COMPUTATION OF THE EXCITABILITY INDEX FOR LINEAR OSCILLATORS

Boris Andrievsky, *Member, IEEE*

Abstract: **The problem of numerical evaluation of the excitability index for oscillating systems is considered. It is proved that the speed-gradient excitation maximizes the full energy of a linear oscillator over the infinite time interval. Upper and lower bounds for the full energy of the system in the steady-state oscillation mode are found. The exact value of the accessible energy for the case of harmonic excitation is found.**

Index Terms: **excitability index, speed-gradient, optimal excitation.**

I. INTRODUCTION

The problem of oscillation excitation via the small external force has various scientific and engineering applications. This problem can be often treated as one of achieving the maximal system energy, which is reachable in the case of the input signal limitations. The so-called *excitability index* (EI), as a quantitative measure of the system excitability property, has been recently introduced by A.L. Fradkov, and the technique of the *excitability analysis*, based on the EI computation, has been also developed [1–3].

Computation of the EI is based on solving the optimal control problem, which is a quite intricate one in the general case. Simplification the EI computation can be achieved in the view of the fact, that the *speed-gradient* (SG) control laws [5] provide the locally-optimal control, and the SG solution approaches to the optimal one for the control action of small magnitude.

In spite of the fact that the excitability analysis is aimed for the study of the properties of the nonlinear systems, the linear systems examination can also give the useful information concerning the excitation problem. In the present paper the excitation of a linear oscillator via the bounded control action over the infinite time interval is considered. It is proved that the SG method gives the optimal solution to the problem of maximization the full energy of the system on the infinite time interval. Therefore, in the considered case the SG method makes possible to find the EI exactly.

The paper is organized as follows. The brief introduction to the excitability analysis is presented in Sec. 2. The SG method for energy-based control laws design and its implementation for excitation the linear oscillator is presented in Sec. 3. The optimal control strategy and its comparison with the SG-control are given in Sec. 4. In Sec. 5 the upper and lower limits of the total energy and the EI are estimated. Section 5 is devoted to the case of the harmonic excitation signal.

II. EXCITABILITY ANALYSIS FOR NONLINEAR SYSTEMS

Excitability index (EI) was introduced in [1] as measure of resonant properties of nonlinear systems. The EI is defined as a family of L_∞ gains for different levels of input. Consider a system described by state-space equations

$$\dot{x} = F(x,u), \quad y = h(x), \qquad (1)$$

where $x \in \mathrm{R}^n$ is the state vector, *u, y* are the scalar input and output, respectively. As it was shown in [1,2], in order to create resonance mode in a nonlinear system (to find small force that leads to significant changes in system behavior), it is possible to solve the following *optimal control problem*

$$Q(\gamma) = \limsup_{\substack{|u(s)|\le\gamma, \\ 0\le s\le t, \\ x(0)=0, \\ t\ge 0}} |y(t)|^2 . \qquad (2)$$

If the system (1) is *bibo* stable and *x=0* is the equilibrium of the unforced system then $Q(\gamma)$ is well defined. Apparently, the signal providing maximum excitation should depend not only on time but also on the system state, i.e. input signal should have a feedback form. The *excitability index* (EI) for the system (1) is introduced as follows:

$$E(\gamma) = \frac{1}{\gamma}\sqrt{Q(\gamma)}, \qquad (3)$$

where $Q(\gamma)$ is the optimum value of the problem (2). For nonlinear systems $E(\gamma)$ is a function of $E(\gamma)$ that characterizes excitability (resonance) properties of the nonlinear system.

The solution to the problem (2) is quite complicated in most cases. It was shown in [2] that approximate locally optimal speed-gradient (SG) solution can be used. The SG solution for the case when output *y* is a system energy is described below.

III. SPEED-GRADIENT METHOD AND ENERGY CONTROL

Let the control goal for the system (1) is expressed as the limit relation

$$y \to 0 \text{ when } t \to \infty . \qquad (4)$$

In order to achieve the goal (4), the following SG-algorithm in the finite form may be applied [3,4]

$$u = -\Psi\left(\nabla_u \dot{h}(x,u)\right), \qquad (5)$$

where $\dot{h} = (\partial h/\partial t)F(x,u)$ is the speed of changing $h(x(t))$ along the trajectories of (1), vector $\Psi(z)$ forms an acute angle with the vector *z*, i.e. $\Psi(z)^{\mathrm{T}} z > 0$ when $z \neq 0$. The first step of the speed-gradient procedure is to calculate the speed $\dot{h}$. The second step is to evaluate the gradient $\nabla_u \dot{h}(x,u)$ with respect to controlling input u. Finally the vector-function $\Psi(z)$ should be chosen to meet the acute angle condition. The choice $\Psi(z)=\gamma z,\ \gamma>0$ yields the *proportional* (with respect to speed-gradient) feedback

$$u=-\gamma \nabla_u \dot{h}(x,u), \tag{6}$$

while the choice $\Psi(z)=\kappa \operatorname{sign} z$ yields the *relay* algorithm

$$u=-\gamma \operatorname{sign}\left(\nabla_u \dot{h}(x,u)\right), \tag{7}$$

The underlying idea of the choice Eqs. (6), (7) is that moving along the antigradient of the speed $\dot{h}$ provides decrease of $\dot{h}$. It may eventually lead to negativity of $\dot{h}$ which, in turn, yields decrease of h and, eventually, achievement of the primary goal (4). However, to prove (4) some additional assumptions are needed, see [3,4].

Consider now the problem of excitation the linear oscillator. The plant (1) equation in this case can be written in the state-space form as

$$\dot{x}=Ax+Bu, \tag{8}$$

where $x=[\varphi,\dot{\varphi}]^{\mathrm{T}}$ is the plant state vector, composed from the oscillator output $\boldsymbol{\varphi}$ and its time derivative, $\boldsymbol{u}$ denotes the control action. The matrices A and B are written in the canonical form of phase variable as

$$A=\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\rho \end{bmatrix},\quad B=\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{9}$$

where ω_0 stands for the natural frequency of the oscillator, $\rho \geq 0$ is the damping gain.

Taking the plant output $y(t)$ in the form $y=h(x)=0.5(H^*-H(x))^2$, where $H(x)$ is the full energy of the system (8), H^* is its prescribed value, one obtains from (6), (7) the following *SG energy-control laws*:

$$u=\gamma x_2\left(H^*-H(x)\right), \tag{10}$$

$$u=\gamma \operatorname{sign}(x_2)\operatorname{sign}\left(H^*-H(x)\right). \tag{11}$$

The total energy $H(x)$ of the system (8), (9) can be written as the following quadratic form

$$H(x)=\frac{1}{2}x^{\mathrm{T}}Gx, \tag{12}$$

where the matrix G is a diagonal one, $G=\operatorname{diag}\{\omega_0^2,1\}$.

By means of settlement the sufficiently large prescribed energy H^* in (11) and $\operatorname{sign}(H^*-H)=1$, one finally obtains the following *SG-excitation* algorithm:

$$u=\gamma \operatorname{sign}(x_2). \tag{13}$$

Let us find now the explicit solution of Eqs. (8), (13). It can be easily shown that if the system (8) is dissipative (i.e. $\rho>0$), the stable limit cycle appears. To find its parameters (the frequency and the amplitude), the *point-mapping* method can be applied [5]. Taking into account that $u=+\gamma$ as far as $x_2>0$ and $u=-\gamma$ for $x_2<0$, one gets the phase trajectories shown in Fig. 1. The point x_∞ corresponds to the steady-state solution of (8). That solution may be obtained by assumption that $u(t)\equiv-\gamma$. The curve $\mathcal{G}$ in Fig. 1 represents the limit cycle.

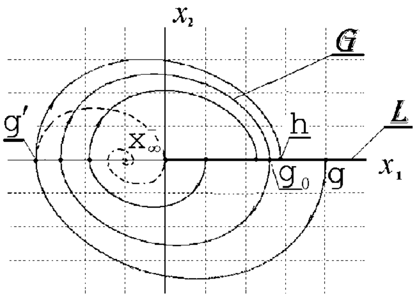

Fig. 1. The phase trajectories

Considering the oscillator (8) equation one can find that the time intervals between two successive intersections the phase plot with the positive semi-axis $\mathcal{L}$ are equal to $T=\frac{2\pi}{\beta}$, where $\beta=\frac{1}{2}\sqrt{4\omega_0^2-\rho^2}$ is the natural oscillations frequency for the plant (8). Therefore the SG control action (13) is a *square waveform* of the period T and some phase shift ψ:

$$u(t)=\gamma \operatorname{sign}(\beta t+\psi). \tag{14}$$

To find the oscillations amplitude, let us build the *Poincaré map*. It is shown in Fig. 2.

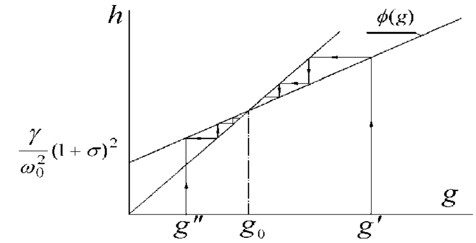

Fig. 2. The Poincaré map

The *fixed point* g_0 of the Poincaré map gives the limit cycle amplitude (see Fig. 1). To find it let us solve the equation $\phi(g_0)=g_0$, where the Poincaré function $\phi(g)$ has a following form:

$$\phi(g)=\gamma\,\omega_0^{-2}(1+\sigma)^2+\sigma^2 g \tag{15}$$

The parameter σ in Eq. (15) is the *overshoot* of the oscillator (8) step response, and can be found as $\sigma=\exp\left(\frac{\alpha}{\beta}\pi\right)$, where $\alpha=-\frac{\rho}{2}$ is the real part of plant model (8) eigenvalues. Note, that $0<\sigma<1$ as $\rho>0$. Finally, one obtains the limit cycle magnitude

$$g_0=\frac{\gamma}{\omega_0^2}\frac{1+\sigma}{1-\sigma} \tag{16}$$

That magnitude can be easily found through the plant model (8) and the control action (13) parameters.

It is known [3,6], that the control law (13) is a locally-optimal one in the sense of maximum of the system (1) output magnitude. As it is shown in the next Section, the law (13) is also globally optimal for the system (8), (9).

IV. Variational Solution for Excitation Problem

Let us apply the *maximum principle* to solve the posed problem of an energy-optimal oscillator excitation. Introduce the *quality index* as

$$J(u) = H(x(t_f)) = \frac{1}{2} x(t_f)^{\mathrm{T}} G x(t_f) ,$$

where t_f stands for the end point, $t \in [0, t_f]$, $x(0) = 0$. Consider the free end optimization problem. The phase variables $x(t)$ and the control action $u(t)$ are connected by means of the differential relation (8). Let the control action $u(t)$ be bounded, $|u(t)| \le \gamma$, where $\gamma > 0$ is a given constant. The quality index $J(u)$ does not depend explicitly on the system trajectory inside the optimization interval, therefore the problem under consideration is a *terminal* one. Let us introduce the *Hamiltonian* $\mathcal{H}(x,u,\lambda)$ as

$$\mathcal{H}(x,u,\lambda) = \lambda(t)^{\mathrm{T}} \left(Ax(t) + Bu(t) \right),$$

where $\lambda(t) \in \mathrm{R}^2$ is the vector of the *dual variables* (the *Lagrangian coefficients*). According to the maximum principle, the optimal control action $u^*(t)$ must be found to satisfy the following conditions:

- extremality on x

$$\dot{\lambda}(t)^{\mathrm{T}} = - \left. \frac{\partial \mathcal{H}(x,u,\lambda)}{\partial x} \right|_{u=u^*} , \qquad (16)$$

- the maximum principle

$$u^*(t) = \max_{|u(t)| \le \gamma} \mathcal{H}(x,u,\lambda) , \qquad (17)$$

- the transversality condition $\lambda(t_f) = -Gx(t_f)$.

As it follows from (16), the dual variables have to satisfy the equation $\dot{\lambda}(t) = -A^{\mathrm{T}} \lambda(t)$, or, componentwise,

$$\begin{cases} \dot{\lambda}_1(t) = \omega_0^2 \lambda_2(t) \\ \dot{\lambda}_2(t) = -\lambda_1(t) + \rho \lambda_2(t). \end{cases} \qquad (18)$$

Condition (17) gives the optimal control signal as

$$u^*(t) = \gamma \ \mathrm{sign} \lambda_2(t). \qquad (19)$$

The pair-wise boundary-value problem (8), (17)–(19) can not be solved explicitly, hence to find the optimal control function $u^*(t)$, let us use some additional arguments.

It follows from Eq. (18) that the dual variable $\lambda_2(t)$ has a form

$$\lambda_2(t) = C \exp(-\alpha\, t) \sin(\beta\, t + \psi), \qquad (20)$$

where the constants C, ψ depend on boundary conditions, α and β are the real and the imaginary parts of plant model (8) eigenvalues. Substitution (20) into (19) yields the optimal control action as

$$u^*(t) = \gamma \ \mathrm{sign}\left(\beta\, t + \psi\right). \qquad (21)$$

It is worth to notice that the value of ψ is not important for the posed problem, because the oscillation parameters in the steady-state mode (as $t \to \infty$) do not depend on ψ. Thus, the optimal solution of the excitation problem is not unique.

Comparison of the formulas (14) and (21) shows that the SG control law (13) is optimal for any $\gamma > 0$.

V. Evaluation of the Total Energy and the Excitability Index

The total energy (12) of the system can be represented as a sum of two terms: the *potential energy* $\Pi(x_1) = \omega_0^2 x_1^2 / 2$ and the *kinetic energy* $\mathrm{K}(x_2) = x_2^2 / 2$. The supremum $\overline{\Pi}$ of the potential energy can be directly found from Eq. (16). By substitution, one obtains $\overline{\Pi}$ as

$$\overline{\Pi} = \frac{\gamma^2 \left(1 + e^{\pi/\mu}\right)^2}{2\omega_0^2 \left(1 - e^{\pi/\mu}\right)^2} \qquad (22)$$

where $\mu = \beta / \alpha = -\sqrt{4\omega_0^2 / \rho^2 - 1}$.

Examination of the first and the second time derivatives of $H_t = H(\mathrm{x}(t))$ along the system (8), (14) trajectories, shows that, asymptotically, $H(\mathrm{x}(t))$ reaches its maximal value $\overline{H}$ as $x_2 = \bar{x}_2$, where $\bar{x}_2 = -\gamma \rho^{-1} \mathrm{sign}\, x_1$. Therefore, $\overline{\Pi}$ is not equal to $\overline{H}$, but $\overline{\Pi}$ can be used as a lower estimate of $\overline{H}$, because it exceeds the lower bound of H_t in the limit cycle. To get the upper estimate of $\overline{H}$ let us consider the arc of the limit cycle between the points $(\bar{x}_1, 0), \bar{x}_1 = g_0$ and $\left(x_1(\bar{t}), \bar{x}_2\right)$, where $H(\bar{t}) = \overline{H}$. The explicit solution for $\bar{t}$ can not be found, then let us assume approximately that $\rho = 0$ in (8). Some computations lead to the following estimate:

$$\overline{\Pi} < \overline{H} < \frac{2\gamma^2}{\omega_0^2 \left(1 - e^{\pi/\mu}\right)^2} \approx \frac{8\gamma^2 \beta^2}{\pi^2 \rho^2 \omega_0^2} \qquad (23)$$

Substituting $\overline{H}$ for $Q(\gamma)$ in Eq. (1) gives the following estimate of the excitability index:

$$E(\gamma) = E < \frac{\sqrt{2}}{\omega_0 \left(1 - e^{\pi/\mu}\right)} \approx \frac{2\sqrt{2}\beta}{\pi \rho \omega_0} \qquad (24)$$

This estimate is more accurate than one obtained by A.L. Fradkov [7] via the *energy balance method*.

Let us consider now the case of the oscillator (8) excitation by means of the harmonic input signal.

VI. Harmonic Excitation

Let us find the excitability index assuming that the admissible input signals are the harmonic ones. Let $u(t)=\gamma\sin(\omega t)$, where γ, ω stand for the excitation magnitude and frequency, respectively. In such a case the steady-state solutions of (8) can be written as $x_1(t)=A(\omega)\sin(\omega t+\psi)$, $x_2(t)=-\omega A(\omega)\cos(\omega t+\psi)$, where $A(\omega)$ is the gain-frequency characteristic of the system (8), $\psi=\psi(\omega)$ is a phase shift between the system input and output signals. For the sake of the present study the value of ψ is unimportant and hereafter it is assumed that ψ=0. Then the steady-state total energy of the system $H_t=H(x(t))$ is as follows:

$$\begin{aligned} H_t &= \gamma^2 A(\omega)^2\left(\omega^2\cos^2(\omega t)+\omega_0^2\sin^2(\omega t)\right)/2 \\ &= \gamma^2 A(\omega)^2\left(\omega^2+(\omega_0^2-\omega^2)\sin^2(\omega t)\right)/2 \end{aligned} \qquad (25)$$

It is seen that H_t oscillates about some constant value with the frequency 2ω, as far as $\omega\neq\omega_0$. If $\omega=\omega_0$, then H_t is constant on t. Let us find the maximum of H_t, $\overline{H}_t=\sup_t H_t$. The time derivative of H_t is $\dot{H}_t=\omega A(\omega)^2\cos(\omega t)\sin(\omega t)(\omega^2-\omega_0^2)$. Hence, the extreme points of H_t are: $t_1=k\pi/\omega$, $t_2=(k\pi+\pi/2)/\omega$, where $k=0,1,2,\ldots$. The corresponding values of H_t in these points are:

$$H_1(\omega)=\omega^2\gamma^2 A(\omega)^2/2,\ H_2(\omega)=\omega_0^2\gamma^2 A(\omega)^2/2\,. \qquad (26)$$

These values depend on the oscillation frequency ω. Let us find the value of ω that ensures the maximization of the total energy of the system for all possible harmonic input waveforms. Recall that it is valid for (8) that $A(\omega)^2=\left((\omega_0^2-\omega^2)^2+\rho^2\omega^2\right)^{-1}$. Differentiating $H_1(\omega), H_2(\omega)$ with respect to ω yields

$$\frac{\partial H_1(\omega)}{\partial\omega}=\gamma^2\omega(\omega_0^4-\omega^4)/R(\omega),$$

$$\frac{\partial H_2(\omega)}{\partial\omega}=\gamma^2\omega_0^2\omega\left(2(\omega_0^2-\omega^2)-\rho^2\right)/R(\omega),$$

where $R(\omega)=\left((\omega_0^2-\omega^2)^2+\rho^2\omega^2\right)^2$. For the oscillator (8) it is valid that $4\omega_0^2>\rho^2$ and $R(\omega)\neq 0$ for all ω. Then there are following extreme points of $H_1(\omega)$, $H_2(\omega)$ as the functions on ω:

$$\omega_1=\omega_0,\ \omega_2=\sqrt{\omega_0^2-\rho^2/2} \qquad (27)$$

By substitution ω in (26) for ω_1, ω_2 from (27) one gets the extreme values for $H_1(\omega), H_2(\omega)$ as

$$\overline{H}_1=\frac{\gamma^2}{2\rho^2},\ \overline{H}_2=\frac{2\omega_0^2\gamma^2}{\rho^2(4\omega_0^2-\rho^2)} \qquad (28)$$

It can be easily verified that $\overline{H}_2>\overline{H}_1$ and therefore,

$$\overline{H}=\frac{2\omega_0^2\gamma^2}{\rho^2(4\omega_0^2-\rho^2)},\ E=\frac{\sqrt{2}\omega_0^2}{\rho\sqrt{4\omega_0^2-\rho^2}}\,. \qquad (29)$$

Note, that ω_2 in (27) can not be found if $\rho>\omega_0\sqrt{2}$. In that case the total energy is a monotone non-increasing function of ω.

Conclusions

In the paper it is shown that the SG excitation maximizes the energy for a linear oscillator. Upper and lower bounds of the full energy of the system and the excitability index are estimated. The exact value of the accessible system energy for the harmonic excitation case is found.

Acknowledgements

The author is grateful to Professor A.L. Fradkov for some ideas behind the paper and the helpful comments.

The work was partly supported by Russian Foundation of Basic Research (grant RFBR 02-01-00765), Complex Program of the Presidium of RAS #19 "Control of mechanical systems"', project 1.4.

References

[1] A.L. Fradkov, Exploring nonlinearity by feedback, Physica D. 1999, vol. 128, No 2-4, pp. 159-168, 1999.

[2] A.L. Fradkov, A nonlinear philosophy for nonlinear systems, Proc. 39th IEEE CDC, Sydney, 2000.

[3] A.L. Fradkov, Cybernetical Physics: Principles and Examples, Nauka, SPb., 208 p. 2003. (in Russian)

[4] A.L. Fradkov, I.V. Miroshnik and V.O. Nikiforov, Nonlinear and Adaptive Control of Complex Systems, Dordrecht, Kluwer, 1999.

[5] L.S. Pontrjagin, Ordinary Differential Equations, Nauka, M., 332 p., 1974. (in Russian)

[6] F.L. Chernousko, Some problems of optimal control with a small parameter, J. Appl. Math. Mech., No.3, 1968.

[7] B.R. Andrievskii, I.I. Blekhman, Yu.A. Bortsov, et al., Control of Mechanical Vibrational Units / Eds. I.I. Blekhman, A.L. Fradkov, Nauka, SPb, 278 p., 2001 (in Russian).

Boris Andrievsky (M'2000) was born in Saint Petersburg (formerly Leningrad), Russia in 1949. He received the control engineer diploma and the Ph.D. from Baltic State Technical University (BSTU), St.Petersburg in 1972 and 1979, respectively. From 1977 he held different positions with the BSTU, where he is currently a part time Associate Professor. From 1995 till 1998 he also was an Associate Professor with St.Petersburg Institute of Machine Building. From 1993 he is a Senior Researcher with the Institute for Problems of Mechanical Engineering of RAS. Dr. Andrievsky gave lectures at the Beijing Institute of Aerospace Industry in 1992, 1999. His research interests are in the areas of adaptive and robust control, control of oscillations, applications to control in the aerospace, control of mechanical systems and signal processing.