Synchronization of Passifiable Lurie Systems via Limited Capacity Communication Channel

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Abstract— Output feedback controlled synchronization problems for a class of nonlinear unstable systems under information constraints imposed by limited capacity of the communication channel are analyzed. A binary time-varying coder-decoder scheme is described and a theoretical analysis for multidimensional master-slave systems represented in Lurie form (linear part plus nonlinearity depending only on measurable outputs) is provided. An output feedback control law is proposed based on the Passification Theorem.

It is shown that the synchronization error exponentially tends to zero for sufficiently high transmission rate (channel capacity). The results obtained for synchronization problem can be extended to tracking problems in a straightforward manner, if the reference signal is described by an external (exogenous) state space model.

The results are illustrated by controlled synchronization of two chaotic Chua systems via a communication channel with limited capacity.

Index Terms— Chaotic behavior, Synchronization, Control, Communication constraints

I. INTRODUCTION

Analysis and control of the behavior of complex interconnected systems and networks has attracted considerable recent interest. The available results significantly depend on models of interconnection between nodes. In some works the interconnections are modeled as delay elements. However, the spatial separation between nodes means that modeling connections via communication channels with limited capacity is more realistic.

Recently the limitations of control under constraints imposed by a finite capacity information channel have been investigated in detail in the control literature, see [1]–[7] and the references therein. It has been shown that stabilization of linear systems under information constraints is possible if and only if the capacity of the information channel exceeds the entropy production of the system at the equilibrium (*Data Rate Theorem*) [3]–[5]. In [8], [9] a general statement was proposed, claiming that the difference between the entropies of the open loop and the closed loop systems cannot exceed

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the information introduced by the controller, including the transmission rate of the information channel.

For nonlinear systems only a few results are available in the literature [5], [10]–[14]. In the above papers only the problems of stabilization to a point are considered. The result of [5] is local, while the papers [10]–[14] deal only with equilibrium stabilization.

In the control literature there is a strong interest in control of oscillations, particularly in controlled synchronization problems [15]–[21]. However, results of the previous works on control systems analysis under information constraints do not apply to synchronization systems since in a synchronization problem trajectories in the phase space converge to a set (a manifold) rather than to a point, i.e. the problem cannot be reduced to simple stabilization. Moreover, the Data Rate Theorem is difficult to extend to nonlinear systems.

The first results on synchronization under information constraints were presented in [22], [23], where the so called observer-based synchronization scheme [24], [25] was considered. The present paper is devoted to the *controlled* synchronization problem. Despite [22], [23], in the present paper we consider an output feedback controlled synchronization scheme for nonlinear systems, where the control signal is computed based on a measurable innovation (error) signal which has been transmitted over a communication channel. The major difficulty with the controlled synchronization problem arises because the system feedback loop involves the limited capacity communication channel. Key tools used to solve the problem are quadratic Lyapunov functions and Passification methods [26], [27]. To minimize technicalities we restrict our analysis to Lurie systems (linear part plus nonlinearity depending only on measurable outputs). It is shown that for the case of an ideal channel and non-corrupted measurements, the proposed controlled synchronization strategy ensures asymptotical vanishing synchronization error if the data transmission rate (channel capacity) exceeds some threshold value. The observer-based synchronization scheme of [22], [23] leads to the limit synchronization error inversely proportional to the transmission rate.

Synchronization of chaotic systems is a focus issue of a substantial number of papers. Starting from the pioneering works [28], [29], application of chaotic synchronization to communications and cryptography has been extensively studied [30]–[33]. Considering possible applications of chaotic synchronization, the controlled synchronization of chaotic Chua systems was chosen in the present work as an illustrative example.

The paper is organized as follows. The controlled syn-

chronization problem is described in Section II. The coding procedure used in the paper, is presented in Sec. III. The main results are presented in Section IV where exponential convergence of the synchronization error to zero is established. An example showing synchronization of the chaotic Chua systems is presented in Section V. Final remarks are given in the Conclusion. Auxiliary results are presented in the Appendices.

II. DESCRIPTION OF CONTROLLED SYNCHRONIZATION SCHEME

Consider two identical dynamical systems modeled in Lurie form (i.e. the right hand sides are split into a linear part and a nonlinear part which depends only on the measurable outputs). Let one of the systems be controlled by a scalar control function $u(t)$ whose action is restricted by a vector of control efficiencies B. The controlled system model is as follows:

$$\dot{x}(t) = Ax(t) + B\varphi(y_1), \; y_1(t) = Cx(t), \tag{1}$$

$$\dot{z}(t) = Az(t) + B\varphi(y_2) + Bu, \; y_2(t) = Cz(t), \tag{2}$$

where $x(t)$, $z(t)$ are n-dimensional (column) vectors of state variables; $y_1(t)$, $y_2(t)$ are scalar output variables; A is an $(n \times n)$-matrix; B is $n \times 1$ (column) matrix; C is an $1 \times n$ (row) matrix, $\varphi(y)$ is a continuous nonlinearity, acting in the span of control; vectors $\dot{x}$, $\dot{z}$ stand for time-derivatives of $x(t)$, $z(t)$ respectively. System (1) is called the *master* (*leader*) *system*, while the controlled system (2) is called the *slave* (*follower*) *system*. Our goal is to evaluate limitations imposed on the synchronization precision by limiting the transmission rate between the systems. The intermediate problem is to find a control function $\mathcal{U}(\cdot)$ depending on the measurable variables such that the synchronization error $e(t)$, where $e(t) = x(t) - z(t)$ becomes small as t becomes large. We are also interested in the value of the output synchronization error $\varepsilon(t) = y_1(t) - y_2(t) = Ce(t)$.

A key difficulty arises because the error signal between the master system and the slave systems is not available directly but only through a communication channel with a limited capacity. This means that the synchronization error $\varepsilon(t)$ must be coded at the transmitter side and codewords then transmitted with only a finite number of symbols per second thus introducing error. We assume that the observed signal $\varepsilon(t)$ is coded with symbols from a finite alphabet at discrete sampling time instants $t_k = kT$, $k = 0, 1, 2, \ldots$, where T is the sampling time. Let the coded symbol $\bar{\varepsilon}[k] = \bar{\varepsilon}(t_k)$ be transmitted over a digital communication channel with a finite capacity. To simplify the analysis, we assume that the observations are not corrupted by observation noise; transmission delay and transmission channel distortions may be neglected. Therefore, the discrete communication channel with sampling period T is considered, but it is assumed that the coded symbols are available at the receiver side at the same sampling instant $t_k = kT$, as they are generated by the coder. Assume that *zero-order extrapolation* is used to convert the digital sequence $\bar{\varepsilon}[k]$ to the continuous-time input of the controller $\bar{\varepsilon}(t)$, namely, that

$\bar{\varepsilon}(t) = \bar{\varepsilon}[k]$ as $kT \leq t < (k+1)T$. Then the *transmission error* is defined as follows:

$$\delta_\varepsilon(t) = \varepsilon(t) - \bar{\varepsilon}(t). \tag{3}$$

On the receiver side the signal is decoded introducing additional error and the controller can use only the signal $\bar{\varepsilon}(t) = \varepsilon(t) - \delta_\varepsilon(t)$ instead of $\varepsilon(t)$. A block diagram of the system is shown in Fig. 1.

We restrict consideration to simple control functions in the form of static linear feedback

$$u(t) = K\varepsilon(t), \tag{4}$$

where $\varepsilon(t) = y_1(t) - y_2(t)$ denotes an output synchronization error and K is a scalar controller gain. The problem of finding static output feedback even for linear systems is one of the classical problems of control theory. Although substantial effort has been devoted to its solution and various necessary and sufficient conditions for stabilizability by static output feedback have been obtained, most existing conditions are not testable practically [34], [35]. In this paper we analyze a natural and relatively broad class of systems for which constructive conditions for output feedback stabilization are known is the class of *passifiable* (or *feedback passive*) systems (for linear systems this was introduced and studied in [26], [36]). Since we are dealing with a nonlinear problem further complicated by information constraints, we restrict our attention to sufficient conditions for solvability of the problem and evaluate upper bounds for synchronization error.

III. CODING PROCEDURES

In the paper [22] the properties of observer-based synchronization for Lurie systems over a limited data rate communication channel with a one-step memory time-varying coder are studied. It is shown that an upper bound on the limit synchronization error is proportional to a certain upper bound on the transmission error. Under the assumption that a sampling time may be properly chosen, optimality of binary coding in the sense of demanded transmission rate is established, and the relationship between synchronization accuracy and an optimal sampling time is found.[1] On the basis of these results, the present paper deals with a binary coding procedure.

Consider the memoryless (static) binary quantizer to be a discretized map $q : \mathbb{R} \to \mathbb{R}$ as

$$q(y, M) = M\text{sign}(y), \tag{5}$$

where $\text{sign}(\cdot)$ is the *signum* function: $\text{sign}(y) = 1$, if $y \geq 0$, $\text{sign}(y) = -1$, if $y < 0$. Parameter M may be referred to as the *quantizer range*. Notice that for a binary coder each codeword symbol contains one bit of information. Therefore the transmission rate is $R = 1/T$. The discretized output of the considered quantizer is given as $\bar{y} = q(y, M)$. We assume that the coder and decoder make decisions based on the same information. The output signal of the quantizer is represented as a one-bit information symbol from the coding alphabet

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[1] For a very special case (the stabilization problem for the first order linear plant) a similar result was obtained in [37].

$\mathscr{S}$ and transmitted over the communication channel to the decoder.

In *time-varying quantizers* [7], [10], [22], [38], [39] the range M is updated with time and different values of M are used at each step, $M = M[k]$. Using such a "zooming" strategy it is possible to increase coder accuracy in the steady-state mode and at the same time, to prevent coder saturation at the beginning of the process [38].

In the present paper we use the following time-based zooming strategy for a quantizer range

$$M[k] = M_0\rho^k, \quad k = 0, 1, \ldots, \tag{6}$$

where $0 < \rho \leq 1$ is the decay parameter. The initial value M_0 should be large enough to capture the region of possible initial values of y_0. Equations (5), (6) describe the coder algorithm. A similar algorithm is realized by the decoder. Namely, the sequence $M[k]$ is reproduced at the receiver node utilizing (6) such that the values of $\overline{y}[k]$ are restored with the given $M[k]$ using the received codeword $s[k] \in \mathscr{S}$.

IV. EVALUATION OF SYNCHRONIZATION ERROR

Let us evaluate the limit synchronization error, taking into account transmission of the error signal over the communication channel and coding procedure. Since the control signal is piecewise constant over sampling intervals $[t_k, t_{k+1}]$, the control law (4) becomes

$$u(t) = K\overline{\varepsilon}(t), \tag{7}$$

where $\overline{\varepsilon}(t) = \overline{\varepsilon}[k]$ as $t_k < t < t_{k+1}$, $\overline{\varepsilon}[k]$ is the result of transmission of the synchronization error signal $\varepsilon(t) = y_1(t) - y_2(t)$ over the channel, $t_k = kT$, $k = 0, 1, \ldots$.

According to the quantization algorithm (5), the quantized error signal $\overline{\varepsilon}[k]$ becomes

$$\overline{\varepsilon}[k] = M[k] \operatorname{sign}(\varepsilon(t_k)), \tag{8}$$

where the range $M[k]$ is defined by (6).

The key point of the approach is application of the so-called *method of continuous models*: analysis of the hybrid nonlinear system via analysis of its continuous-time approximate model [40], [41], see also [42].

In order to analyze the synchronization error we make two assumptions:

A1. Nonlinearity $\varphi(y)$ is Lipschitz continuous:

$$|\varphi(y_1) - \varphi(y_2)| \leq L_\varphi |y_1 - y_2| \tag{9}$$

for all y_1, y_2 and some $L_\varphi > 0$.

A2. The linear part of (1) is strictly passifiable: according to the Passification Theorem [26], [27] (see Appendix I), this means that the numerator $\beta(\lambda)$ of the transfer function $W(\lambda) = C(\lambda I - A)^{-1}B = \beta(\lambda)/\alpha(\lambda)$ is a Hurwitz (stable) polynomial of degree $n - 1$ with positive coefficients (the so-called hyper-minimum-phase (HMP) property).

It follows from condition A2 and the Passification Theorem (see Appendix I), that the stability degree η_0 of the polynomial $\beta(\lambda)$ (a minimum distance from its roots to the imaginary axis) is positive and for any $\eta: 0 < \eta < \eta_0$ there exist a positive

definite matrix $P = P^{\mathrm{T}} > 0$ and a number K such that the following matrix relations hold:

$$PA_K + A_K^{\mathrm{T}}P \leq -2\eta P, \quad PB = C^{\mathrm{T}}, \quad A_K = A - BKC. \tag{10}$$

Any sufficiently large real number can be chosen as the value of K.

The main result of this Section is formulated as follows.

Theorem 1. Let A1, A2 hold, the controller gain K satisfies passivity relations (10) and the coder parameters ρ, T be chosen to meet the inequalities

$$\exp(\eta T)\big(\exp(L_F T) - 1\big) \leq \frac{L_F}{\|C\|\big(K\|B\| + L_F\big)},$$
$$\exp(-\eta T) < \rho < 1, \tag{11}$$

where $L_F = \|A\| + L_\varphi \|B\| \cdot \|C\|$, η is from (10). Let the coder range $M[k]$ be specified as

$$M[k] = M_0\rho^k. \tag{12}$$

Then for all initial conditions $e(0)$ such that $e(0)^{\mathrm{T}}Pe(0) \leq M_0^2$ the synchronization error decays exponentially

$$|\varepsilon[k]| \leq \|e[k]\| \leq M_0\rho^k. \tag{13}$$

In addition, $|\varepsilon(t)| \leq |\varepsilon[k]|$ for $t_k \leq t \leq t_{k+1}$.

Proof. Choose K, ρ, T satisfying (10) and (11). Then the following inequality is valid:

$$L_F T \leq \ln\left(1 + \frac{\rho L_F}{\|C\|\big(K\|B\| + L_F\big)}\right).$$

Taking into account the stepwise shape of the control function in (7), rewrite the controller model in the following form:

$$u(t) = K\varepsilon(t) - K\delta(t), \tag{14}$$

where $\delta(t) = \delta_q(t) + \delta_s(t)$ is a *total error*, $\delta_q(t) = \varepsilon(t_k) - \overline{\varepsilon}[k] = Ce(t_k) - \overline{\varepsilon}[k]$ is a *quantization error*, $\delta_s(t) = \varepsilon(t) - \varepsilon(t_k) = Ce(t) - Ce(t_k)$ is a *sampling error*.

It is seen from the quantization procedure (8) that if the value $\varepsilon(t_k)$ satisfies the inequality $|\varepsilon(t_k)| \leq 2M[k]$, then the quantization error does not exceed $M[k]$: $\delta_q(t)| \leq M[k]$.

Let us evaluate the sampling error $\delta_s(t)$. To this end, apply the following auxiliary statement, proved in Appendix II.

Lemma 1. Consider the system (1), (2), (4) for $t_k \leq t \leq t_{k+1}$. Let the nonlinearity $\varphi(y)$ be Lipschitz continuous with the constant L_φ, and the initial values $\delta_s(t)$, $e(t_k)$ satisfy the inequalities $|\delta_s(t_k)| \leq M[k]$, $\|e(t_k)\| \leq M[k]$, and

$$T < \frac{1}{L_F}\ln\left(1 + \frac{\rho L_F}{\|C\|\big(K\|B\| + L_F\big)}\right). \tag{15}$$

Then the inequality $|\delta_s(t_{k+1})| \leq \rho M[k]$ holds.

Apparently, conditions of Lemma 1 are valid. It follows from the Lemma that $|\delta_s(t_{k+1})| \leq M[k+1] = \rho M[k]$ and $|\delta_s(t)| \leq M[k+1]$ for $t_k \leq t \leq t_{k+1}$.

The key point of the proof is comparison of the hybrid system in question with an auxiliary continuous-time system (the *continuous model*) possessing useful stability and passivity properties [40], [41].

Rewrite the error equation in the following form:

$$\dot{e} = Ae + B\zeta(\varepsilon, t) - Bu, \quad \varepsilon = Ce, \tag{16}$$

where $\zeta(\varepsilon,t) = \varphi\big(y(t)\big) - \varphi\big(y(t) - \varepsilon\big)$ satisfies the inequality $|\zeta(\varepsilon,t)| \le L_\varphi |\varepsilon|$.

Substituting (14) into (16) we obtain

$$\dot{e} = A_K e + B\zeta(\varepsilon,t) + BK\delta(t), \qquad (17)$$

where $A_K = A - BKC$. The last term in (17) is considered as an error term with respect to the continuous-time model (4), (16). Employing the HMP condition A2 and the Passification Theorem, see Appendix I, pick up the $(n \times n)$-matrix $P = P^{\mathrm{T}} > 0$ and the positive number K such that $PA_K + A_K^{\mathrm{T}} P \le -2\eta P$, $PB = C^{\mathrm{T}}$, and choose the Lyapunov function candidate $V(e) = \frac{1}{2} e^{\mathrm{T}} Pe$. Introducing a new nonlinearity $\xi(\varepsilon,t) = \zeta(\varepsilon,t) + L_\varphi \varepsilon$, satisfying the sector condition $\xi\varepsilon \ge 0$ and making the change $K \to K + L_\varphi$, transform equation (17) to the form

$$\dot{e} = A_K e + B\xi - B(K + L_\varphi)\delta(t). \qquad (18)$$

The time derivative of $V(e)$ is evaluated as follows:

$$\dot{V} = e^{\mathrm{T}}\big(PA_K + A_K^{\mathrm{T}} P\big)e - e^{\mathrm{T}} PB\big(K + L_\varphi\big)\delta(t),$$

or, after simple algebra

$$\dot{V} \le -2\eta V + |\varepsilon||(K + L_\varphi)||\delta(t)|. \qquad (19)$$

Integrating inequality (19) over $[t_k, t_{k+1}]$ and taking into account the Lemma, we get

$$V_{k+1} \le \exp(-2\eta T)V_k + a\rho^{2k}, \qquad (20)$$

where $a = 2(K + L_\varphi)\rho M_0^2$. Any solution of inequality (20) is majorized by the solution of the difference equation $\bar{V}_{k+1} = \exp\big(-2\eta T\big)\bar{V}_k + aM[k]^2$ with the same initial condition. Therefore

$$V_{k+1} \le \exp(-2\eta T)V_0 + \frac{a\rho^{2k}}{\rho^2 - \exp(-2\eta T)}$$
$$\le \left(V_0 + \frac{2(K + L_\varphi)\rho M_0}{\rho^2 - \exp\big(-2\eta T\big)}\right)\rho^{2k}.$$

The proof is completed.

Based on Theorem 1, the following design method is proposed. First, the value of accessible stability degree η of the continuous model (18) should be found based on the solution of LMI (10) for the chosen control gain K (the maximal value of η corresponds to the stability degree η_0 of the numerator $\beta(\lambda)$ of the transfer function $W(\lambda) = C(\lambda \mathrm{I} - A)^{-1}B$). Then the transmission rate T should be chosen from the first inequality of (11) and ρ should be chosen to meet the second inequality of (11).

Remark 1. The first inequality of (11) gives an upper bound on the sampling time $T_{\max}$ and a lower bound on the channel capacity $R_{\min} = 1/T_{\max}$. It is always solvable for sufficiently small T (i.e. for sufficiently large capacity of the communication channel) and an approximate value of the channel capacity for small T ($0 < T << 1/L_F$) is as follows:

$$R_{\min} \approx \|C\|\big(K\|B\| + L_F\big)/\rho. \qquad (21)$$

Remark 2. In a stochastic framework the estimates of the mean square value of the synchronization error can be obtained. There is a significant body of work in which the quantization error signal $\delta(t)$ is modeled as an extra additive white noise. This assumption, typical for digital filtering theory, is reasonable if the quantizer resolution is high [43], but it needs modification for the case of a low number of quantization levels [7].

Remark 3. For practice, it is reasonable to choose the coder range $M[k]$ separated from zero. The following zooming strategy for a quantizer range may be recommended instead of (6) [22]:

$$M[k] = (M_0 - M_\infty)\rho^k + M_\infty, \quad k = 0, 1, \ldots, \qquad (22)$$

where $0 < M_\infty < M_0$ stands for the limit value of $M[k]$.

V. EXAMPLE. SYNCHRONIZATION OF CHAOTIC CHUA SYSTEMS

Let us apply the above results to synchronization of two chaotic Chua systems coupled via a channel with limited capacity.

Master system. Let the master system (1) be represented by the following *Chua system*:

$$\begin{cases} \dot{x}_1 = p(-x_1 + \varphi(y_1) + x_2), & t \ge 0, \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -qx_2, \end{cases} \qquad (23)$$
$$y_1(t) = x_1(t),$$

where $y_1(t)$ is the master system output, p, q are known parameters, $x = [x_1, x_2, x_3]^{\mathrm{T}} \in \mathbb{R}^3$ is the state vector; $\varphi(y_1)$ is a piecewise-linear function, having the form:

$$\varphi(y) = m_0 y + m_1(|y + 1| - |y - 1|), \qquad (24)$$

where m_0, m_1 are given parameters.

Evidently, Chua system (23) may be represented in Lurie form (1) with the matrices:

$$A = \begin{bmatrix} -p & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{bmatrix}, \quad B = \begin{bmatrix} p \\ 0 \\ 0 \end{bmatrix}, \quad C = [1, 0, 0]. \qquad (25)$$

It is easy to check that the linear part of the Chua system satisfies the HMP condition. Indeed, for the triple (A, B, C) from (25) the transfer function $W(\lambda) = C(\lambda \mathrm{I} - A)^{-1}B = \beta(\lambda)/\alpha(\lambda)$ is as follows:

$$W(\lambda) = \frac{p(\lambda^2 + \lambda + q)}{\lambda^3 + (1 + p)\lambda^2 + q\lambda + pq}.$$

The numerator $\beta(\lambda) = p(\lambda^2 + \lambda + q)$ is a Hurwitz polynomial of degree 2, i.e. the HMP condition holds for all $p > 0$, $q > 0$.

Slave system. Correspondingly, the slave system equations (2) for the considered case becomes

$$\begin{cases} \dot{z}_1 = p\big(-z_1 + \varphi(y_2) + z_2 + u(t)\big), & t \ge 0, \\ \dot{z}_2 = z_1 - z_2 + z_3 \\ \dot{z}_3 = -qz_2, \end{cases} \qquad (26)$$
$$y_2(t) = z_1(t),$$

where $y_2(t)$ is the slave system output, $z = [z_1, z_2, z_3]^{\mathrm{T}} \in \mathbb{R}^3$ is the state vector, $\varphi(y_2)$ is defined by (24).

Controller has a form (7), where the control gain K is a design parameter.

Coding procedure has a form (6), (8). The input signal of the coder is $\varepsilon(t)$. The error signal $\bar{\varepsilon}(t)$ of the controller (7) is found by holding the value of $\bar{\varepsilon}[k]$ over the sampling interval $[kT, (k+1)T)$, $k = 0, 1, \ldots$. The initial value M_0 of the coder range and the decay factor ρ in (6) are design parameters.

The following parameter values were used for the simulation:

- Chua system parameters: $p = 10$, $q = 15.6$, $m_0 = 0.33$, $m_1 = 0.945$. For chosen parameter values the system behavior is chaotic;
- the controller gain $K = 10$. The gain K is chosen according to relations (10) in Theorem 1. Feasibility of relations (10) for this value of K and the given matrices A, B, C is checked by means of YALMIP package [44];
- the sampling time T was taken from the interval $T \in [0.02, 0.1]$ s for different simulation runs (a corresponding interval for the transmission rate R is $R \in [10, 50]$ bit/s);
- the coder parameter $M_0 = 5$ is chosen to cover the region of the initial values of y_0. This region is found based on the maximum value of $|y_0|$ over Chua system attractor;
- the coder parameter is taken for each sampling interval T as $\rho = \exp(-\eta T)$, where parameter $\eta = 0.3$. This value is chosen according to Theorem 1 to ensure inequalities (11);
- the initial conditions for the master and slave systems were: $x = [3, -1, 0.3]^{\mathrm{T}}$, $z = 0$;
- the simulation final time $t_{\mathrm{fin}} = 1000$ s.

The normalized state synchronization error

$$Q = \frac{\max\limits_{0.8t_{\mathrm{fin}} \leq t \leq t_{\mathrm{fin}}} \|e(t)\|}{\max\limits_{0 \leq t \leq t_{\mathrm{fin}}} \|x(t)\|}, \tag{27}$$

where $\delta_y(t) = y_1(t) - \bar{y}_1(t)$, $e(t) = x(t) - z(t)$ was calculated.

Simulation results are plotted in Figs. 2–8.

Synchronization performance may be evaluated based on time histories of the state variables $x_1(t)$, $z_1(t)$, $x_3(t)$, $z_3(t)$ and the synchronization errors $e_1(t)$, $e_3(t)$. Typical trajectories are depicted in Figs. 2, 3 (respectively). As seen from the plots, the synchronization transient time is about 15 seconds, which agrees with the chosen value of the coder parameter η.

Details of the synchronization process are demonstrated in Fig. 4, where the time histories of the output synchronization error $\varepsilon(t)$, quantized error $\bar{\varepsilon}(t)$ signal and the transmitted binary sequence $s[k]$ (for $\mathscr{S} = \{-1, 1\}$, $\eta = 0.3$, $R = 25$ bit/s) at the first 5 seconds of the process are plotted.

The logarithmic graphs of the output synchronization error $|\varepsilon(t)|$ and the coder range $M(t_k) = M[k]$ time histories for different values of the coder parameter η, $\eta \in \{0.1, 0.3, 0.5\}$ and $R = 25$ bit/s are plotted in Figs. 5–7. These plots demonstrate that the synchronization error vanishes exponentially, and its decay rate coincides with the coder decay parameter $\eta > 0$, but synchronization fails if η exceeds a threshold value $\eta_{\max}$.

The logarithmic graph of the normalized synchronization error Q as a function of the transmission rate R is shown in Fig. 8. The *minimal bound* $R_{\min}$ is obtained numerically by intensive simulation, based on evaluation of the normalized state synchronization error (27). The simulations were made for 500 values of R, taken from the logarithmically split range $[10, 50]$ bit/s. It is seen from this plot that if the transmission rate exceeds the minimal bound $R_{\min} \approx 24$ bit/s, the proposed controlled synchronization strategy ensures asymptotical vanishing synchronization error. If the transmission rate is less that the bound $R_{\min}$, the synchronization is not always possible.

In the above example the identical parameters of the master and slave systems were taken. Let us take into account parameter mismatches. Let the parameters $\tilde{p} = (1 + \partial)p$ and $\tilde{q} = (1 + \partial)q$ of the slave system (26) be taken instead of p and q (respectively) where ∂ is a given *relative mismatch*. For the case of non-identical master and slave systems, vanishing synchronization error can not been ensured. Then the zooming strategy (6) leads to the synchronization loss, as it is demonstrated in Fig. 9 for the case of $\partial = 0.03$. The choice of the coder range $M[k]$ separated from zero makes possible achieving the synchronization of systems with parameter mismatches. Apply the zooming strategy (22) for a quantizer range instead of (6). To study an effect of the parameter mismatch numerically, the number of N simulations were accomplished. The relative mismatches ∂_j for each simulation run $j = 1, 2, \ldots, N$ were taken as a uniformly distributed random numbers from the interval $[-0.3, 0.3]$. The corresponding normalized state synchronization errors Q_j were calculated and the quantity $Q = \sup\limits_{1 \leq j \leq N} Q_j$ was taken as an overall assessment of the system performance. The simulations were made for $\partial = 0.03$, $N = 100$. The logarithmically scaled output synchronization errors $|\varepsilon_j(t)|$ and the coder range $M[k]$ are plotted in Fig. 10. Normalized synchronization error Q as a function of the transmission rate R is shown in Fig. 11. The simulation results demonstrate that the synchronization occurs if $M_\infty = 0.1$, confirming the aforesaid suggestion of usability of the zooming strategy (22).

Remark 4. An idealized problem has been considered in this paper to highlight the effect of the data-rate limitations in the closed-loop synchronization of nonlinear systems. In real-world problems external disturbances, measuring errors and channel imperfections should be taken into account. Evidently in the presence of irregular nonvanishing disturbances, asymptotic convergence of the master and slave systems trajectories cannot be achieved.

Remark 5. Similar results are obtained if the control signal is also subjected to information constraints.

Remark 6. Optimality of the binary coder for synchronization under information constraints was established in [22] for the case when the master system output $y_1(t)$ rather than the output synchronization error $\varepsilon(t)$ is transmitted over the channel. The problem of coder optimization for the considered case is under investigation.

Remark 7. The results obtained for synchronization problem can be extended to tracking problems in a straightforward manner, if the reference signal is described by an *external* (*exogenous*) state space model [45]–[47]. In the disturbance free case an asymptotically exact tracking will be ensured with a finite transmission rate if the linear part of the external model is passifiable.

VI. CONCLUSION

Limit possibilities of controlled synchronization systems under information constraints imposed by limited information capacity of the coupling channel are evaluated. It is shown that the framework proposed in [22], is suitable not only for observer-based synchronization but also for controlled master-slave synchronization via a communication channel with limited information capacity.

We propose a simple coder-decoder scheme and provide theoretical analysis for multi-dimensional master-slave systems represented in Lurie form. An output feedback control law is proposed based on the Passification Theorem [26], [27]. It is shown that the synchronization error exponentially tends to zero for sufficiently high transmission rate (channel capacity). The key point of the synchronization analysis is comparison of the hybrid system in question with an auxiliary continuous-time system (*the continuous model*) possessing useful stability and passivity properties. Such an approach was systematically developed in the 1970s under the name of *the Method of Continuous Models* [40], [41].

The results are applied to controlled synchronization of two chaotic Chua systems via a communication channel with limited capacity. Simulation results illustrate and confirm the theoretical analysis.

Unlike many known papers on control of nonlinear systems over a limited-band communication channel, we propose and justify a simple coder/decoder scheme, which does not require transmission of the full system state vector over the channel. A constructive design method for controller and coder/decoder pair is proposed and estimates of the convergence rate are given. The results obtained for synchronization problem can be extended to tracking problems in a straightforward manner, if the reference signal is described by an external (exogenous) state space model.

Future research is aimed at examination of more complex system configurations, where channel imperfections (drops, errors, delays) will be taken into account.

APPENDIX I
PASSIFICATION THEOREM

Consider a linear system

$$\dot{e} = Ae + B\xi(t), \quad \varepsilon = Ce \qquad (28)$$

with transfer function $W(\lambda) = C(\lambda I - A)^{-1}B = b(\lambda)/a(\lambda)$, where $b(\lambda)$, $a(\lambda)$ are polynomials, degree of $a(\lambda)$ is n, degree of $b(\lambda)$ is not greater than $n-1$. The system is called *hyper-minimum phase* (HMP), if $b(\lambda)$ is Hurwitz polynomial of degree $n-1$ with positive coefficients. To find existence conditions for a quadratic Lyapunov function we need the following result.

Passification Theorem [26], [27]. There exist positive-definite matrix $P = P^{\mathrm{T}} > 0$ and a number K such that

$$PA_K + A_K^{\mathrm{T}}P < 0, \quad PB = C^{\mathrm{T}}, \quad A_K = A - BKC \qquad (29)$$

if and only if $W(\lambda)$ is HMP.

Consider a linear system with feedback

$$\dot{e} = A_K e + B\xi(t), \quad \varepsilon = Ce, \quad A_K = A - BKC. \qquad (30)$$

Remark A1. It follows from the Passification Theorem that there exist a quadratic form $V(e) = e^{\mathrm{T}}Pe$ and a number K such that time derivative $\dot{V}(e)$ of $V(e)$ along trajectories of (30) satisfies relation

$$\dot{V}(e) < 0 \quad \text{for} \ \xi\varepsilon \geq 0, \ x \neq 0 \qquad (31)$$

if and only if $W(\lambda)$ is HMP. Indeed, assume that K is fixed. Relation (31) is equivalent to existence of the matrix $P = P^{\mathrm{T}} > 0$ such that $e^{\mathrm{T}}P(A_K e + B\xi) + \xi Ce < 0$ for $x \neq 0$. Since ξ is arbitrary, the latter in turn, is equivalent to matrix relations $PA_K + A_K^{\mathrm{T}}P < 0$, $PB = C^{\mathrm{T}}$ and, by Passification Theorem, to HMP condition.

Remark A2. It also follows from Passification Theorem that if HMP condition holds then K satisfying (29) can be chosen sufficiently large. Besides, zero matrix in the right hand side of the inequality in (29) can be replaced by matrix $-\mu P$ for sufficiently small $\mu > 0$.

APPENDIX II
PROOF OF LEMMA 1

Let $F_k(e,t) = Ae + B\bar{\varphi}(\varepsilon,t) + BKM[k]\,\mathrm{sign}\,\delta_s(t)$ be the right hand side of the error equation

$$\dot{e} = Ae + B\bar{\varphi}(\varepsilon,t) + BKM[k]\,\mathrm{sign}\,\delta_s(t), \qquad (32)$$

where $\bar{\varphi}(\varepsilon,t) = \varphi(y(t)) - \varphi(y(t) - \varepsilon)$. Then $F_k(e,t)$ is Lipschitz in e with the constant $L_F = \|A\| + L_\varphi\|B\| \cdot \|C\|$. Denote $\bar{e}(t) = e(t) - e(t_k)$. Then $\|\bar{e}(t)\| \leq (t - t_k)\|F_k(e(t_k),t_k)\| + \int_{t_k}^{t} L_F\|\bar{e}(t)\|\mathrm{dt}$.

By a version of Gronwall–Bellman inequality,

$$\|e(t) - e(t_k)\| \leq \|F_k(e(t_k),t_k)\|\frac{\exp(L_F(t-t_k)) - 1}{L_F}.$$

and, therefore,

$$\delta_s(t_{k+1}) \leq \|C\| \cdot \left(M[k]k\|B\| + L_F\|e(t_k)\|\right)\frac{\exp(L_F(t_{k+1} - t_k)) - 1}{L_F}. \qquad (33)$$

Hence, inequality $|\delta_s(t_{k+1})| < \rho M[k]$ holds if the right hand side of (33) does not exceed $\rho M[k]$: $\|C\|\left(M[k]K\|B\| + L_F\|e(t_k)\|\right)\frac{\exp(L_F T) - 1}{L_F} \leq \rho M[k]$. Recalling condition $\|e(t_k)\| \leq M[k]$ and resolving for T completes the proof. Apparently the inequality $|\delta_s(t)| \leq \rho M[k]$ holds as well.

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