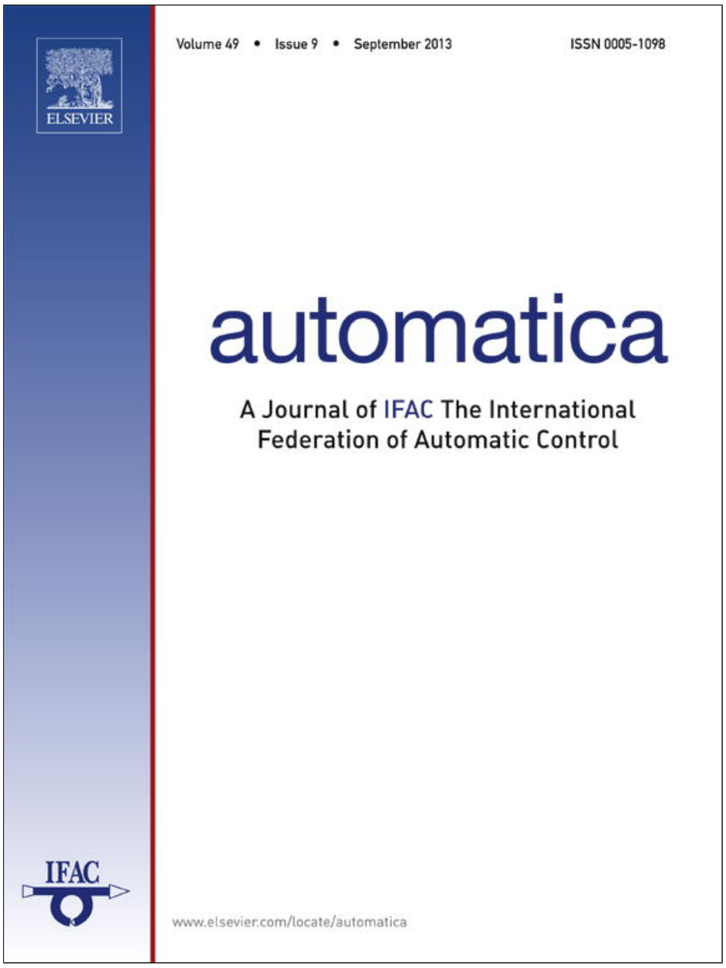
Volume 49 • Issue 9 • September 2013
ISSN 0005-1098
ELSEVIER
automatica
A Journal of IFAC The International
Federation of Automatic Control
IFAC
www.elsevier.com/locate/automatica

Automatica 49 (2013) 2928–2932

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

Average consensus in networks with nonlinearly delayed couplings and switching topology[☆]

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ARTICLE INFO

Article history:
Received 10 October 2012
Received in revised form
14 March 2013
Accepted 1 June 2013
Available online 6 July 2013

Keywords:
Complex networks
Synchronization
Delays
Nonlinear control

ABSTRACT

The paper addresses consensus under nonlinear couplings and bounded delays for multi-agent systems, where the agents have the single-integrator dynamics. The network topology is undirected and may alter as time progresses. The couplings are uncertain and satisfy a conventional sector condition with known sector slopes. The delays are uncertain, time-varying and obey known upper bounds. The network satisfies a symmetry condition that resembles the Newton's Third Law. Explicit analytical conditions for the robust consensus are offered that employ only the known upper bounds for the delays and the sector slopes.

1. Introduction

Enormous attention from the research community has recently been paid to various regular cooperative behaviors in multi-agent systems that are achieved via local interactions between the agents. *Consensus* or synchronization among the agents is one of the simplest yet important samples of such a behavior, which lies in the heart of many natural phenomena and engineering designs (including flocking, swarming and stabilization of formations) and was the subject of extensive research. We refer the reader to Mesbahi and Egerstedt (2010) and Ren and Beard (2008) for its excellent survey.

Despite the overall progress, some issues in this area still await further research even for agents with the simplest single integrator dynamics. Among them, there is consensus robustness against uncertainties in interactions between the agents, including time delays. These delays inevitably occur in applications and are well known as potential sources of instability. However the effect of delays on consensus was studied only for a few special situations (see e.g. Münz, 2010 for a survey).

A relatively well studied case is where each agent has access to its current output, whereas the influence of the neighbors may be delayed, see e.g., Angeli and Bliman (2006), Moreau (2004) and Papachristodoulou, Jadbabaie, and Muënz (2010). Then consensus tolerates arbitrarily large time-varying delays, provided that they remain bounded. This fact can be established by the research technique elaborated for non-delayed systems and based on the shrinking property of the convex hull of the current states of the agents (Angeli & Bliman, 2006; Lin, Francis, & Maggiore, 2007), whose hull should be replaced by the convex hull of all states observed during a sufficiently large time interval under the circumstances. However this property does not hold in the case of self-delays, which may arise due to e.g., delayed self-actuation (Tian & Liu, 2008) or delayed effect of the neighbors due to relative measurements. In general, large enough delays of such a kind do cause instability; so disclosing the delay margin below which the consensus is maintained is a real concern. In the case of linear time-invariant networks, tight margins can be established on the basis of Nyquist-type criteria and other frequency-domain techniques (Bliman & Ferrari-Trecate, 2008; Lestas & Vinnicombe, 2010; Muënz, Papachristodoulou, & Allgöwer, 2010; Tian & Liu, 2008). However for networks with time-varying topologies, only sufficient margins, typically based on high-dimensional LMI's, are known for only restrictive classes of delays, with the constant delays and delays whose derivatives do not exceed 1 being the basic examples (Lin & Jia, 2011; Sun & Wang, 2009). Another restriction of the previous relevant research is that it deals with only linearly coupled networks. However nonlinear couplings are intrinsically inherent in many applications (Ren & Beard, 2008). For example, synchronization between oscillators is often arranged via their periodic

☆ Supported by RFBR grants 11-08-01218, 12-01-00808, 13-08-01014 and Federal Program "Cadres" (agreement 8855). The material in this paper was partially presented at the IFAC Workshop Adaptation and Learning in Control and Signal Processing (ALCOSP 2010), August 26–28, 2010, Antalya, Turkey; IFAC Workshop on Time-Delay Systems (TDS 2013), February 4–6, 2013, Grenoble, France; and European Control Conference (ECC 2013), July 17–19, 2013, Zurich, Switzerland. This paper was recommended for publication in revised form by Associate Editor Jie Chen under the direction of Editor André L. Tits.

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http://dx.doi.org/10.1016/j.automatica.2013.06.007

coupling (Ren & Beard, 2008); range-restricted communication is another typical source of nonlinearities in couplings (Lin et al., 2007). In general, linear control algorithms may acquire nonlinearities due to actuator saturations and measurement distortions, analog-to-digital conversions and quantization effects.

Unlike the previous research, this paper deals with nonlinear couplings, switched topology, and time-varying delays, including self-delays. Assumptions about the delays are primarily inspired by the situation where the self-delay is incorporated into delayed relative measurement response from neighbors. We limit ourselves to networks of first-order agents with undirected topologies and symmetric delayed time-varying couplings. Both delays and couplings may be uncertain; the available data comes to upper bounds on the delays and a sector containing the graphs of the couplings. The paper offers an explicit analytic criterion for not only consensus but also its robustness against this uncertainty.

2. Preliminaries and the problem setup

We first recall some concepts from the graph theory. A *graph* is a pair $G = (V, E)$ constituted by the finite set of *nodes* V and the set of *arcs* $E \subset V \times V$. The graph is said to be *undirected* if $(v, w) \in E \Leftrightarrow (w, v) \in E\ \forall v, w \in V$; a sequence of its nodes $v_1, v_2, \dots, v_k$ with $(v_i, v_{i+1}) \in E\ \forall i$ is called the *path* between v_1 and v_k; the graph is said to be *connected* if a path exists between any two nodes.

We consider the networked systems governed by the equations of the form:

$$\dot{x}_j(t) = \sum_{k=1}^{N} a_{jk}(t)\varphi_{jk}(t, y_{jk}(t - \tau_{jk}(t))),\qquad y_{jk}(t) := x_k(t) - x_j(t). \tag{1}$$

Here $t \ge 0, j = 1, \dots, N, x_j(t) \in \mathbb{R}^n$ stands for the state of the j-th node, the maps $\varphi_{jk}(t, y)$ are called *couplings*, $\tau_{jk}(t) \ge 0$ are time-varying delays, and $a_{jk}(t) \ge 0$ are weighting coefficients. They define the time-varying interaction graph: the k-th node influences the j-th one at time t if and only if $a_{jk}(t) > 0$. When dealing with discontinuous at $t = 0$ solutions of (1), we assume them right-continuous at $t = 0$ for the definiteness. We also assume that the initial functions $x_j(t), t < 0$, are bounded.

The objective is to disclose conditions under which the system comes to consensus in the following sense.

Definition 1. The networked system (1) *reaches the consensus* if $|x_k(t) - x_j(t)| \xrightarrow{t\to\infty} 0$ for any initial data and j, k, and the *average consensus* if for any initial data,

$$x_j(t) \xrightarrow{t\to\infty} \frac{1}{N}(x_1(0) + x_2(0) + \cdots + x_N(0)) \quad \forall j. \tag{2}$$

In general, consensus does not imply the average consensus or even existence of the limits $\lim_{t\to+\infty} x_j(t)$.

3. Main assumptions

Our starting assumption concerns the network topology.

Assumption 2. (i) The matrix $A(t) = (a_{jk}(t))$ is symmetric and Lebesgue measurable. (ii) There exist $\varepsilon > 0$ and $T > 0$ such that the graph $(V_N, \mathcal{E}_t)$ with the set of nodes $V_N = \{1, \dots, N\}$ and that of arcs $\mathcal{E}_t = \left\{(j, k) : \int_t^{t+T} a_{jk}(s)ds > \varepsilon\right\}$ is connected for all $t \ge 0$.

By (i), the interaction graph is undirected. The property (ii) is often referred to as the *uniform connectivity* of the network and is commonly adopted in the literature along with its analogs for directed graphs (Lin et al., 2007; Moreau, 2004; Papachristodoulou et al., 2010). It prohibits disintegration of the network into separated clusters and is acknowledged as nearly necessary for consensus.

Assumption 3. For any $t \ge 0, x \in \mathbb{R}^n, j, k \in V_N, j \ne k$, one has $\varphi_{jk}(t, x) = -\varphi_{kj}(t, -x)$ and $\tau_{jk}(t) = \tau_{kj}(t)$.

A basic motivation behind this assumption comes from various applications where coupling is due to the transmission of a physical energy between nodes, which may cause mutual direct influence of nodes or be the basis for the evaluation of inter-nodes characteristics to feed the control law, like the time-of-flight of an electromagnetic or acoustic pulse in radars, sonars, and hydroacoustic sensors. In many cases, the transmission time not only substantially exceeds the internal time scales of the nodes, thus making the coupling delay a real concern, but also is basically determined by the inter-node distance and environmental condition. Since they are typically the same for "j-to-k" and "k-to-j" transmissions, symmetry of related delays in Assumption 3 is a reasonable option. The same argument explains symmetry of couplings: the phenomenon whose refined expression in the case of direct inter-node influence is given by the Newton's Third Law. In turns, many popular sensor-based coupling protocols mimic physical laws at least by the symmetry property. Examples include, but are not limited to, chaotic lasers coupled by their laser beams (Schöll & Schuster, 2008), neurons in the brain interacting through exchange of signals, electronic circuits, where inductors and transmission lines delay a signal to another end, etc. Similar in flavor the *microscopic traffic flow model* will be considered in detail in Section 4.

Remark 4. Under Assumption 3, consensus implies the average consensus (2).

Indeed, summing up (1) over $j = 1, \dots, N$ yields $\dot{s} \equiv 0$ for $s(t) := N^{-1}\sum_j x_j(t)$. So $s(t) \equiv s(0)$ and $x_k(t) - x_j(t) \to 0\ \forall j, k \Longrightarrow x_j(t) \to s(0)\ \forall j$.

We are concerned with the situation where the couplings φ_{jk} are not completely known, which may be caused by e.g., various uncertainties in the medium transmitting energy between nodes. By following the common approach adopted in the robust control theory, we assume that the available knowledge about couplings can be boiled down into a certain *quadratic inequality*: for a known constant $\gamma > 0$, they belong to the set $\mathfrak{S}(\gamma)$ of continuous maps $\varphi : [0; +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\gamma x^T \varphi(t, x) \ge |\varphi(t, x)|^2 \quad \forall t \ge 0,\ x \ne 0; \tag{3}$$

$$\inf_{t\ge 0, |x|>\delta} |\varphi(t, x)| > 0 \quad \forall \delta > 0, \qquad \varphi(t, 0) \equiv 0. \tag{4}$$

If the state is scalar ($n = 1$), (3) expresses the conventional sector condition: the graph of $\varphi(t, \cdot)$ lies between the lines $y = \gamma x$ and $y = 0$. Inequality (4) prevents the couplings from decay to zero as time progresses. For $n \ge 2$, the inclusion $\varphi_{jk} \in \mathfrak{S}(\gamma)$ may be viewed as a multi-variable analog of the sector condition.

The system uncertainties are completed by those in the coupling weights a_{jk} and delays τ_{jk}: they are unknown Lebesgue measurable functions of time. At the same time, their upper bounds are available:

$$a_{jk}(t) \le \bar{a}_{jk}, \qquad \tau_{jk}(t) \le \bar{\tau}_{jk}, \qquad d_j(t) \le \bar{d}_j \quad \forall t \ge 0, \tag{5}$$

where $d_j(t) := \sum_{k=1}^N a_{jk}(t)$. A consensus criterion should be given in terms of these bounds $\bar{a}_{jk}, \bar{\tau}_{jk}, \bar{d}_j \le \sum_k \bar{a}_{jk}$ and the "sector slope" γ, but not the couplings, weights, and delays themselves. Such a criterion in fact ensures the *robust* consensus in the sense that consensus holds for all uncertainties satisfying the above requirements.

4. Main results

Theorem 5. *Let (5) and Assumptions 2 and 3 hold and $\varphi_{jk} \in \mathfrak{S}(\gamma)$ for some $\gamma > 0$. The network (1) reaches the average consensus whenever*

$$\frac{1}{2\gamma} - \left(\bar{d}_j \sum_{k=1}^{N} \bar{a}_{jk}\bar{\tau}_{jk}^2\right)^{1/2} > 0 \quad \forall j. \tag{6}$$

The proof of Theorem 5 is given in Section 6.

4.1. Consensus in linear networks

The identical coupling maps $\varphi_{jk}(x) = x$ evidently satisfy Assumption 3 and belong to the class $\mathfrak{S}(1)$. Since $\bar{d}_j \le \sum_{k=1}^{N} \bar{a}_{jk}$, we arrive at the following.

Corollary 6. *Let $\varphi_{jk}(x) = x$, Assumptions 2, 3 and (5) hold, and the following inequality be valid*

$$\max_k \bar{\tau}_{jk} \le \left(2\sum_{m=1}^{N} \bar{a}_{jm}\right)^{-1} \quad \forall j. \tag{7}$$

Then the network (1) achieves consensus.

This corollary is in touch with Bliman and Ferrari-Trecate (2008), Muënz et al. (2010) and Tian and Liu (2008), where, unlike this paper, only linear couplings and fixed topology were addressed. For constant delays and interaction graph, Tian and Liu (2008, Section IV, Remark 4) claims that (7) implies consensus even if the graph is directed with oriented spanning tree and Assumption 3 is dropped. Tight estimates of the maximal tolerable delay level were obtained in Bliman and Ferrari-Trecate (2008) and Muënz et al. (2010) for constant delays. Under Assumption 3, these estimates take the form Bliman and Ferrari-Trecate (2008) $\bar{\tau}_{jk} \le \pi/(2\lambda_{\max}(L))$ $\forall j, k$, where $\lambda_{\max}(L)$ is the maximal eigenvalue of the Laplacian matrix L of the weighted graph given by (a_{jk}). Here the right-hand side exceeds that of (7) since $\lambda_{\max} < 2\sum_{k=1}^{N} a_{jk}$ by the Gershgorin theorem (Mesbahi & Egerstedt, 2010). So for fixed topology and trivial couplings, (7) is only sufficient but not necessary for the consensus. It should be emphasized that the techniques from Bliman and Ferrari-Trecate (2008), Muënz et al. (2010) and Tian and Liu (2008) are not applicable to switching networks and nonlinear couplings, which are among the main concerns of this paper.

4.2. Application to microscopic traffic flow models

In the face of considerable economic and ecological losses due to traffic accidents and congestions, the study of the vehicular traffic dynamics still represents a real challenge for the research community. *Microscopic* traffic flow models are closely related to models of self-propelled particle ensembles (Helbing, 2001) and commonly adopted as simple but instructive tools for traffic analysis. Since the pioneering work Chandler, Herman, and Montroll (1958), the *delay* in drivers reaction has been recognized as a crucial factor participating into the overall flow dynamics, see e.g. Michiels, Morarescu, and Niculescu (2009) and Sipahi, Atay, and Niculescu (2007) and references therein. A model of such kind (Chandler et al., 1958; Helbing, 2001; Sipahi et al., 2007) deals with N vehicles, indexed 1 through N, following along a common circular single lane road. In doing so, every vehicle tries to equalize its velocity with that of its predecessor:

$$\dot{v}_j(t) = K(v_{j\oplus 1}(t-\tau) - v_j(t-\tau)).$$

Here $v_j(t)$ is the velocity of the j-th vehicle, τ is the delay in the driver's action, $\oplus$ is the summation modulus N, and K stands for the driver's "sensitivity" to alterations of the relative velocity of the vehicle in front of him.

A key issue addressed via this model is that of stability of the constant-velocity solutions $v_1 \equiv \cdots \equiv v_N \equiv$ const. The respective results of Michiels et al. (2009) deal with fixed observation topology, which assumed that a driver may watch not only its predecessor but also other vehicles and the constant common for all drivers' delay, which may be both discrete and distributed, taking thus into account special effects in human memory (Sipahi et al., 2007).

Now we extend the results of Michiels et al. (2009) and Sipahi et al. (2007) on the more realistic case where the delays in driver reactions depend on time, and the observation topology alters over time since a driver may lose or acquire sight of the companions depending on the relief and weather conditions. Furthermore, unlike (Michiels et al., 2009; Sipahi et al., 2007), we do not assume the acceleration to be linear function of relative velocity (e.g. the actions of drivers may be subjected to saturation).

Specifically, we assume that the driver of the j-th vehicle adjusts the velocity v_j based on the relative velocities of $p \le N-1$ preceding and p following vehicles:

$$\dot{v}_j(t) = K\sum_{m=-p}^{p} a^m(t)\varphi(v_{j\oplus m}(t-\tau^m(t)) - v_j(t-\tau^m(t))). \tag{8}$$

Here $a^0(t) := 0$ and $\varphi(\cdot) \in \mathfrak{S}(\gamma)$. This model takes into account that the reaction on the nearest neighbors may be faster then on distant ones (τ^m depends on m). This "order-based" determinism leads to the assumption that the response to the m-th predecessor and the m-th follower are equally sharp and fast: $a^m = a^{-m}$ and $\tau^m = \tau^{-m}$ for any m.

Applying Theorem 5 gives rise to the following.

Theorem 7. *Let $a^m(t) = a^{-m}(t) \ge 0$ and $\tau^m(t) = \tau^{-m}(t) \ge 0$. Suppose that $T, \varepsilon > 0$ exist such that $\int_t^{t+T} a^1(t)dt > \varepsilon$ for any $t \ge 0$. Let $\bar{a}^m := \sup_{t\ge 0} a^m(t)$, $\bar{\tau}^m := \sup_{t\ge 0}\tau^m(t)$ satisfy the inequality*

$$\max_{m=-p,\dots,p} \bar{\tau}^m \le \left(2\gamma\sum_{m=-p}^{p}\bar{a}^m\right)^{-1}.$$

Then the system (8) achieves the average consensus, i.e., $v_j(t) \to N^{-1}(v_1(0) + \cdots + v_N(0))$ as $t \to \infty$ for all j.

Proof. We introduce the coupling weights a_{jk} and delays τ_{jk} by putting $a_{j,j\oplus m} := a^m$, $\tau_{j,j\oplus m} := \tau^m$ for $m = -p, \dots, p$, and $a_{jk} := 0$, $\tau_{jk} := 0$ for any other pair j, k. It is easy to see that Assumptions 2 and 3 hold. After this, the proof is completed similarly to that of Corollary 6 since $\varphi(\cdot) \in \mathfrak{S}(\gamma)$. □

By Theorem 7, the vehicles travel with asymptotically equal velocities and traffic jams are impossible provided that the drivers react with small enough delays.

5. Simulations

To illustrate Theorem 5, two simulation tests were performed for $N = 4$ scalar ($n = 1$) agents. The first test concerns the continuous-time network (1) with saturated couplings: $\varphi_{jk}(x) = 0.5(|x+0.5| - |x-0.5|)$ ($\gamma = 1$) and periodic weights $a_{12}(t) = a_{21}(t)$ (shown in Fig. 1), $a_{23}(t) = a_{32}(t) := a_{12}(t+1)$, $a_{34}(t) = a_{43}(t) := a_{12}(t+0.5)$, $a_{14}(t) = a_{41}(t) := a_{12}(t+1.5)$ (thus $\bar{a}_{12} = \bar{a}_{23} = \bar{a}_{34} = \bar{a}_{14} = 1$ and $d_j = 2\,\forall j$), and the other weights a_{jk} being zero. The delays are $\tau_{12} = \tau_{21} = \tau_{34} = \tau_{43} = 0.1$ and $\tau_{23} = \tau_{32} = \tau_{14} = \tau_{41} = 0.339$, thus (6) holds. Fig. 2 displays the result of the simulation with the initial data $x(0) = [10, -10, 5, 2]^T$ and $x(t) \equiv 0$ for $t < 0$.

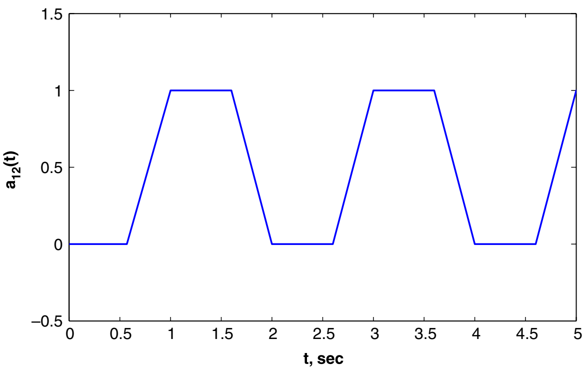

Fig. 1. Weight function a_{12}.

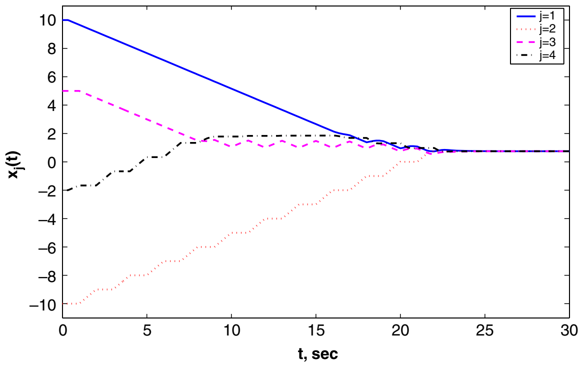

Fig. 2. Continuous-time dynamics with saturation couplings.

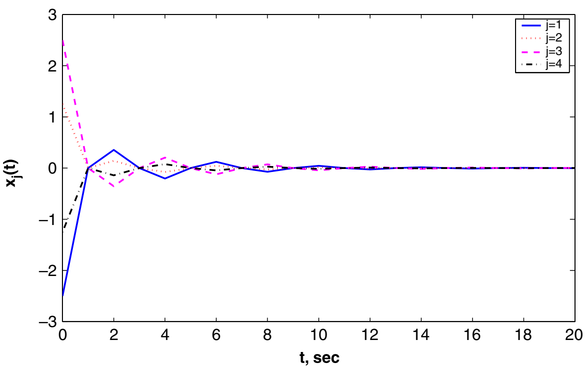

Fig. 3. Discrete-time dynamics, $\varphi_{jk}(x) = x(1.01 + \cos x)$.

The second test demonstrates the application of Theorem 5 to discrete-time systems of the form

$$x_j(t+1) = x_j(t) + \sum_{k=1}^{N} a_{jk}\varphi_{jk}(y_{jk}(t-\tau_{jk}(t))), \qquad (9)$$
$$y_{jk}(t) := x_k(t) - x_j(t).$$

Here $t \geq 0$ and $\tau_{jk}(t) \geq 0$ take only integer values. Adopting the approach from Mikheev, Sobolev, and Fridman (1988), one may consider the solutions of (9) as sample sequences of a continuous-time system (1) with $a_{jk}(t) := a_{jk}$, $\varphi_{jk}(t,x) := \varphi_{jk}(x)$ and delays of special type $\tau_{jk}(t) := t - \lfloor t \rfloor + \tau_{jk}(\lfloor t \rfloor)$, where $\lfloor t \rfloor$ denotes the integer floor of t (so $\bar{\tau}_{jk} = 1 + \sup_{t=0,1,\ldots} \tau_{jk}(t)$). Fig. 3 shows the corresponding simulation result for the network (9) with $\varphi_{jk}(x) = x(1.01 + \cos x)$ ($\gamma = 2.01$), $\tau_{jk}(t) \equiv 2$ for $t = 0, 1, 2\ldots$ and $a_{12} = a_{23} = a_{34} = a_{41} = a_{21} = a_{32} = a_{43} = a_{14} = K := 0.041$ (thus $d_j = 2K\ \forall j$), which is easily shown to satisfy (6).

Both simulation tests have confirmed that the consensus is achieved in accordance with Theorem 5.

6. Proof of Theorem 5

From now on, its assumptions are supposed to hold. The key part of the proof is contained in the following.

Lemma 8. *Any solution $x_j(t), j = 1, \ldots, N$ of (1) is bounded, can be extended on $[0, \infty)$, and is such that*

$$\int_0^\infty a_{jk}(t)|\xi_{jk}(t)|^2\,dt < \infty, \quad \forall j, k, \qquad (10)$$

where $\xi_{jk}(t) := \varphi_{jk}(t, x_k(t-\tau_{jk}(t)) - x_j(t-\tau_{jk}(t)))$.

Proof. Given ensembles $f = \{f_{jk}(\cdot)\}_{j,k=1}^N, g = \{g_{jk}(\cdot)\}_{j,k=1}^N$ of equally dimensioned vector functions, we put $f^\dagger := \{f_{kj}(\cdot)\}_{j,k=1}^N$ and

$$\langle f, g\rangle_t := \sum_{j,k}\int_0^t a_{jk}(s)f_{jk}(s)^* g_{jk}(s)ds, \qquad \|f\|_t := \sqrt{\langle f, f\rangle_t}.$$

Since $a_{jk} = a_{kj}$ and $\xi_{jk} = -\xi_{kj}$ by Assumption 3,

$$\langle \xi, f^\dagger\rangle_t = \sum_{j,k} a_{jk}\xi_{jk}f_{kj} = -\sum_{j,k} a_{kj}\xi_{kj}f_{kj} = -\langle \xi, f\rangle_t. \qquad (11)$$

We denote $\tau^0_{jk}(t) := \min(t; \tau_{jk}(t))$, $\eta_{jk}(t) := x_j(t-\tau_{jk}(t))$, $\eta^0_{jk}(t) := x_j(t-\tau^0_{jk}(t))$, $\rho_{jk}(t) := \eta_{jk}(t) - \eta^0_{jk}(t)$, $\chi_{jk}(t) := x_j(t)(\forall k)$.

Since $\tau_{jk} = \tau_{kj} \Rightarrow \xi_{jk}(t) = \varphi_{jk}(t, \eta_{kj}(t) - \eta_{jk}(t))$, the definition of $\mathfrak{S}(\gamma)$ yields that

$$\xi_{jk}(t)^*(\eta_{kj}(t) - \eta_{jk}(t)) - \gamma^{-1}|\xi_{jk}(t)|^2 \geq 0.$$

After multiplication by a_{jk} and integration, we get

$$0 \geq \gamma^{-1}\|\xi\|_t^2 - \langle \xi, \eta^\dagger - \eta\rangle_t \overset{(11)}{=} 2\langle \xi, \eta\rangle_t + \gamma^{-1}\|\xi\|_t^2$$
$$= 2\left[\langle \xi, \chi\rangle_t + \langle \xi, \eta - \eta^0\rangle_t + \langle \xi, \eta^0 - \chi\rangle_t + (2\gamma)^{-1}\|\xi\|_t^2\right]. \qquad (12)$$

Now we observe that (1) can be shaped into

$$\dot{x}_j(t) = \sum_{k=1}^N a_{jk}(t)\xi_{jk}(t) \qquad (13)$$

and $\chi_{jk} = x_j$ by definition. Hence

$$2\langle \xi, \chi\rangle_t = 2\sum_{j,k}\int_0^t a_{jk}\xi^*_{jk}x_j dt \overset{(13)}{=} \sum_j (|x_j(t)|^2 - |x_j(0)|^2). \qquad (14)$$

We are going to show that there exists $\varepsilon > 0$ such that

$$\langle \xi, \eta^0 - \chi\rangle_t + (2\gamma)^{-1}\|\xi\|_t^2 \geq \varepsilon\|\xi\|_t^2 \quad \forall t. \qquad (15)$$

Indeed, thanks to the Cauchy–Schwartz inequality,

$$|\dot{x}_j|^2 \overset{(13)}{=} \left|\sum_k a_{jk}^{1/2}(a_{jk}^{1/2}\xi_{jk})\right|^2 \leq \sum_k a_{jk}\sum_m a_{jm}|\xi_{jm}|^2 \overset{(5)}{\leq} \bar{d}_j\sum_m a_{jm}|\xi_{jm}|^2, \qquad (16)$$

$$|x_j(t) - \underbrace{\eta^0_{jk}(t)}_{x_j(t-\tau^0_{jk}(t))}|^2 = \left|\int_{t-\tau^0_{jk}(t)}^t \dot{x}_j(s)\,ds\right|^2 \leq \left|\int_{t-\tau^0_{jk}(t)}^t 1\cdot|\dot{x}_j(s)|ds\right|^2 \leq \tau^0_{jk}(t)\int_{t-\tau^0_{jk}(t)}^t |\dot{x}_j(s)|^2 ds. \qquad (17)$$

Let $\nu_T(s)$ be the Lebesgue measure of the set $\{t \in [0;T] : t - \tau_{jk}(t) \le s \le t\} \subset [s; s+\bar{\tau}_{jk}]$, and so $\nu_T(s) \le \bar{\tau}_{jk}$. By (16) and (17), we have for sufficiently small $\varepsilon > 0$

$$\begin{aligned}\|\chi - \eta^0\|_T^2 &\overset{(5),(17)}{\le} \sum_{j,k} \bar{a}_{jk}\bar{\tau}_{jk} \int_0^T \int_{t-\tau^0_{jk}(t)}^t |\dot{x}_j(s)|^2 ds dt \\ &= \sum_{j,k} \bar{a}_{jk}\bar{\tau}_{jk} \int_0^T \nu_T(s)|\dot{x}_j(s)|^2 ds \\ &\le \sum_{j,k} \bar{a}_{jk}\bar{\tau}^2_{jk} \int_0^T |\dot{x}_j(t)|^2 dt \\ &\overset{(16)}{\le} \sum_j \left(\bar{d}_j \sum_k \bar{a}_{jk}\bar{\tau}^2_{jk}\right) \sum_m \int_0^T a_{jm}(t)|\xi_{jm}(t)|^2 dt \\ &\overset{(6)}{\le} \left(\frac{1}{2\gamma} - \varepsilon\right)^2 \|\xi\|_T^2.\end{aligned}$$

The proof of (15) is completed by the estimates

$$|\langle \xi, \eta^0 - \chi\rangle_t| \le \|\xi\|_t \|x_j - \eta^0_{jk}\|_t \le \left(\frac{1}{2\gamma} - \varepsilon\right)\|\xi\|_t^2.$$

To estimate $\langle \xi, \eta - \eta^0\rangle_t$ in (12), we observe that $\eta_{jk}(t) - \eta^0_{jk}(t) = x_j(t-\tau_{jk}(t)) - x_j(0)$ if $t - \tau_{jk}(t) < 0$ and $\eta_{jk}(t) - \eta^0_{jk}(t) = 0$ otherwise (in particular, for $t \ge \bar{\tau}_{jk}$). This implies that $\sup_t \|\eta - \eta^0\|_t < \infty$ and

$$2|\langle \xi, \eta - \eta^0\rangle_t| \le \varepsilon\|\xi\|_t^2 + \varepsilon^{-1}|\eta - \eta^0|_t^2 \le \varepsilon\|\xi\|_t^2 + C, \tag{18}$$

where $C > 0$ depends on the initial data and delays only. Substituting (14), (15), and (18) into (12) assures that

$$\sum_j |x_j(t)|^2 + \varepsilon\|\xi\|_t^2 \le C_1,$$

where C_1 depends on the initial data only but not on t. The last statement implies that the solution remains bounded and thus can be extended on $[0, \infty)$. Moreover, $\|\xi\|_t^2 \le \varepsilon^{-1}C_1$ for any $t > 0$, which implies (10). □

Proof of Theorem 5. Suppose to the contrary that consensus does not hold. Then there exist a solution of (1), a number $\delta > 0$, and a sequence $t_m \uparrow +\infty$ such that $\max_{j,k} |x_j(t_n) - x_k(t_n)| > 3\delta N$. Without any loss of generality, it can be assumed that $t_{m+1} - t_m > T_0$, where T_0 is taken from Assumption 2. Then the sets $\Delta_m = [t_m; t_m + T_0]$ are disjoint. Since the graph $(V_N, \mathcal{E}_{t_m})$ is connected for any m, an arc $(j_m, k_m) \in \mathcal{E}_{t_n}$ exists such that $|x_{j_m}(t_m) - x_{k_m}(t_m)| > 3\delta$. Thanks to (10) and (16), $\int_0^\infty |\dot{x}_j(t)|^2 dt < \infty\ \forall j$; hence $|x_j(t'') - x_j(t')| \to 0$ as $t'', t' \to +\infty$ and $|t'' - t'| \le M < \infty$, where we denote $M := T_0 + \max_{j,k} \bar{\tau}_{jk}$. Hence $|x_{j_m}(t - \tau_{j_m k_m}(t)) - x_{j_m}(t_m)| \le \delta$ and $|x_{k_m}(t - \tau_{j_m k_m}(t)) - x_{k_m}(t_m)| \le \delta$ for $t \in \Delta_m$ and large m, thus $|x_{j_m}(t - \tau_{j_m k_m}(t)) - x_{k_m}(t - \tau_{j_m k_m}(t))| \ge \delta$. Thanks to (4), there exists $\nu > 0$ such that $|\xi_{j_m k_m}(t)| \ge \nu\ \forall t \in \Delta_m$ and thus $\sum_{j,k} \int_{\Delta_m} a_{jk}(t)|\xi_{jk}(t)|^2 dt \ge \nu^2 \sum_{j,k} \int_{t_m}^{t_m+T} a_{jk}(t) dt \ge \varepsilon\nu^2$ by the definition of $\mathcal{E}_{t_m}$. Thus we arrive at the contradiction with (10), which completes the proof. □

7. Conclusion

The state consensus among first-order agents was addressed for a network with switching undirected topology and nonlinear uncertain couplings. The network satisfies the uniform connectivity assumption, a symmetry condition similar in flavor to Newton's Third Law, and a sector condition with known slopes. Interaction between the agents is corrupted by uncertain time-varying delays with known upper bounds. A new criterion for robust consensus is obtained; its applicability is confirmed by simulations. Its extensions on the leader–follower and reference-tracking consensus, and agents with higher-order dynamics are the subjects of ongoing research.

Acknowledgment

The author is grateful to Prof. Alexey Matveev for his invaluable help in improving the paper.

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