchronization problem, as it is shown in [13] by example of the chaotic Chua-like systems synchronization.

B. Evaluation of synchronization error for tree-like network under communication constraints

Extending the communication scheme of Section III-A to the case of the line network topology yields the following equations of the network dynamics:

$$\begin{aligned}
&\dot{x}_0 = Ax_0 + B\varphi(y_0), \qquad y_0 = Cx_0,\\
&\dot{x}_1 = Ax_1 + B\varphi(y_1) + BK_1(\bar{y}_0 - y_1),\\
&\qquad\qquad y_1 = Cx_1, \quad \bar{y}_0 = y_0 + \delta_0(t),\\
&\dot{x}_2 = Ax_2 + B\varphi(y_2) + BK_2(\bar{y}_1 - y_2),\\
&\qquad\qquad y_2 = Cx_2, \quad \bar{y}_1 = y_1 + \delta_1(t),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\dot{x}_N = Ax_N + B\varphi(y_N) + BK_N(\bar{y}_{N-1} - y_N),\\
&\qquad\qquad y_N = Cx_N, \quad \bar{y}_{N-1} = y_{N-1} + \delta_N(t), \quad (18)
\end{aligned}$$

where $x_i \in \mathbb{R}^n$ is the state vector of the ith node in the network, y_i is the ith output, $\bar{y}_i$ is the signal arriving to $(i+1)$th node through the ith communication channel, $\delta_i(t)$ is an equivalent error.

Let all nonlinearities be scalar and enter the jth dynamics equation of the network node with the gain b_i, i.e., $\varphi(y) = B\varphi_0(y)$, where $B = [b_1, \dots, b_n]^{\mathrm{T}}$. Introduce the *state error* $e_i = x_{i-1} - x_i$ and the *output errors* $\varepsilon_i = y_{i-1} - y_i$, $i = 1, \dots, N$. Then the error dynamics are modeled as (see (12) to compare)

$$\begin{aligned}
&\dot{e}_1(t) = A_{K,1}e_1(t) + B\big(\varphi(y_1(t) - \varphi(y_0(t))\big) - BK\delta_{y,1}(t),\\
&\dot{e}_2(t) = A_{K,2}e_2(t) + B\big(\varphi(y_2(t) - \varphi(y_1(t))\big) - BK\delta_{y,2}(t),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\dot{e}_{N-1}(t) = A_{K,N-1}e_{N-1}(t) + B\big(\varphi(y_{N-1}(t) - \varphi(y_{N-2}(t))\big)\\
&\qquad\qquad - BK\delta_{y,N-1}(t). \qquad (19)
\end{aligned}$$

The problem of controlled synchronization systems design is to find the gains $K_1, \dots, K_N$ providing small values of the asymptotic synchronization errors

$$Q_i = 0.9 \limsup_{t\to\infty} \|e_i(t)\| \qquad (20)$$

for bounded transmission errors: $|\delta_i| \leq \Delta$.

For $N = 1$ the problem coinsides with the one considered above in Section III-B, see also [13]. It is shown that if the following assumptions are valid: the transfer function $W(\lambda) = C(\lambda I - A)^{-1}B$ of the linear part of the system (2) is Hyper-Minimum Phase (HMP), the trajectories of the drive system $x_0(t)$ are bounded and the nonlinearity $\varphi_0(y)$ is Lipschitz continuous; the gain K_1 is chosen properly (i.e. the modified Lyapunov inequality $PA_K + A_K^{\mathrm{T}} P \leq -\mu P$ is valid for some $\mu > 0$ and $P = P^{\mathrm{T}} > 0$), then the synchronization error satisfies inequality (14).

We will show that for $N > 1$ the following inequality holds:

$$Q_i \le \mathcal{K}_i/\bar{R}_i, \quad 1 \le i \le N, \tag{21}$$

where $\bar{R}_i = \min\limits_{1\le j\le i} R_j$ and $\mathcal{K}_i > 0$.

To solve the problem the induction method is used. For $i = 1$ the problem is reduced to one solved in Section III-B and inequality (14) can be rewritten as

$$Q_1 \le \mathcal{K}_1/R_1, \tag{22}$$

where R_1 is a transmission rate (capacity) of the first channel.

Assuming that (21) holds for $i = k$ we need to prove that it holds for $i = k+1$. To this end, choose $K_k \equiv K_1$ such that $PA_{K,1} + A_{K,1}^{\mathrm{T}} P \le -\mu P$ is valid. Then the error equation for e_{k+1} reads

$$\dot{e}_{k+1}(t) = A_{K,k+1} e_{k+1}(t) + B\xi_{k+1}(t), \tag{23}$$

where $\xi_{k+1}(t) = \varphi(y_k + K\delta_k(t)) - \varphi(y_{k-1} + K\delta_{k-1}(t))$. Since $\xi_{k+1}(t) \le |L|\big(|e_k| + K|\delta_{k-1}(t)|\big)$, the error obeys a linear stable equation with a bounded input. To finish the proof the following simple statement is helpful.

Lemma. Let $w_0(t)$, $w_1(t)$ be nonnegative continuous functions on $[0,\infty)$ and $w_1(t)$ be differential. Let the following inequalities hold:

$$\limsup_{t\to\infty} w_0(t) \le A,$$
$$\dot{w}_1 \le -\alpha w_1 + \beta_0 w_0(t) + \beta_1.$$

Then $\limsup\limits_{t\to\infty} w_1(t) \le \big(\beta_0 A + \beta_1\big)/\alpha$.

Employing Lemma with $w_0(t) = \sqrt{e_k^{\mathrm{T}} P_k e_k}$, $w_i(t) = \sqrt{e_i^{\mathrm{T}} P_i e_i}$, where P is the solution of the Lyapunov equation $PA_1 + A_1^{\mathrm{T}} P = -\lambda P$ for some $\lambda > 0$ and taking into account inequalities $|\delta_k(t)| \le rL_y/R_k$ for $i = k$, and relation (22), we obtain (21) for $i = k+1$.

Summarizing, asymptotic synchronization error in the linear network topology is inversely proportional to the minimum channel capacity. The same conclusion holds for the tree configuration.

IV. Example. Synchronization in line network of chaotic Chua systems

Let us apply the above results to synchronization in line network of chaotic Chua systems coupled via a channel with limited capacity. The system model parameters are taken from [13], where an example of controlled synchronization of a pair master–slave Chua systems vis limited capacity communication link is presented.

Leader system. Let the leader system (1) be represented as the following *Chua system*:

$$\begin{cases} \dot{x}_1 = p(-x_1 + \varphi(y_0) + x_2), \quad t \ge 0, \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -qx_2, \end{cases} \tag{24}$$
$$y_0(t) = x_1(t),$$

where $y_0(t)$ is the leader system output (to be transmitted over the communication network), p, q are system parameters, $x = [x_1, x_2, x_3]^{\mathrm{T}} \in \mathbb{R}^3$ is the state vector; $\varphi(y_0)$ is a piecewise-linear function, having the form:

$$\varphi(y) = m_0 y + m_1(|y+1| - |y-1|), \tag{25}$$

where m_0, m_1 are given parameters.

Coding procedure has a form (6)–(8) and the binary coder (5) is used. The input signal of the coder is $y_1(t)$. The reference input $\bar{y}_1(t)$ for controller (27) is found by holding the value of $\bar{y}_1[k]$ over the sampling interval $[kT, (k+1)T)$, $k = 0, 1, \ldots$.

Network topology.

We considered a synchronization problem over the network having line topology and consisting of $N = 20$ nodes. The network edges represent limited capacity communication links. We assume that the sampling intervals for each edge are equal, i.e. that $T_i = T$, $i = 1, \ldots, N$.

Slave systems.

The slave systems equations (1) for each node $i = 1, \ldots, N$ become

$$\begin{cases} \dot{z}_{1,i} = p\big(-z_{1,i} + \varphi(y_i) + z_{2,i} + u(t)\big), \quad t \ge 0, \\ \dot{z}_{2,i} = z_{1,i} - z_{2,i} + z_{3,i} \\ \dot{z}_{3,i} = -qz_{2,i}, \end{cases} \tag{26}$$
$$y_i(t) = z_{1,i}(t),$$

where $y_i(t)$ is the ith system output, $z_{1,i}$, $z_{2,i}$, $z_{3,i}$ are the state variables, $\varphi(y)$ is defined by (25), $i = 1, \ldots, N$ is the node number.

Controller has a form

$$u_i(t) = -K\varepsilon_i(t), \tag{27}$$

where $\varepsilon_i(t) = y_i(t) - \bar{y}_{i-1}(t)$; $\bar{y}_{i-1}(t)$, $i = 1, \ldots, N$ is a master system output, restored from the transmitted codeword by the receiver at the slave system node, $y_0(t)$ is the leader system output; the gain K is a design parameter.

The following parameter values were taken for the simulation: Chua system parameters: $p = 10$, $q = 15.6$, $m_0 = 0.33$, $m_1 = 0.945$; controller gain $K = 20$; the bound L_y for the rate of $y_0(t)$ was evaluated by numeric integration of (1) over the time interval $t \in [0, t_{\text{fin}}]$, $t_{\text{fin}} = 1000$ s, as $L_y = 45$; parameter $\Delta = 1$; the sample interval T was found from (11) as $T = 0.14$ s; the coder (8) parameters : $M_0 = 5$, $M_\infty = \Delta/2 = 0.5$, $\rho = \exp(-0.1T) = 0.9987$; initial conditions for leader system $x_1 = x_2 = x_3 = 0.3$, all the slave systems had zero initial conditions, $z_{i,j} = 0$ $(j = 1, 2, 3)$.

Simulation results are depicted in Figs. 1, 2. Time histories of the variables $y_5(t)$, $y_{10}(t)$, $y_{20}(t)$ (the outputs of 5th, 10th and 20th nodes) are shown in Fig. 1 demonstrating synchrotization process over the network.

Some details may be viewed in Fig. 2 where the time histories of $y_0(t)$ (leader system output) and $y_{20}(t)$ (the last node of the network) during the time niterval $t \in [35, 45]$ s are plotted. It is seen that due to coding procedure there exists

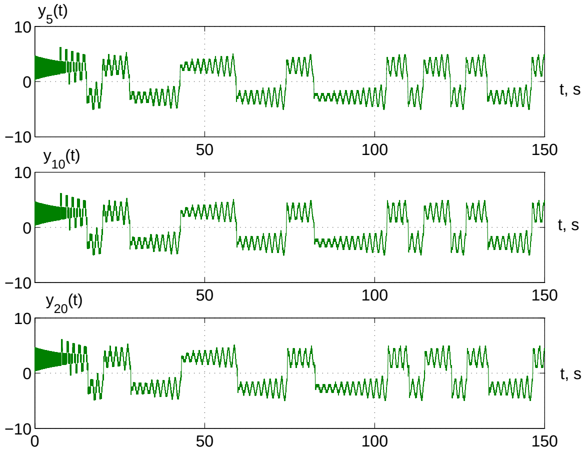

Fig. 1. Time histories of $y_5(t)$, $y_{10}(t)$, $y_{20}(t)$.

a certain time lag τ between the processes in the outputs of systems, which depends on the "distance" beetween the nodes i and j as $\tau = (j-i)T$ for the case of $T_i = T$, or as $\tau = \sum_{k=i}^{j} T_k$ in a general case of different T_i.

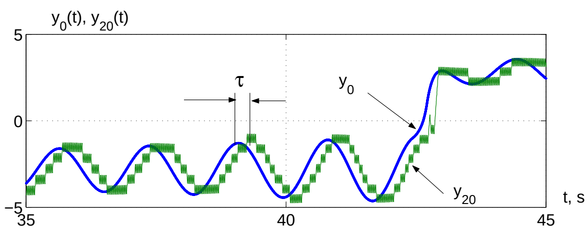

Fig. 2. Time histories of $y_0(t)$, $y_{20}(t)$. The time lag τ between the system outputs is shown.

V. Conclusions

Limit possibilities of controlled synchronization systems under information constraints imposed by limited information capacity of the coupling channel are evaluated. We propose a simple first order coder-decoder scheme and provide theoretical analysis for multi-dimensional master-slave systems represented in the Lurie form by means of passification method [15], [17], [27]. It is shown that the upper and lower bounds of limit synchronization error are proportional to the maximum rate of the coupling signal and inversely proportional to the information transmission rate (channel capacity). The analysis is extended to networks having a linear, star, or "comeit" topology. Adaptive chaotic synchronization under information constraints is analyzed. The results are applied to controlled synchronization of two chaotic Chua systems coupled via a channel with limited capacity.

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