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Singular Perturbations of Systems Controlled by Energy-Speed-Gradient Method

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Abstract— **Existing results on partial stability of differential form of speed-gradient control for singularly perturbed systems are extended to the case of speed-gradient control in finite form. The results are illustrated by example of energy and synchronization control of two coupled pendulums, taking into account inertia of the motor. It is shown that the perturbations stemming from inertia of the coupling link and the perturbations stemming from inertia of the motor provide different influence upon the perturbed system behavior.**

I. INTRODUCTION

An important problem for control of complex nonlinear systems is dealing with unmodeled dynamics, particularly with singularly perturbed systems. It is well known that unmodeled dynamics may not only prevent from achieving the control goal, but also cause unboundedness of control system trajectories [1], [2], [3]. Conditions for stability of singularly perturbed speed-gradient based control systems were proposed in [4], see also [3]. These conditions are well suited for adaptive control systems where stability with respect to only a part of variables may be observed. However, the conditions of [4] are not fulfilled for energy control problems, since (A) the energy-based Lyapunov function is not radially unbounded, (B) speed-gradient algorithms in finite rather than in differential form are used for energy control and (C) an unperturbed system possesses weaker stability properties, namely, partial stability with respect to a function rather than stability with respect to a part of variables. At the same time control of system energy is an important problem having different applications to control of mechanical and electromechanical systems [5], [6]. An efficiency of general approach to energy control based on speed-gradient method has been demonstrated, see [5], [7], [8], [9]. Therefore studying speed-gradient energy control of singularly perturbed Hamiltonian systems seems important.

In this paper the results of [4] are extended to encompass the problems of speed-gradient based energy control of singularly perturbed Hamiltonian systems. In Section II new stability results for singularly perturbed nonlinear systems are presented and applied to energy control of Hamiltonian systems. An example of application to controlled synchronization of two coupled pendulums, taking into account inertia of the actuating motor is studied in the Section III by computer simulation.

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II. SPEED-GRADIENT METHOD FOR SINGULARLY PERTURBED SYSTEMS

Consider the following plant model

$$\dot{x}_1 = f_1(x_1, x_2, u) \tag{1}$$

$$\varepsilon \dot{x}_2 = f_2(x_1, x_2, u), \tag{2}$$

where u is the control action, $x_1 \in \mathbb{R}^{n_1}$ is the vector of slow variables, $x_2 \in \mathbb{R}^{n_2}$ is the vector of fast variables, and $f_1(\cdot)$, $f_2(\cdot)$ are the vector functions of appropriate dimensions.

Let the control objective be the fulfillment of the relation

$$\lim_{t\to\infty} Q(x_1(t)) = 0, \tag{3}$$

where $Q(x_1)$ is a scalar smooth objective function, $x = (x_1, x_2)^{\mathrm{T}}$.

To design a simplified control law, the initial system (1), (2) is replaced by a reduced-order one obtained by substitution $\varepsilon = 0$ as follows:

$$\dot{x}_1 = \overline{f}(x_1, u), \quad \overline{x}_2 = \eta(x_1, u), \tag{4}$$

where $\overline{x}_2 = \eta(x_1, u)$ is a root of the equation $f_2(x_1, x_2, u) = 0$ (the root is assumed to exist and be unique), $\overline{f}(x_1, u) = f_1(x_1, \eta(x_1, u), u)$.

Then the speed-gradient control algorithm for the reduced-order system model is designed:

$$u = -\Gamma \nabla_u \omega(x_1, u), \tag{5}$$

where $\Gamma = \Gamma^{\mathrm{T}} > 0$ is a positive definite matrix and

$$\omega(x_1, u) = \frac{\partial Q}{\partial t} + \left(\frac{\partial Q}{\partial x}\right) \overline{f}(x_1, u) \tag{6}$$

(it is assumed that (5) can be solved in u: $u = u(x_1)$, where $u(x_1)$ is a smooth vector function). The final stage of the design is to verify stability properties of the closed-loop system. To ensure the fulfillment of the control goal (3) for the reduced-order system (4), (5), it suffices, e.g. to assume that the function $\omega(\cdot)$ is convex in u and there exists a bounded vector-function $u_*(x_1)$ such that the system (4) with substituted $u = u_*(x_1)$ is asymptotically stable with respect to function Q, i.e., $\omega(x_1, u_*) \le -\rho(Q)$ where $\rho(Q) > 0$ for $Q > 0$. However, the fulfillment of the control objective for the reduced-order system does not guarantee the same for the initial system (1), (2), (5), see [1]. Therefore application of the above design method requires additional conditions assuming, in particular, small value of the parameter ε. These conditions are introduced below.

Known classical results concerning the method of singular perturbations either deal with a finite period of time (Tikhonov's theorem, the first theorem of Bogoliubov) or require uniform asymptotic stability of the reduced-order system (the second theorem of Bogoliubov [10], the Hoppensteadt theorem [11]).

The uniform asymptotic stability condition was weakened in [4] and the result was extended to the case when $x_1 = (z, \theta)$ and the reduced system is asymptotically stable with respect to z, i.e. asymptotically stable with respect to the part of the state variables. Such a case is important for adaptive control where θ is the vector of adjustable parameters. However, in the energy control problems the system (4), (5), generically, exhibits only partial asymptotic stability with respect to some function of state variables $x_1(t)$ and the previous results cannot be applied. Besides, the energy based goal function Q does not meet standard assumptions of radial unboundedness.

Below the results of [4] are modified to the form, suitable for speed-gradient energy control.

Theorem 1. *Given a system (1), (2), (5) and $R > 0$, let the functions $f_1(\cdot)$, $Q(\cdot)$ and $f_2(\cdot)$ be twice continuously differentiable and satisfy the following conditions:*

A1) *The functions $\overline{f}(x_1, u)$, $Q(x_1)$, and their first and second derivatives are bounded in the set $\Omega_R = \{x_1 : Q(x_1) \le R\}$ for all u satisfying (5).*

A2) *For any $x_1 \in \Omega_R$ and u satisfying (5) there exists a unique root $\overline{x}_2 = \eta(x_1, u)$ of the equation $f_2(x_1, x_2, u) = 0$ and the function $\eta(x_1, u)$ is twice continuously differentiable;*

A3) *For any $x_1 \in \Omega_R$ the function ω is convex in u, and there exist a bounded vector-function $u_*(x_1)$, a nonnegative scalar function $\nu(x_1) \ge 0$ and a positive number $\bar{\alpha} > 0$ such that*

$$\omega(x_1, u_*(x_1)) \le -\nu(x_1)$$

$$\left| \frac{\partial Q(x_1)}{\partial x_1} \big[f_1(x_1, x_2, u) - \bar{f}(x_1, u) \big] \right|^2 \le \bar{\alpha}\nu(x_1) |\tilde{x}_2|^2,$$

where $\tilde{x}_2 = x_2 - \eta(x_1, u)$.

A4) *There exist a continuously differentiable function $V_2(\widetilde{x}_2) \ge 0$ and numbers $\beta_i > 0$ ($i = 0, 1, 2$) such that*

$$\partial_2 V_2 \ \le \ -\beta_0 |\widetilde{x}_2|^2, \quad \beta_1 |\widetilde{x}_2| \ \le \ |\nabla V_2| \ \le \ \beta_2 |\widetilde{x}_2|,$$

where $\widetilde{x}_2 = x_2 - \eta(x_1, u)$,

$$\partial_2 V_2 = (\nabla V_2(\widetilde{x}_2))^{\mathrm{T}} f_2(x_1, \widetilde{x}_2 + \eta(x_1, u), u).$$

Then for any bounded set D of the initial states $x_2(0)$ there exists a number $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the solutions of the system (1), (2) and (5) with $x_1(0) \in \Omega_R$ satisfy the following relations:

$$\lim_{t\to\infty} \nu\big(x_1(t)\big) = 0, \ \ \lim_{t\to\infty} \Big(x_2(t) - \eta(x_1(t), u(t)) \Big) = 0.$$

Additionally, let $B_R = \Omega_R \times \mathbb{R}^{n_2}$ and the values

$$L_1(R) = \sup_{B_R} \Big| \frac{\partial f_1(x_1, x_2, u(x_1))}{\partial x_2} \Big|,$$

$$L_2(R) = \sup_{\Omega_R} \Big| \frac{\partial \eta(x_1)/\partial x_1 \ \bar{f}_1(x_1, u(x_1))}{\nu(x_1)^{1/2}} \Big|$$

be finite. Let

$$L_3(R) = \sup_{\Omega_R} \Big| \frac{\partial}{\partial x_1} \eta(x_1, u) \Big|.$$

Then for $x(0) \in D_R$, where

$$D_R = \left\{ (x_1, x_2) : Q(x_1) + \frac{\bar{\alpha} L_1(R)}{\beta_1 L_2(R)} V_2(\tilde{x}_2) \ \le \ R \right\}. \quad (7)$$

the value of $\varepsilon_0 = \varepsilon_0(R)$ can be estimated as follows:

$$\varepsilon_0(R) = \frac{\beta_0 L_2(R)}{\bar{\alpha} L_1(R) L_2(R) + L_1(R) L_3(R)}. \quad (8)$$

The theorem can be summarized as follows. If the fast subsystem (2) is exponentially stable in x_2 for bounded x_1 and for u satisfying (5) and the reduced-order system (4), (5) is partially asymptotically stable with respect to the function $\nu(x_1)$, then the algorithm (5) ensures partial stabilization of the initial system (1), (2) with respect to the function $\nu(x_1)$ for sufficiently small value of the parameter $\varepsilon > 0$. The particular value of ε depends on initial conditions $x(0)$, since the right-hand sides of (1), (2) may be locally, rather than globally Lipschitz. In other words, the control algorithm (5) is robust with respect to unmodeled fast dynamics (singular perturbations).

Proof of Theorem 1. Fix $\varrho > 0$ and consider the Lyapunov function candidate

$$V_\varrho(x_1, x_2) = Q(x_1) + \varrho V_2(\tilde{x}_2),$$

where $\tilde{x}_2 = x_2 - \eta(x_1), \eta(x_1) = \eta(x_1, u(x_1))$, $u(x_1)$ is the solution of (5). Choose $\bar{R} > 0$ such that the set

$$D(\bar{R}, \varrho) = \{(x_1, x_2) : \ Q(x_1) + \varrho V_2(x_2 - \eta(x_1)) \le \bar{R}\},$$

covers the set $\Omega_R \times D$. Evaluate the time derivative $\dot{V}_\varrho(x_1, x_2)$ with respect to initial system (1), (2), (5) for $(x_1, x_2) \in D(R, \varrho)$. Evaluation of $\dot{Q}(x_1)$ yields

$$\dot{Q} = \partial_1 Q(x_1) + \frac{\partial Q}{\partial x_1}(x_1)[f(x_1, x_2, u(x_1)) - f_1(x_1, \eta(x_1))] \quad (9)$$

or

$$\dot{Q} \le -\nu(x_1) + \bar{\alpha} \left(\nu(x_1)\right)^{1/2} |\tilde{x}_2|. \quad (10)$$

Evaluating $\dot{V}_2$ we get

$$\dot{V}_2(x_1, x_2) = \left(\nabla V_2(\tilde{x}_2)\right)^{\mathrm{T}} [\dot{x}_2 - \dot{\eta}(x_1, x_2)] = \varepsilon^{-1} \partial_2 V_2(x_1, x_2) - \left(\nabla V_2(\tilde{x}_2)\right)^{\mathrm{T}} \dot{\eta}(x_1, x_2). \quad (11)$$

Note that the relation (9) remains true if the function $Q(x_1)$ is replaced by an arbitrary smooth vector function of x_1, for example $\eta(x_1)$. Taking into account that $\partial_1\eta(0,x_2)=0$ we obtain

$$\dot V_2(x_1,x_2)\le -\varepsilon^{-1}\beta_0|\tilde x_2|^2+(\nabla V_2(\tilde x_2))^{\mathrm T}\partial_1\eta(x_1)+ |\nabla V_2(\tilde x_2)|\left|\frac{\partial\eta(x_1)}{\partial x_1}\right|L_1(R,\varrho)|\tilde x_2|, \quad (12)$$

or

$$\dot V_2\le -[\varepsilon^{-1}\beta_0-\beta_1L_3(R)L_1(R,\varrho)]|\tilde x_2|^2+ \beta_1|\tilde x_2|L_2(R)\nu(x_1)^{1/2}, \quad (13)$$

where

$$L_1(R,\varrho)=\sup_{D(R,\varrho)}\left|\frac{\partial f_1(x_1,x_2)}{\partial x_2}\right|.$$

Multiplying (13) by ϱ and adding (10) yields

$$\dot V_\varrho\le -\nu(x_1)+[\bar\alpha L_1(R,\varrho)+\varrho\beta_1L_2(R)](\nu(x_1))^{1/2}|\tilde x_2| -\varrho[\varepsilon^{-1}\beta_0-\beta_1L_1(R,\varrho)L_3(R)]|\tilde x_2|^2. \quad (14)$$

The right-hand side of (14) is the quadratic form of variables $(\nu(x_1))^{1/2}$, $|\tilde x_2|$, which is negative definite for

$$4\varrho[\varepsilon^{-1}\beta_0-\beta_1L_1(R,\varrho)L_3(R)]>(\bar\alpha L_1(R,\varrho)+\varrho\beta_1L_2(R))^2,$$

i.e., for $0<\varepsilon<\varepsilon_0$, where

$$\varepsilon_0=\frac{4\varrho\beta_0}{(\bar\alpha L_1(R,\varrho)+\varrho\beta_1L_2(R))^2+4\varrho\beta_1L_1(R,\varrho)L_3(R)}. \quad (15)$$

If $0<\varepsilon<\varepsilon_0$, then $\dot V_\varrho(x_1,x_2)\le 0$ for $V_\varrho(x_1,x_2)\le R$. Therefore, $(x_1(0),x_2(0))\in D(R,\varrho)$ implies that $(x_1(t),x_2(t))\in D(R,\varrho)$ for any $t\ge 0$. In this case also

$$\partial V_\varrho(x_1,x_2)\le -\delta(\nu(x_1)+|\tilde x_2|^2)$$

for some $\delta>0$. Then the Barbalat lemma (see e.g.[3]) yields $\nu(t)\to 0$, $\tilde x_2(t)\to 0$ for $t\to\infty$. Choosing the value $\varrho=\bar\alpha L_1(R)/(\beta_1L_2(R))>0$, maximizing the right-hand side of (15) ends the proof of the theorem.

Consider control problem for system in Hamiltonian form

$$\dot p_i=-\frac{\partial H(q,p,u)}{\partial q_i},\ \dot q_i=\frac{\partial H(q,p,u)}{\partial p_i},\ i=1,\dots,n, \quad (16)$$

where $p=\mathrm{col}(p_1,\dots,p_n)$, $q=\mathrm{col}(q_1,\dots,q_n)$ are the vectors of generalized coordinates and momenta, $H=H_0(q,p)+H_1(q,p)u$ is the Hamiltonian function, and $u=u(t)\in\mathbb R^m$ is the input (generalized force) with the control goal

$$H_0(p(t),q(t))\to H_*\quad\text{as}\quad t\to\infty. \quad (17)$$

The speed-gradient algorithm is as follows:

$$u=-\gamma(H_0-H_*)[H_0,H_1]^{\mathrm T}, \quad (18)$$

where $\gamma>0$ is the gain,

$$[f,g]=\sum_{i=1}^n\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}-\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right).$$

stands for the Poisson bracket of smooth functions $f(p,q)$ and $g(p,q)$ (if the functions f,g are the vector functions then the Poisson bracket is defined componentwise).

To apply Theorem 1 introduce $Q(x_1)=\frac12\big(H_0(x_1)-H_*\big)^2$, where $x_1=\mathrm{col}(p,q)$, x_2 is a state vector of the perturbation system (2). The condition A3 of the theorem holds for $\nu(x_1)=[H_0(x_1),H_1(x_1)]^2Q(x_1)$. It follows from Theorem II that $\nu(x_1(t))\to 0$ as $t\to\infty$, with the control algorithm (18) for initial conditions from B_R and sufficiently small $\varepsilon>0$, i.e. the goal (3) is achieved if $[H_0(x),H_1(x)]\ne 0$ for $x\in D_R$ for some $R>0$. However, the inequality $[H_0(x),H_1(x)]\ne 0$ does not hold for many oscillatory systems. The following theorem may help.

Theorem 2. *Let conditions A1–A4 of Theorem 1 hold with an additional condition*

A5) *function $\nu(x_1)$ can be factorized as follows: $\nu(x_1)=\nu_1(x_1)Q(x_1)$ and no whole trajectories of the free system (with $u=0$) belong to the set $\{x_1:\nu_1(x_1)=0, Q(x_1)>0\}$.*

Then the statement of Theorem 1 hold, i.e. for any bounded set D of the initial states x_2 there exists $\varepsilon_0>0$ such that the goal (3) holds for $0<\varepsilon<\varepsilon_0$ for all bounded solutions of the system (1), (2) and (5) with $x_1\in\Omega_R$.

Proof of Theorem 2 goes along lines of the corresponding theorems in [8], [9]. The condition of boundedness of the trajectories can be weakened similarly to [8], [9].

III. Example. Flexibly Coupled Pendulums

A. Model of the controlled system

Consider two pendulums coupled by the spring. The both ends of the spring are connected with the pendulum rods. The control torque is produced by the motor and applied to the first pendulum. Such a system is the special case of the diffusively coupled oscillators model, which is often used for modeling various physical and mechanical systems. In the case of the first-order model of motor dynamics, the full-order system can be modeled as

$$\begin{cases} J\ddot\varphi_1+R_p\dot\varphi_1+mgl\sin\varphi_1=K(\mu-\varphi_1)+M(t),\\ J\ddot\varphi_2+R_p\dot\varphi_2+mgl\sin\varphi_2=K(\mu-\varphi_2),\\ \varepsilon\dot M+M=k_mV(t),\end{cases} \quad (19)$$

where $\varphi_i(t)$ are the pendulum deflection angles ($i=1,2$); $M(t)$ is the torque, applied to the first pendulum; m, l are the mass and the length of the rod for each pendulum; J is the pendulum moment of inertia ($J=ml^2$ for the point mass pendulum); g is the acceleration of gravity; K is the coupling parameter (the stiffness of the spring); R_p is the viscous friction coefficient; V is the control voltage, applied to the motor drive winding; k_m, ε stand for the motor gain and time constant. The mass and the damping of the coupling unit are neglected.

Rewrite plant model (19) as follows:

$$\begin{cases} \ddot\varphi_1+\rho\dot\varphi_1+\omega^2\sin\varphi_1+k\big(\varphi_1-\varphi_2\big)=\mu(t),\\ \ddot\varphi_2+\rho\dot\varphi_2+\omega^2\sin\varphi_2+k\big(\varphi_2-\varphi_1\big)=0,\\ \varepsilon\dot\mu+\mu=u(t),\end{cases} \quad (20)$$

where $\mu=M/J$ is the rescaled driving torque; $u(t)=V(t)/k_m$ is the rescaled control voltage; ω is the natural frequency of small oscillations for the uncoupled pendulums, $\omega^2 = mgl/J$; $k = K/J$ is the coupling coefficient; $\rho = R/J$ is the friction parameter.

Consider the problem of excitation oscillations with the desired amplitude, understood as achieving the given energy level. An additional goal may be posed as the requirement that pendulums have either coinciding or opposite phases of oscillation (in-phase or anti-phase synchronization). To design the control law we use speed-gradient method for the reduced plant model, neglecting the coupling dynamics.

B. Plant model reduction

Assuming that the coupling dynamics are "fast", consider the motor time constant ε in Eq. (19) as a "small parameter". Evidently, Eq. (20) is a particular case of Eqs. (1), (2), where $x_1 = \mathrm{col}\,(\varphi_1, \dot\varphi_1, \varphi_2, \dot\varphi_2) \in \mathbb{R}^4$ is the vector of slow variables and $x_2 = \mu \in \mathbb{R}$ is the fast variable.

To design a control law, the full-order initial system (20) is replaced by a reduced-order one, obtained by substitution $\varepsilon = 0$ and described by the following equations

$$\begin{cases} \ddot\varphi_1 + \rho\dot\varphi_1 + \omega^2 \sin\varphi_1 + k(\varphi_1 - \varphi_2) = u(t), \\ \ddot\varphi_2 + \rho\dot\varphi_2 + \omega^2 \sin\varphi_2 + k(\varphi_2 - \varphi_1) = 0, \end{cases} \tag{21}$$

(the notations are given above, see Eqs. (19), (20).)

The simplified model (21) is used further on for energy control law design.

C. Control law design for the reduced plant model

The total energy $H(x_1)$ of the system (21) can be written as follows

$$\begin{aligned} H(x_1) = \frac{1}{2}\dot\varphi_1^2 + \omega^2(1-\cos\varphi_1) + \frac{1}{2}\dot\varphi_2^2 \\ + \omega^2(1-\cos\varphi_2) + \frac{k}{2}\big(\varphi_1 - \varphi_2\big)^2, \end{aligned} \tag{22}$$

$x_1(t) = \mathrm{col}\,(\varphi_1, \dot\varphi_1, \varphi_2, \dot\varphi_2)$.

In order to apply the speed-gradient procedure of Sec. II, introduce the two objective functions as follows:

$$\begin{aligned} Q_\varphi(\dot\varphi_1, \dot\varphi_2) &= \frac{1}{2}\big(\delta_\varphi\big)^2 \\ Q_H(x_1) &= \frac{1}{2}(H(x_1) - H_*)^2. \end{aligned} \tag{23}$$

where $\delta_H = H\big(x_1\big) - H_*$ is referred to as *energy error*; $\delta_\varphi = \dot\varphi_1 + \sigma\dot\varphi_2$ is referred to as *synchronization error*; $\sigma \in \{-1, 1\}$ is a reference phase-shift parameter; H_* is the prescribed value of the total energy.

Apparently, minimization of Q_H means achievement of the desired oscillations magnitude. The minimum value of the function Q_φ allows, additionally, to meet the "in-phase/antiphase" requirement (at least for small initial phases $\varphi_1(0), \varphi_2(0)$) : $Q_\varphi(\dot\varphi_1, \dot\varphi_2) \equiv 0$ if and only if $\dot\varphi_1 \equiv -\sigma\dot\varphi_2$. Hence option $\sigma = 1$ sets the *antiphase* desired pendulums oscillations, while $\sigma = -1$ sets the *inphase* ones.

To design the control algorithm, the objective function $Q(x_1)$ as the weighted sum of Q_φ and Q_H is introduced:

$$Q(x_1) = \alpha Q_\varphi(\dot\varphi_1, \dot\varphi_2) + (1-\alpha) Q_H(x_1), \tag{24}$$

where $\alpha \in [0, 1]$ is a *weighting coefficient*.

The speed-gradient procedure of Sec. II leads to the following control laws:
–the *proportional form*

$$u = -\gamma\big(\alpha\delta_\varphi + (1-\alpha)\delta_H\dot\varphi_1\big), \tag{25}$$

–the *relay form*

$$u = -\gamma\,\mathrm{sign}\,\big(\alpha\delta_\varphi + (1-\alpha)\delta_H\dot\varphi_1\big), \tag{26}$$

where $\gamma > 0$ is a gain factor. The case $\alpha = 0$ corresponds to the energy control problem. It follows from Theorem 2 that for $\alpha = 0$ the control goal $Q(x_1(t)) \to 0$ is achieved if the desired level of energy does not exceed the value $H_* = 2\omega^2$, corresponding to the upper equilibrium of one pendulum and the lower equilibrium of the other one.

It is worth noting that the control law (25) for the system (21) with $\sigma = 1$ has been proposed and numerically examined in [12]. In the paper [13] the case of $\sigma = –1$ and nonlinear coupling function in (21) is considered and the results of analytical, numerical and experimental study of the closed-loop system are presented. Numerical analysis of the system with the control law (26) for the both $\sigma = \{0, 1\}$ and for the cases of conservative ($\rho = 0$) and dissipative ($\rho > 0$) oscillators are presented in [14].

D. Comparative examination of the closed-loop full-order and reduced systems

In this subsection some numerical results of examination the closed-loop system with the proportional control law (25) are presented. The parameter values are as follows:

– the control law parameters: $\gamma = 1$, $\alpha = 0.7$, $\sigma = 1$ (the anti-phase steady-state oscillations are required), $H_* = 10\ \mathrm{s}^{-2}$;

– the plant model parameters: $\omega^2 = 10\ \mathrm{s}^{-2}$, $k = 1.75\ \mathrm{s}^{-2}$, $\varepsilon = 0.1$ s.

The phase variables have zero initial values. Two cases of the damping parameter ρ are studied: $\rho = 0$ (the conservative system), and $\rho = 0.1\ \mathrm{s}^{-1}$ (the dissipative system).

The simulation results are depicted in Figs. 1–6. First consider the energy control problem ($\alpha = 0$) for conservative case ($\rho = 0$). It is seen from Fig. 1 and Fig. 2 that the energy approaches the desired value both for reduced and for the full-order systems. This result confirms the theoretical statements.

Now consider more complex problem of energy control with synchronization ($\alpha > 0$). Since the existing theoretical results do not apply, simulation is the only way of its analysis. It is seen that for the lossless case ($\rho = 0$) and the "ideal" plant model ($\varepsilon = 0$) the control goal is achieved: both pendulums fall in anti-phase oscillatory mode, the total energy $H(x_1(t))$ tends to the desired value H_*, and the control torque $u(t)$ tends to zero, see Fig. 3. Note, that the relation between transient times for H and for Q_φ can be changed by means of changing the weight coefficient α. In the lossless case the control amplitude can be arbitrarily decreased by means of decreasing the gain γ.

The simulation results for both energy control and the anti-phase oscillations excitation by means of the SG-control law (25) for the full plant model (20) are pictured in Figs. 2–6. As it is seen, the closed-loop system behaviour close to that of the system (25), (21), where the reduced plant model is used. It also should be noticed that for the system with loss ($\rho > 0$) there exists the "threshold value" $\bar{\varepsilon}$ of the motor time constant ε such that the excitation does not occur, or existing oscillation relaxes, as far as $\varepsilon > \bar{\varepsilon}$. The threshold $\bar{\varepsilon}$ depends on the system parameters. The areas of oscillations quenching are found by means of linearization of the closed-loop system model (25), (20) near the origin and applying the Hermite–Mikhailov frequency-domain criterion. These results have been confirmed via the numerical simulations. Exemplary bifurcation diagrams on the (ε, H_*) plane for $\alpha = 0.7$ are pictured in Fig. 7. For the looseless case ($\rho = 0$), the oscillations with the desired energy H_* are excited for all $\varepsilon \geq 0$.

The simulations show that the examined speed-gradient energy control law (25) possesses the robustness with respect to unmodeled dynamics of the driving motor. Using the combined goal function allows to achieve, additionally, in-phase or anti-phase synchronization and this property is also robust with respect to dynamical disturbances.

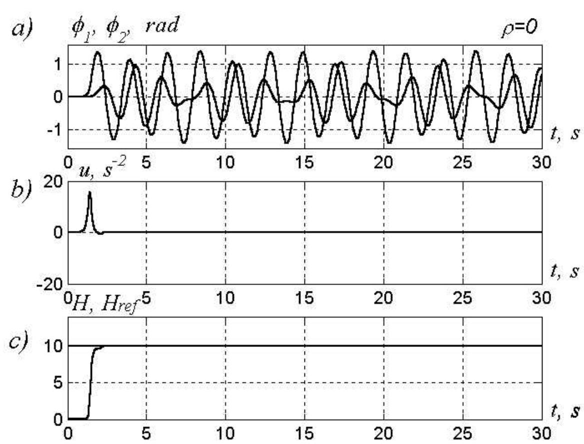

Fig. 1. Energy control; reduced plant model (21), $\rho = 0$.

IV. CONCLUSIONS

In the present work an energy speed-gradient control of singularly perturbed Hamiltonian systems is studied both theoretically and by computer simulation. Previous results on stability of speed-gradient control of singularly perturbed systems are extended to the case of partial stability. Quantitative results are obtained for synchronization of two coupled pendulums, taking into account the driving motor dynamics.

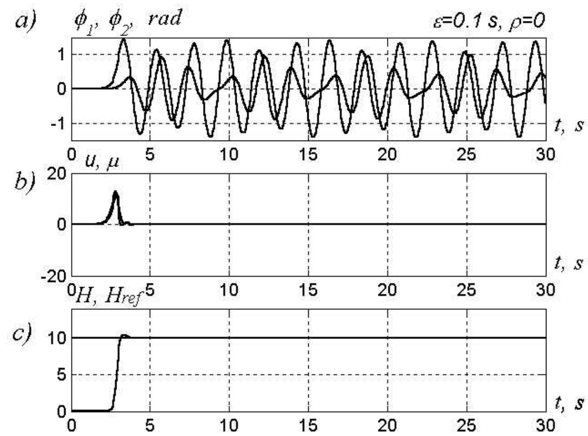

Fig. 2. Energy control; full plant model (20), $\rho = 0$, $\varepsilon = 0.1$ s.

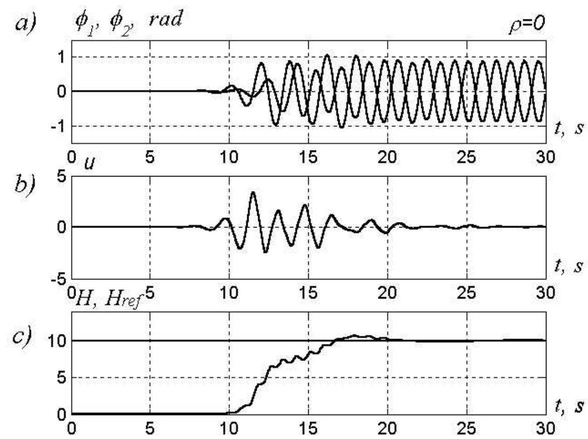

Fig. 3. Excitation of anti-phase oscillations; reduced plant model (21), $\rho = 0$.

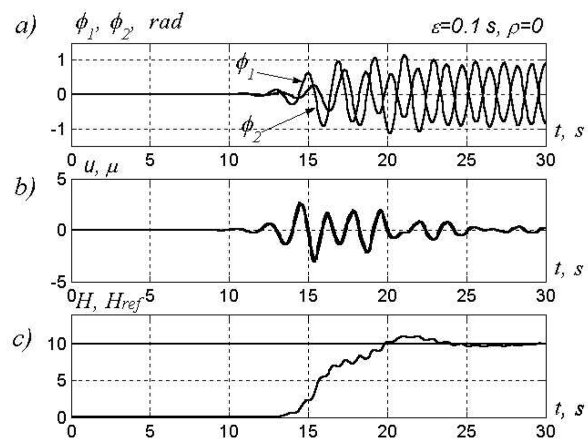

Fig. 4. Excitation of anti-phase oscillations; full plant model (20), $\rho = 0$, $\varepsilon = 0.1$ s, $\alpha = 0.7$.

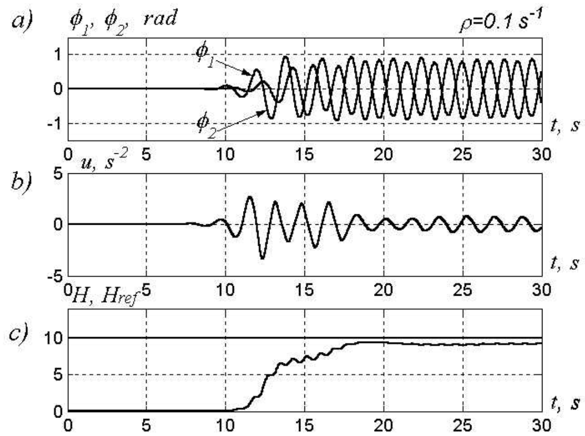

Fig. 5. Excitation of anti-phase oscillations; reduced plant model (21), $\rho = 0.1\ \mathrm{s}^{-1}$.

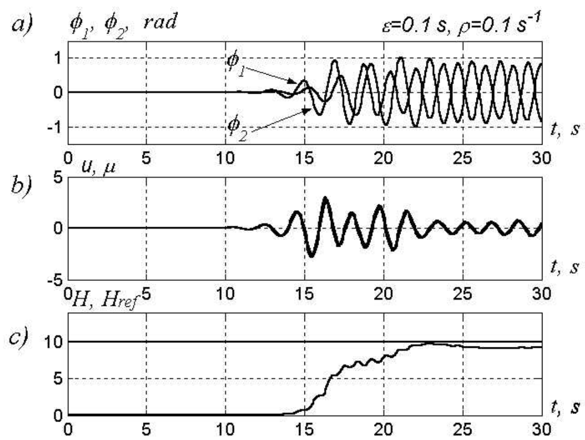

Fig. 6. Excitation of anti-phase oscillations; full plant model (20), $\rho = 0.1\ \mathrm{s}^{-1}$, $\varepsilon = 0.1$ s, $\alpha = 0.7$.

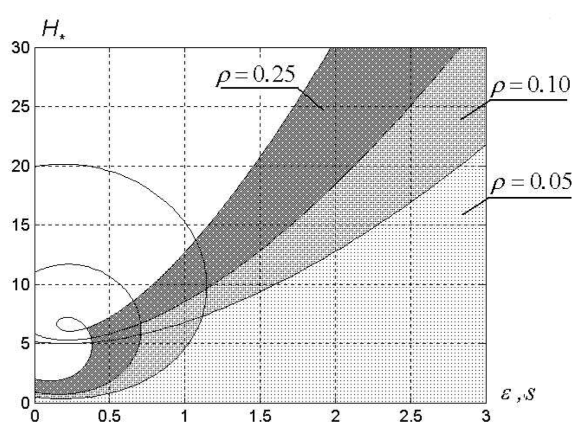

Fig. 7. Areas of the excitation death for $\alpha = 0.7$, $\rho = 0.05$, 0.10, $0.25 s^{-1}$.

The simulations show that the examined speed-gradient energy control law possesses the robustness with respect to unmodeled dynamics of the motor. Moreover, using more sophisticated goal function allows to achieve, additionally, in-phase or anti-phase synchronization and this property is also robust with respect to dynamical disturbances. Comparison with the results of [15] shows that the perturbations stemming from inertia of the coupling link and the perturbations stemming from inertia of motor provide different influence upon the perturbed system behavior.

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