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journal homepage: www.elsevier.com/locate/wavemotiSolitary wave interactions and reshaping in coupled systems[☆]A.V. Porubov^a, G.A. Maugin^{b,*}, B.R. Andrievsky^a^a Institute of Problems in Mechanical Engineering, Bolshoy av. 61, V.O., Saint-Petersburg 199178, Russia^b Université Pierre et Marie Curie (Paris 6), Institut Jean le Rond d'Alembert, UMR 7190 CNRS, Case 162, Tour 55, Place Jussieu 4, F-75252 Paris Cedex 05, France

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ABSTRACT

The interaction of the components of composite solitary waves governed by nonlinear coupled equations is studied numerically. It is shown how predictions of the known exact traveling wave solutions may help in understanding and explaining the process of reshaping seen as head-on and take-over collisions of individual solitary waves. The most interesting results concern the switch in the sign or the periodic modulation of the amplitude of the solitary wave and the direction of its propagation due to collisions.

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1. Introduction

The interaction of solitary waves has attracted much attention because it emphasizes the principle difference from the behavior of linear waves. Thus, the familiar take-over interaction of solitons of the Korteweg-de Vries (KdV) equation, see, e.g., [1], yields a one-hump structure for a considerable difference in the incident amplitudes, while a two-hump structure appears for the waves with close amplitudes. In both cases the highest peak in the interaction zone is less than the sum of the amplitudes of the incident waves, and the linear superposition principle does not work. The integrable Gardner equation demonstrates another specific phenomenon when the solitary wave suffers sharp short time variations in the sign of the amplitude during take-over interaction with another solitary wave [2,3]. Similar nonlinear processes have been revealed only numerically for interactions of the solitary wave solutions of non-integrable equations [4–8]. At the same time inelastic (but almost elastic) collision for nonintegrable equations may be studied analytically to some extent [9,10]. These equations contain nonlinear and dispersive terms since bell-shaped solitary waves usually arise as a result of balance between nonlinearity and dispersion. When the relevant equation also contains dissipative terms, this balance is broken yielding decay of the solitary waves. However, addition of active terms makes possible compensation for dissipative ones. This in turn reinstates a balance between nonlinearity and dispersion, and bell-shaped solitary waves may propagate and collide [11,12].

The description of solitary wave interactions becomes more complicated when the process is governed by coupled equations. Sometimes one can exactly or asymptotically decouple these equations yielding a single governing equation for one function and the equations connecting it with other functions. A familiar example is the shallow water problem where a single governing equation for the free surface elevation is the KdV equation or a variant of the Boussinesq equation [8,13]. In this case the equation of connection between fluid velocity and surface elevation does not impose considerable restrictions on the solution, and the interaction of solitary waves is studied using the methods developed for single equations.

Also, the collisions of solitary waves are studied analytically or numerically without decoupling. In particular, Euler equations are solved in [14], and conservation laws are employed to account for solitary wave interactions for estimations of deviations from

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pure elastic interactions. The collisions between solitary surface waves (for only one variable of the Euler equations) were studied in [14] while the wave interactions in the coupled systems may be between the waves described by different variables. On the one hand, there are no wave interactions here in a strict sense because the solutions for the two variables always co-exist. There is only one solitary wave with two degrees of freedom. On the other hand – a line we follow here –, one can assume that single solitary waves of each function of the coupled equations do interact. In this line of thought one can note the coupled Nonlinear Schrödinger equations (CNLS) where such coupling give rise to new exact solutions, see [15] and references therein. Both take-over and head-on collisions of the solitary wave solutions of the CNLS equations were studied numerically by C.I. Christov and co-workers in Refs. [16–18]. They have found that linear and/or nonlinear coupling in the CNLS equations causes variations in the features of the waves due to collisions. In particular, a nonlinear effect on interactions has been obtained in Ref. [16] when coupled waves are gradually excited during the interaction, and then the wave of one function carries away trapped by the wave of another function.

The nonlinearity and dispersion exist in each of the CNLS equations even without coupling. Another case arises when both longitudinal and transverse displacements are taken into account in a continuum limit of a lattice model [19,20]. It is governed by a generalization of a Boussinesq-like equation with higher-order nonlinearity by coupling with another equation, and coupling does provide nonlinearity and dispersion only for one of the functions, while the equation for another one contains these terms without coupling. Numerical simulations of these equations performed in [19,20] did not reveal trapped waves due to collisions.

In this paper we consider equations where coupling is also the decisive factor affecting nonlinearity and dispersion for one of the functions. However, their solitary wave interaction exhibits not only trapping but variations in the amplitude of the wave and its direction of propagation. The equations under study come from the theory of microstructures where coupling between macro- and micro-strains happens. Thus, the rotatory molecular groups were added to the usual one in atomic chains in Refs. [21], and large rotations were considered. A more complicated internal motion is modeled in Refs. [22–24], where translational internal motion is considered.

The governing equations in both problems are similar. Here we consider the one-dimensional limit of equations from Refs. [22–24] accounted for by the coupled equations of the form

$$\rho v_{tt} - E v_{xx} = S (\cos(u))_{xx}, \tag{1}$$

$$\mu u_{tt} - \kappa u_{xx} = (Sv - p) \sin(u). \tag{2}$$

where $v = U_x$ describes macro strains, u accounts for a non-dimensional relative micro-displacement, S is a coupling or a striction coefficient, and p accounts for an activation energy of interatomic connections [22–24]. Choice of the trigonometric function allows us to describe translational symmetry of the crystal lattice, while the trigonometric terms account both for nonlinearity and coupling. There are no nonlinear terms for v in Eq. (1). Without coupling, the linear wave equations holds for v when $S = 0$. Further it will be seen that the nonlinear equation with dispersion holds for the function v thanks to the coupling.

Often a formal analysis based on the use of conservation laws may be employed for solitary wave interactions, see, e.g., [14]. However, these laws rarely exist for coupled systems, in particular, the authors of [16] failed to obtain more than the conservation of mass for the CNLS equations. It seems this is also the case of our equations, and we shall use particular exact solutions for interpreting the numerical results. There exist traveling solitary wave solutions to Eqs. (1), (2) [24]. Presumably these equations are non-integrable, and the interaction of the wave solutions has to be studied numerically. However predictions obtained on the basis of particular exact solutions may be employed to develop and check numerical solutions. Particular exact solutions and balance laws were often used by C.I. Christov to perform numerical investigation even outside the formal applicability of particular solutions. Such a combined approach was also developed in our previous works [4,5,12] where its efficiency has been demonstrated for single non-integrable nonlinear equations.

Considerable progress has been achieved in numerical studies of nonlinear equations. In particular, various algorithms have been developed for the numerical study of single and coupled nonlinear equations by C.I. Christov and co-workers [8,11,16,17]. A realization of algorithms is now possible within available symbolic programs. Thus we shall solve our equations numerically by means of a standard ODE solver using the standard MATLAB routine *ode45* [25]. Also calculations based on the numerical facilities of Mathematica 7, will be performed. Both calculations are compared to avoid errors caused by the scheme.

The plan of the paper is as follows. First some known exact solitary wave solutions of Eqs. (1), (2) are presented. These solutions are used further to perform a numerical study of the solitary wave interactions. The new effects caused by couplings will be revealed studying both head-on and take-over collisions of the waves v and u . An influence of initial velocity on the wave interactions will be investigated.

2. Exact solitary wave solutions

To obtain exact traveling wave solutions to Eqs. (1), (2) an exact decoupling is used, not an asymptotic one like in the shallow water problem. It is possible to do it for traveling wave solutions depending on the phase variable $\theta = x - V t$. Going to the new variable, one can solve Eq. (1) for u as

$$\cos(u) = 1 - \frac{(E - \rho V^2)v - \sigma}{S}, \tag{3}$$

where σ is a constant of integration. Then Eq. (2) is integrated once, multiplied by u_θ and integrated again. Finally, use of Eq.(3) allows us to obtain an ordinary differential equation (ODE) for the function $v(\theta)$,

$$v_\theta^2 = a_0 + a_1 v + a_2 v^2 + a_3 v^3 + a_4 v^4. \tag{4}$$

The expression of the coefficients may be found in Ref. [24]. Similar findings may be obtained for coupled equations from Ref. [21]. One can note that the well-known integrable Gardner equation [3,28],

$$v_t + q_1 v v_x + q_2 v^2 v_x + q_3 v_{xxx} = 0, \tag{5}$$

possesses the same ODE for its traveling wave solutions. This is also true for the equation

$$v_{tt} - av_{xx} - c_1 (v^2)_{xx} - c_2 (v^3)_{xx} + b_1 v_{xxtt} - b_2 v_{xxxx} = 0. \tag{6}$$

obtained in Refs. [4,5,26] for longitudinal nonlinear strain waves. Dispersion terms $q_3 v_{xxx}$ in Eq. (5) and $b_1 v_{xxtt}, b_2 v_{xxxx}$ in Eq.(6) ensure the term v_θ^2 in ODE (4). In our case this dispersion term arises thanks to the coupling.

Only when $a_0=0, a_1=0$ does the ODE Eq. (4) possess known exact bell-shaped localized solutions. These conditions are achieved for two particular values of $\sigma, \sigma=0$ and $\sigma=-2S$ [24]. The solution is of two types that may be obtained by direct integration [27,28],

$$v_1 = \frac{A}{Q \cosh(k \theta) + 1}, \tag{7}$$

and

$$v_2 = -\frac{A}{Q \cosh(k \theta) - 1}. \tag{8}$$

The coefficients for $\sigma=0$ read

$$A = \frac{4 S}{\rho(c_0^2 + c_l^2 - V^2)}, Q_\pm = \pm \frac{c_l^2 - V^2 - c_0^2}{c_l^2 - V^2 + c_0^2}, k = 2\sqrt{\frac{p}{\mu(c_l^2 - V^2)}} \tag{9}$$

where $c_l^2 = E/\rho, c_0^2 = \kappa/\mu, c_0^2 = S^2/(p \rho)$. The corresponding set for $\sigma = -2S$ may be found in Ref. [24].

The bounded solutions (7), (8) may co-exist for the Gardner equation and Eq. (6) [3,28], and their amplitudes are always of either sign. However, an analysis of the reality of the parameters (9) gives rise to the conclusion that simultaneous existence of these bounded solutions is impossible [24], and the sign of the solitary wave v is defined by the value of the phase velocity V .

This is not true for the waves u as follows from Eq. (3). The shape of u depends upon the value of the first derivative at $\theta=0$ in the r.h.s. of Eq. (3). Inverting the \cos -function for derivation of an explicit expression for u , one has to avoid the points where the first derivative does not exist. This breaking happens for $\theta=0$ at $\sigma=0$ and for $Q=Q_+$. Therefore, the solution for u obtained using both Eqs. (7) and (8) should be written as

$$u = \pm \arccos\left(\frac{(\rho V^2 - E)v}{S} + 1\right) \text{ for } \theta \leq 0, \tag{10}$$

$$u = \pm \pi \mp \arccos\left(\frac{(\rho V^2 - E)v}{S} + 1\right) \text{ for } \theta > 0, \tag{11}$$

However, the first derivative is zero for $Q=Q_-$ at $\theta=0$, and the expression for u then reads

$$u = \pm \arccos\left(\frac{(\rho V^2 - E)v}{S} + 1\right). \tag{12}$$

The solutions (10), (11) accounts for the kink-shaped profile of the wave, while solution (12) describes the bell-shaped solitary wave. The velocity intervals when one or another profile exists as well as an analysis for $\sigma = -2S$ may be found in [24,29]. The sign \pm in the above expressions means that one wave v may be accompanied by two different waves u . Finally, one can note that traveling wave solutions should satisfy specific initial conditions in the form of these solutions at $t=0$ and the maxima/minima of the solitary wave solutions for v and u should coincide.

In the following we consider only the velocity interval $(c_l^2 - c_0^2; c_l^2)$ for $\sigma=0$ that corresponds to the solitary waves Eqs. (7) and (12) for v and u respectively [24,29].

3. Solitary wave interactions

The analysis based on the exact solutions is not sufficient due to the above mentioned dependence on the initial conditions. But are they useful as a matter of fact? To see this we perform numerical simulations of our governing equations varying conditions required for the existence of the exact traveling wave solutions (7), (12). The most important ingredient for collisions is the difference in relative position of the initial conditions. This problem was the focus of Ref. [30] where it was shown that the difference in the relative position of the waves moving in the *same* direction yields considerable disturbances in the shape of u . However, further it recovers the shape of the exact solution (12) and propagates together with the solitary wave v according to exact solutions (7), (12).

The time of recovering is rather large. Our previous calculations in Ref. [29] demonstrate it on an example of evolution of motionless Gaussian input for u with initial exact profile Eq. (7) and initial velocity V directed to the right chosen for v . The formation of a traveling solitary wave u is accompanied by oscillating variations in the amplitude which are shown by the dotted line in Fig. 1. Finally (times 340–400 in Fig. 1), the amplitudes and velocities of the solitary waves u and v do not vary anymore, and traveling solitary waves with constant velocity propagate together according to the exact solutions (7), (12). Similarly, a negative amplitude solitary wave u arises when an initial Gaussian profile for u is chosen with negative amplitude. It is not a good idea, for the sake of visibility, to show both processes of solitary wave interaction and further recovering of their shapes in a single figure. That is why the last process will be skipped in the following figures.

3.1. Head-on collision

The shapes' initial conditions are chosen in the form of exact solutions (7), (12) but the positions of their peaks, see Fig. 2, do not coincide. The initial velocities are chosen equal to the V of the exact solution but directed to the right for v and to the left for u .

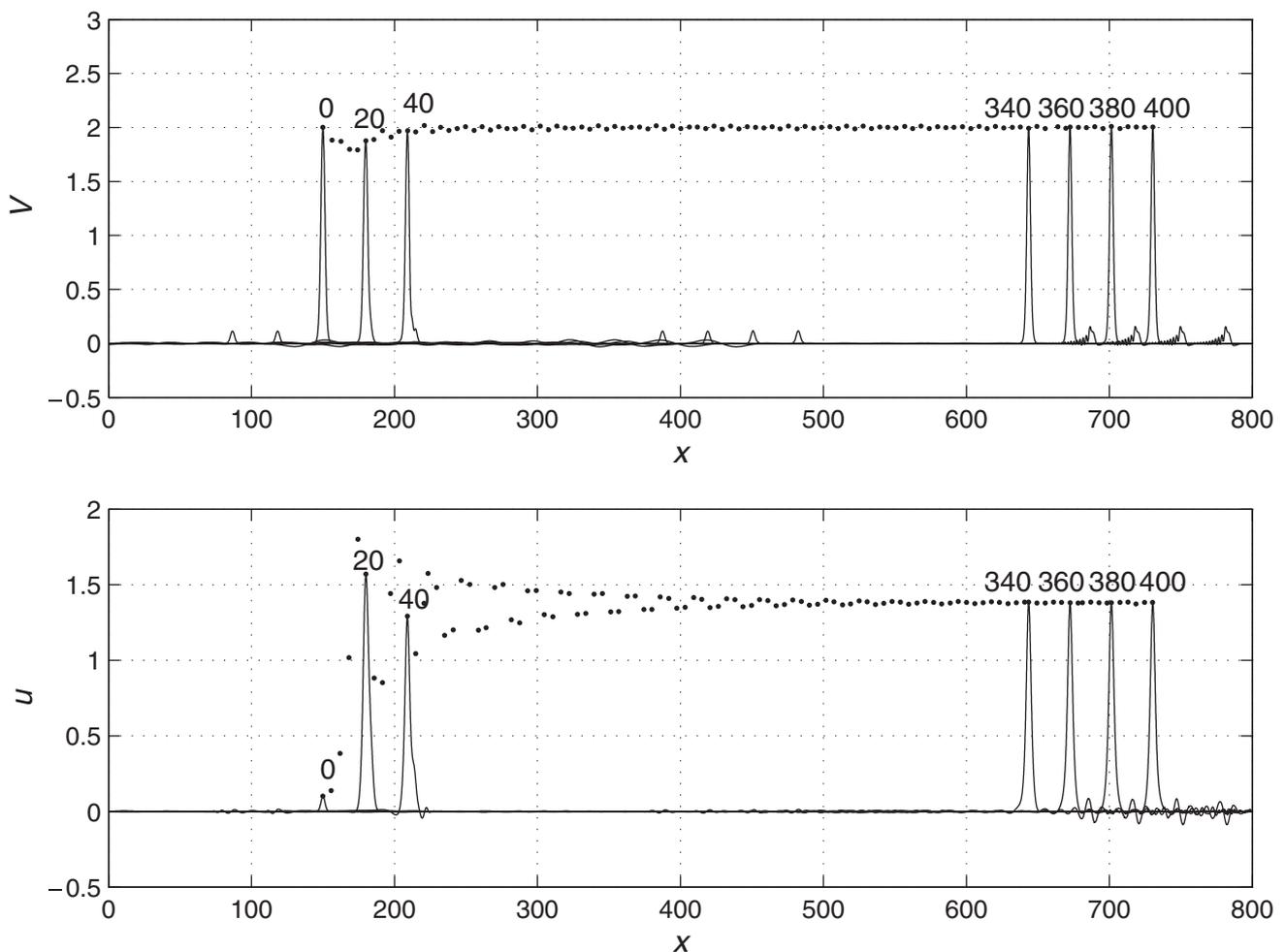


Fig. 1. Generation of solitary wave u from motionless Gaussian input. Points of time are marked at the neighboring peaks. Dotted oscillating lines account for the amplitude variations.

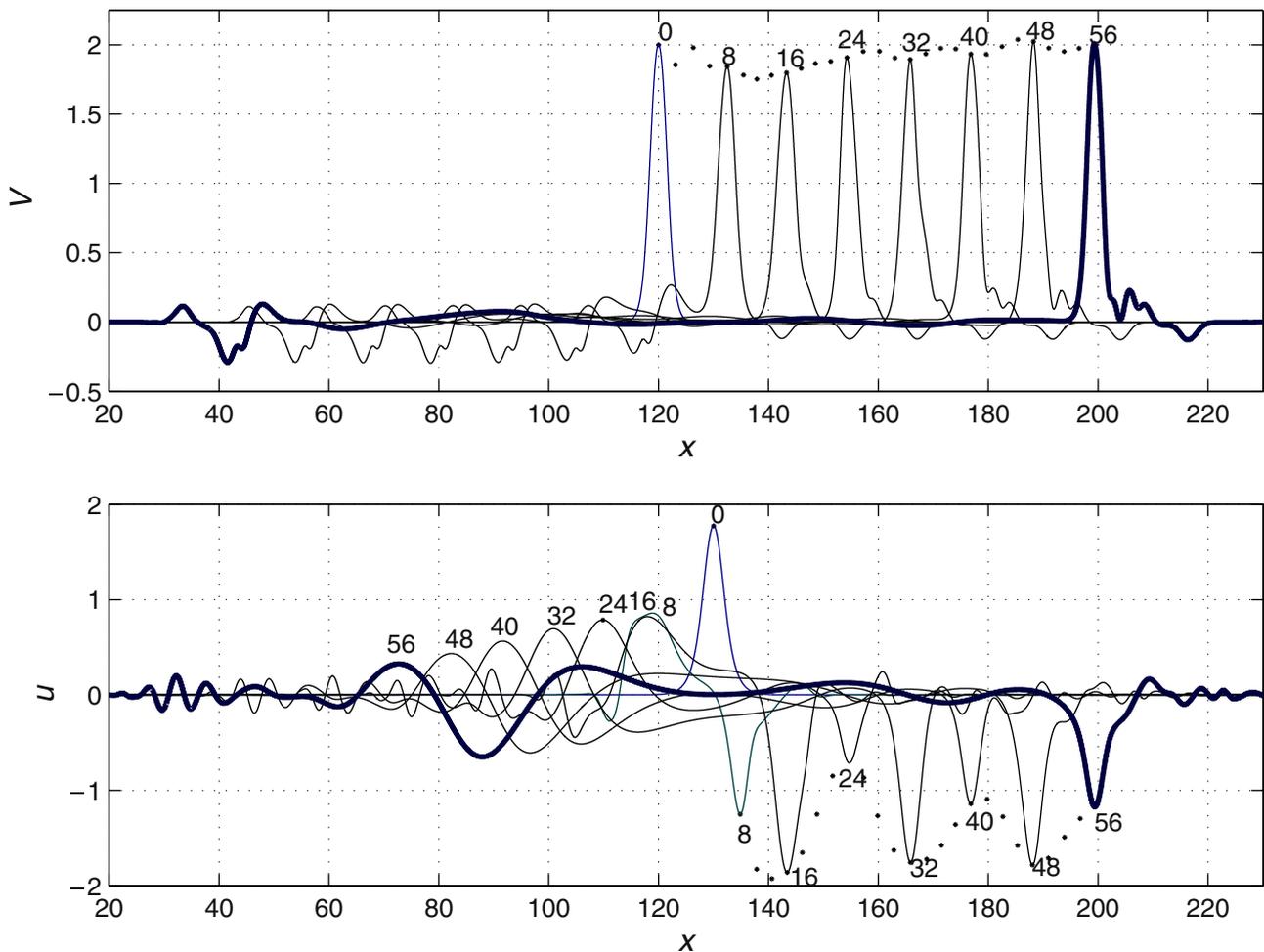


Fig. 2. Head-on collision of the solitary waves when initial position of u equal to 130 units ahead of that of v equal to 120 units. Points of time are marked at the corresponding peaks and holes. Dotted lines illustrate variations in the amplitude. Bold curves mark final positions of the waves.

As a result the waves v and u move towards each other and collide, as shown in Fig. 2. It is known, that solitary waves usually restore their shapes and velocity after interaction and suffer only phase shift in integrable systems governed by single nonlinear equations. Non-integrability usually yields some small oscillating tails around solitary waves still propagating with permanent shapes and velocity. Fig. 2 demonstrates a completely different scenario. Following the time marks one can see that the wave v evolves more or less like the waves for single non-integrable equations. There is a tail that propagates to the left but the initial solitary wave v suffers only oscillations in its amplitude. On the contrary, the wave u is trapped by the wave v and changes its direction of propagation to the opposite one. Moreover, the sign of the amplitude of u is switched as seen already for time equal to 8. One can see that the incident wave is split into two parts. One part of positive amplitude continues moving to the left. The second one of negative amplitude is trapped, and its minimum coincides with the corresponding maximum of v . Further oscillations in the amplitude of the latter part of u are observed (see dotted line in Fig. 2 for u) but the position of the minimum of u always coincides with the maximum of v . The part of u moving to the left is gradually delocalized as seen in Fig. 2 due to the time marks. Thus, the solitary wave u of initial positive amplitude evolves into a solitary wave of negative amplitude.

The dominant role of v over u cannot be explained by exact solutions. However, the fact that the waves propagate together and the extremal points coincide is in an agreement with Eqs. (7), (12). From this point of view trapping means a tendency of the exact solution that “decides” that traveling waves u and v cannot propagate in another way than together. The waves moving to the right in Fig. 2 propagate with the same and permanent velocity within the interval prescribed for bell-shaped waves. Also change of the sign of the amplitude does not contradict the exact solution (12). As we noted before the permanent amplitude of u is achieved further for a considerable time, see Fig. 1.

Trapping does not depend on the relative distance between inputs. The latter causes drastic variations in the evolution of u as shown in Fig. 3. The larger – than in Fig. 2 – initial distance between inputs yields a trapped wave u propagating together with the wave v but now it keeps its incident positive amplitude. One can note that the waves propagate with the same velocity in Figs. 2, 3. The evolution of the part of u moving to the left in Fig. 3 remains the same as that shown in Fig. 2.

Further increase in the relative distance demonstrates almost periodic appearance of the scenarios shown in Figs. 2 and 3. Thus, keeping the initial position of the peak of v and moving the position of the peak of the input for u within the interval 145–165, we

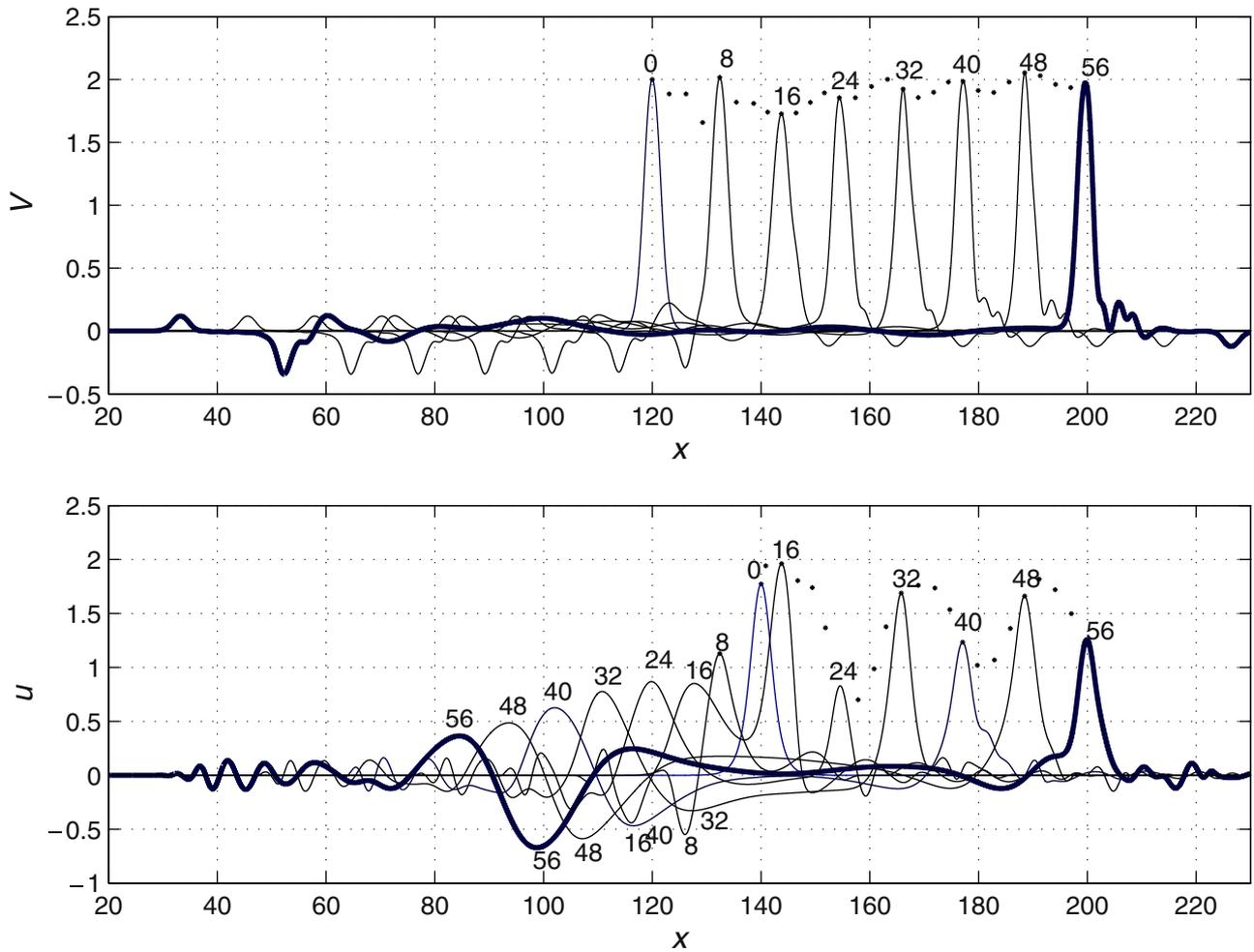


Fig. 3. Head-on collision of the solitary waves when initial position of u is equal to 140 units ahead of that of v equal to 120 units. Points of time are marked at the corresponding peaks and holes. Dotted lines mark variations in the amplitude. Bold curves mark final positions of the waves.

obtain a wave evolution with the switch of the sign of the amplitude of u like in Fig. 2. However, further increase in the 165–180 interval yields an evolution without switch as shown in Fig. 3. Larger distance again gives rise to variation in the sign etc.

Variation in the initial velocity V of u does not yield considerable variations in the waves' evolution in comparison with those shown in Figs. 2,3. On the contrary, variation in the initial velocity V of v results in new scenarios for the evolution of the wave u . One can see in Fig. 4 that trapped part of u moving to the right does not change the sign of the amplitude first- see times 16 and 24. In the following, besides change of the sign of the amplitude as in Fig. 2, decrease in the initial velocity gives rise to a deeper variation in the wave amplitude. We see that it decreases almost to zero, then recovers, and finally again decreases. The process looks almost as periodic.

Further decrease in the initial velocity of v yields a new kind of propagation of the solitary wave u whose amplitude is modulated and changes periodically as shown in Fig. 5. Position of both peak and hole of the wave u at each time coincides with the position of the peak of the wave v . The process of modulation also looks almost as periodic.

3.2. Take-over collision

The shapes of initial conditions are chosen the same as in the previous section. Recently, an influence of the relative positions of the inputs (7), (12) on the waves' evolution was studied in [30]. It was found that co-directional waves v and u interact even if their initial velocities are equal. Contrary to the case of head-on collisions, there exists only one interval of the values of relative positions where the switch of the amplitude of u occurs. Otherwise, the solitary wave u with initial polarity propagates together with the solitary wave v . The velocity of the wave u and the positions of its peak or trough of it finally coincide with those of v . Certainly the direction of propagation of u does not switch to the opposite one as it does for head-on collisions.

An influence of the initial velocity is studied when at least one of the velocities of u and v is increased or decreased in order to realize take-over of the waves. When the initial velocity of v is increased this results in the growth of the amplitude of the ahead propagating wave u at the initial stage. Very quickly the positions of the peaks of v and u come to coincide. Decrease in the initial velocity of v yields more variations in the shape of u . At small decrease switch of the sign of the amplitude of u happens, and

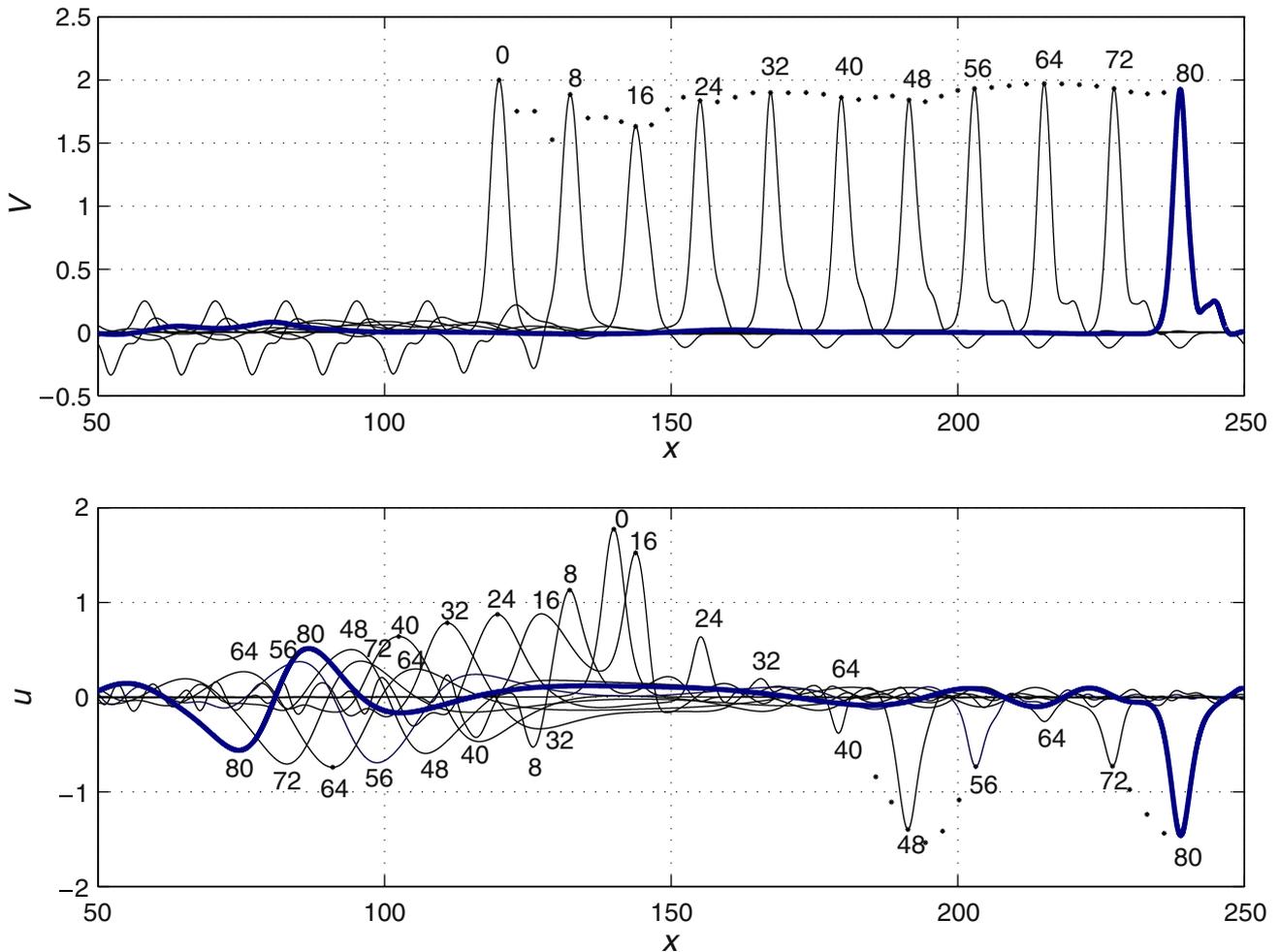


Fig. 4. Head-on collision of the solitary waves when initial position of u is equal to 140 units ahead of that of v equal to 120 units, but the initial velocity for v is chosen equal to $0.85 \cdot V$. Points of time are marked at the corresponding peaks and holes. Dotted lines mark variations in the amplitude. Bold curves mark final positions of the waves.

solitary wave with opposite polarity propagates together with the wave v , it looks similar to Fig. 2. Further decrease in the velocity gives rise to the appearance of a periodically modulated solitary wave u as shown in Fig. 5. At smaller velocity, periodicity fails and a process of delocalization develops at smaller value of the initial velocity.

Variations in the initial velocity of u do not result in considerable deviations in the profile of u similar to the head-on collisions.

4. Conclusions

The studied coupled systems give rise to a new nonlinear wave behavior caused by the wave interactions. It concerns the interaction of waves co-propagating with equal velocities, switch of the wave direction and of the sign of its amplitude, and propagation of bell-shaped solitary wave with periodically modulated amplitude. These phenomena are not observed for solitary waves governed by single nonlinear equations.

The wave trapping was numerically found by C.I. Christov et al. [16,17] for the CNLS equations. In their works the incident wave of one function traps a wave of another function. However, the last type of waves arises in [16–18] due to the incident wave interaction. In our case no new waves are generated, and the trapping of one incident wave by another one occurs. A possible reason for this is that there is no symmetric coupling between Eqs. (1) and (2) in contrast to what happens in the CNLS equations.

Due to the lack of a sufficient number of conservation laws for Eqs. (1) and (2) we employed their particular exact traveling wave solutions to interpret the numerical results. Exact solutions (7), (12) predict the existence of the solitary wave u with either sign of the amplitude but they do not describe switch of the sign. The tendency of the peaks and/or troughs of v and u to take place at the same position at each time is in agreement with the exact solutions. However, solutions (7), (12) cannot predict the periodic modulation of the amplitude of u shown in Fig. 5.

Concerning the view of the dynamical interactions as a reshaping of the solitary wave solution for a two-degrees-of-freedom system, we note that Figs. 2–5 show how close is a reshaping of initial data to the exact solutions. However, the bold profiles in these figures are not exactly Eqs. (7), (12), but clearly some important features of these solutions are hereby revealed.

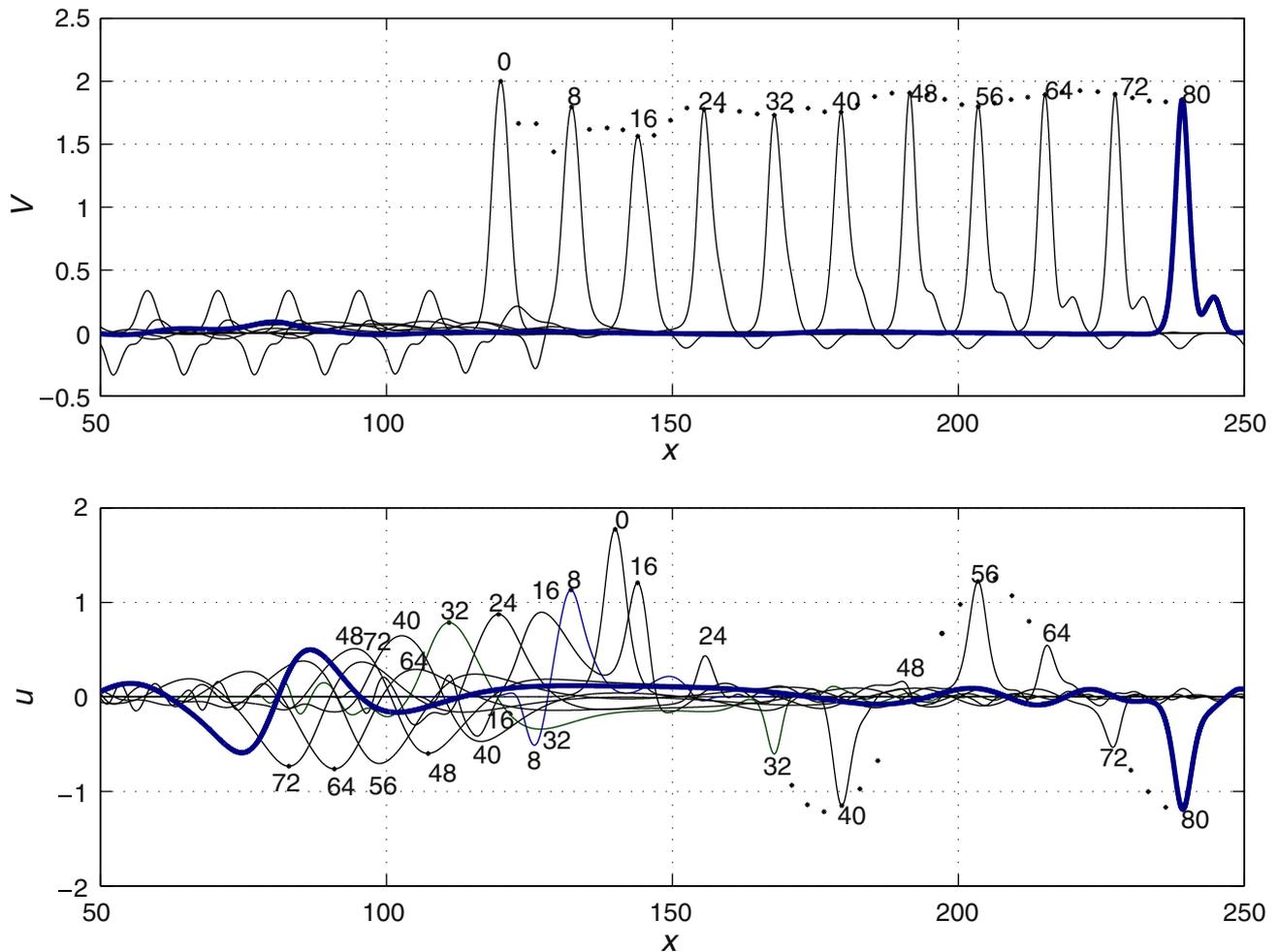


Fig. 5. Head-on collision of the solitary waves when initial position of u is equal to 140 units ahead of that of v equal to 120 units, but the initial velocity for v is chosen equal to $0.75 \cdot V$. Points of time are marked at the corresponding peaks and holes. Dotted lines mark variations in the amplitude. Bold curves mark final positions of the waves.

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