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Electronic Notes in DISCRETE MATHEMATICS

Electronic Notes in Discrete Mathematics 51 (2016) 7–14

www.elsevier.com/locate/endm

Consensus robustness against inner delays

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Abstract

Time-delays, inevitable in a large scale dynamical network, may considerably deteriorate its behavior up to the loss of stability. Given a network of agents that apply a distributed algorithm (or protocol) in order to reach some common goal, the delays may be caused by both communication and inner dynamics of the agents. We examine robustness of a cooperative behavior against delays of the second type. Using consensus problem over switching interaction graph as a case study, an analytic criterion for robustness is offered.

Keywords: Networks, robustness, delays, consensus, synchronization.

1 Introduction.

Many complex systems arising in engineering and natural sciences may be treated as teams of self-dependent *agents*, that are interconnected in accordance with some

[1] The work was supported by the ERC (grant ERCStG- 307207) and RFBR (grants 13-08-01014 and 14-08-01015). The main result (Theorem 3.1) is supported solely by RSF, grant 14-29-00142.

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http://dx.doi.org/10.1016/j.endm.2016.01.002

distributed *complex* in order to reach some desired cooperative behavior and constitute thus a dynamical network. Examples include e.g. regular behavior of animal populations (flocks, swarms etc.), ensembles of neuronal or genetic oscillators, social networks, smart power grids etc. A basic cooperative goal is synchronism or *consensus*, underlying many natural phenomena and engineering designs, see [1,2] and references therein for the survey of main historical milestones and applications.

An extremely important issue concerning the design of large scale networks is their robustness against time-delays that may seriously deteriorate the cooperative behavior up to the loss of stability of a network. Despite the enormous overall progress in analysis and control of networks, delay robustness problems are not satisfactorily studied even for the simplest first-order consensus algorithms. It is known that consensus tolerates arbitrary bounded *communication* delays which affect only data the agent receives from neighbors but not its own state [3,4]. Such a robustness fails under *self-delays*, affecting own states of the agents. Self-delays may be induced by retarded *relative* measurements [5–7] or be inherent for the agents themselves, arising from data processing and computations, delayed self-actuation ("input delays" [8]), or the operator response lag in a human-machine system (e.g. driver reaction time in traffic models [9]). In this paper, we deal with delays of the second type, which we refer to as *inner* delays. Unlike communication delays, large self-delays destabilize the network, and hence the concern is to find the delay margin below which consensus is preserved. For *time-invariant* networks tight estimates for this margin have been obtained by virtue of the frequency-domain analysis [4,5,8]. Under time-varying topology the existing consensus conditions address mainly continuous-time case [10] and employ systems of linear matrix inequalities (LMI), whose complexity is proportional to the numbers of different delays and different graphs among which the interaction topology switches. Generally the latter value grows as $O(N^2 2^{N^2/2})$ where N is the number of agents which makes the result practically inapplicable for switching large-scale networks.

In the present paper we derive an *analytic* condition for consensus robustness against heterogeneous inner delays in discrete-time networks with undirected time-varying graphs. To simplify matters, we confine ourselves with fixed delays and linear couplings. Notice that our criterion cannot be derived from those in the previous works of the first author [6,7] which, firstly, deal with a different structure of delays providing *average consensus* and, secondly, address only continuous-time case. Like the results from [6,7], our criterion is easily extended to *nonlinear* protocols with symmetric couplings (see Section 3.2); its extension to more general classes of networks (involving both inner and communication delays, directed topology and asymmetric couplings) is subject of ongoing research.

2 Preliminaries and the problem setup.

Throughout the paper, $\underline{N}$ stands for the set $\{1,2,\dots,N\}$. A graph $G = (\underline{N}, E)$, where nodes are identified with numbers from $\underline{N}$ and E stands for the set of arcs is said to be *compatible* to a matrix $A \in \mathbb{R}^{N\times N}$ if $a_{jk} \neq 0 \Leftrightarrow (j,k) \in E$. For two graphs $G_1 = (\underline{N}, E_1)$ and $G_2 = (\underline{N}, E_2)$ let $G_1 \cup G_2 = (\underline{N}, E_1 \cup E_2)$ stand for their union.

We consider a team of discrete-time agents indexed 1 through N

$$y_j(t+1) = y_j(t) + u_j(t-\tau_j),\, t = 0,1,\dots, \quad j = 1,2,\dots,N, \tag{1}$$

associated to the nodes of some switching undirected graph $G(t) = (\underline{N}, E(t))$; they aim to reach *consensus*: $y_j(t) - y_k(t) \xrightarrow[t\to\infty]{} 0$ and apply a conventional protocol [3]

$$u_j(t) = \sum_{k=1}^{N} a_{jk}(t)[y_k(t) - y_j(t)], \quad a_{jk}(t) \geq 0. \tag{2}$$

Here $y_j(t) \in \mathbb{R}, u_j(t) \in \mathbb{R}$ stand respectively for the state and input of the j-th agent, and integer $\tau_j \geq 0$ is the inner delay. The graph $G(t)$ is compatible to the matrix $A(t)$ for any $t = 0,1,\dots$, which may be chosen symmetric $a_{jk} = a_{kj}$ since $G(t)$ is undirected. Without loss of generality let $a_{jj}(t) \equiv 0$. A solution of (1),(2) is uniquely defined by the initial conditions $y_j(0), u_j(-1), \dots, u_j(-\tau_j)$ $(j = 1,\dots,N)$.

Below we disclose conditions on the delays, which guarantee consensus establishing under arbitrary coupling gains $a_{jk}(t)$ and switching topology $G(t)$, satisfying the conditions of non-degeneracy, boundedness and *repeated joint connectivity*.

Assumption 2.1 *There exist* $\varepsilon \geq 0$, $\bar{d}_1, \dots, \bar{d}_N \in (0;1)$ *and integer* $T \geq 1$ *such that*

$$\sum_{k=1}^{N} a_{jk}(t) \leq \bar{d}_j,\;\; a_{jk}(t) = a_{kj}(t) \in \{0\} \cup [\varepsilon; +\infty) \quad \forall t \geq 0, \tag{3}$$

and the union of graphs $G[t] \cup G[t+1] \cup \ldots \cup G[t+T]$ *is connected for any* $t \geq 0$.

It was shown in [3] that Assumption 2.1 with $\bar{d}_j < 1$ (so that $1 - \sum_{k=1}^{N} a_{jk}(t) \geq 1 - \max_j \bar{d}_j > 0$) implies consensus for undelayed agents (1) ($\tau_j = 0 \forall j$), which in fact holds under bounded *communication* delays and directed topology [3]. For undirected topology the repeated connectivity may be further relaxed to the infinite connectivity [3, Assumption 2]. Below we give a criterion which provides consensus for delayed agents (1), ensuring thus consensus *robustness* against inner delays $\tau_j \geq 0$ and uncertainty in the coupling gains and the switching network topology.

An important example of network (1),(2) (and its continuous-time analogs) is *microscopic traffic flow model* which has been studied since 1950s and earned the reputation of a simple yet and instructive model for traffic flow analysis. A model of such kind [11] deals with N vehicles on a circular single lane road. A driver controls the acceleration of his/her car to equalize its velocity with that of its predecessor:

$$\frac{d}{dt}v_j(t) = u_j(t-\tau), \quad u_j(t) = K(v_{j\oplus 1}(t) - v_j(t)).$$

Here $v_j(t)$ is the velocity of the j-th vehicle, τ is the average driver reaction delay, $\oplus$ is the summation modulus N (i.e. $N \oplus 1 = 1$), and K stands for the driver's "sensitivity". A key issue is the velocity consensus $v_j(t) - v_k(t) \xrightarrow[t\to\infty]{} 0$, implying the absence of jams. Conditions for such a consensus from [11] were extended in [9] to more general protocols (2) with fixed gains (assuming that a driver may watch several vehicles) and distributed delays, reflecting the features of human memory. Network (1),(2) is a discrete-time analog of the model from [9], where each driver may have individual reaction time and lose or acquire the sight of companions due to relief or weather conditions and hence the graph and gains are time-varying.

3 Main Result.

The following condition for consensus robustness is the main result of the paper.

Theorem 3.1 *Under Assumption 2.1, suppose the inequalities hold as follows*

$$\bar{d}_j(1+2\tau_j) < 1 \quad \forall j \in \underline{N}. \tag{4}$$

Then protocol (2) *establishes consensus:* $y_j(t) - y_k(t) \xrightarrow[t\to\infty]{} 0$ *for any solution.*

As shown in [8, Theorem 2], under *time-invariant* topology $A(t) \equiv A$ the result of Theorem 3.1 retains its validity also for directed topology; and consensus, besides inner delays, tolerates also constant communication delays (in the right-hand side of (2) each term $y_k(t)$, $k \neq j$, may be replaced with $y_k(t-\tau_{jk})$). In the case where communication delays are absent and $A = A^\top$ condition (4) in fact may be tightened [8] as $\bar{d}_j < \sin\frac{\pi}{2(1+2\tau_j)}\, \forall j$, which is weaker than (4) since, as may be easily shown, $\alpha \leq \sin\frac{\pi\alpha}{2}\, \forall \alpha \in [0;1]$. So condition (4) in general is only sufficient but not necessary for consensus. It should be noticed that results of [8] are not applicable to the case of switching topology the present paper is focused on.

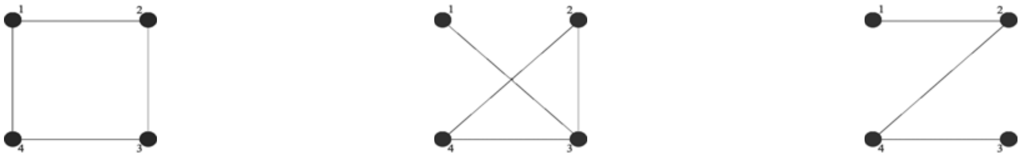

Fig. 1. $G(t)$ attains 3 values; $a_{jk} \in \{0; \varkappa\}$; $\bar{d}_1 = \bar{d}_2 = \bar{d}_4 = 2\varkappa$, $\bar{d}_3 = 3\varkappa$.

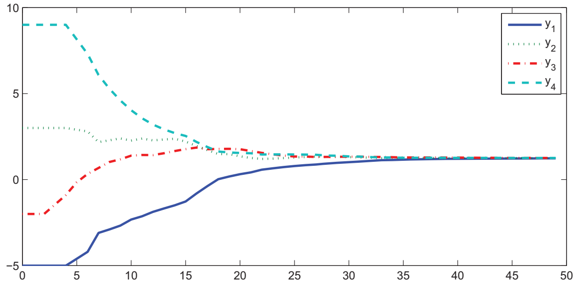

Fig. 2. Dynamics of $y_j(t), 0 \le t \le 49$, with $y(0) = (-5, 3, -2, 9)^\top$, $\tau_1 = \tau_2 = \tau_4 = 4$, $\tau_3 = 3$.

3.1 Numerical example

To demonstrate the applicability of Theorem 3.1, we simulated a team of $N = 4$ agents (1), coupled via protocol (2). Here $G(t)$ randomly switches between the graphs from Fig. 1 and $a_{jk} \in \{0; \varkappa\}$, hence $\bar{d}_1 = \bar{d}_2 = \bar{d}_4 = 2\varkappa$, $\bar{d}_3 = 3\varkappa$. Choosing $\varkappa = 0.05$, one easily finds the maximal delays under which consensus is provided: $\tau_1 = \tau_2 = \tau_4 = 4$, $\tau_3 = 3$. Fig. 2 illustrates convergence to consensus under initial conditions $y(0) = (-5, 3, -2, 9)^\top$, $u(t) < 0$ for $t < 0$ and these "worst-case" delays.

3.2 Extension: nonlinear protocols

As was mentioned in Introduction, in fact our result is applicable to more general agents with n-dimensional state $y_j(t) \in \mathbb{R}^n$ and nonlinear protocols

$$u_j(t) = \sum_{k=1}^{N} a_{jk}(t)\varphi_{jk}[y_k(t) - y_j(t)]. \tag{5}$$

Here $\varphi_{jk} : \mathbb{R}^n \to \mathbb{R}^n$ stand for some nonlinearities, describing the interaction law between the agents. A closer analysis of the proof of Theorem 3.1 reveals that it remains valid if the couplings φ_{jk} satisfy assumptions as follows.

Assumption 3.2 *The functions φ_{jk} are anti-symmetric: $\varphi_{jk}(y) := -\varphi_{kj}(-y)$ and, furthermore $0 < |\varphi_{jk}(y)|^2 \le \varphi_{jk}(y)^\top y$ whenever $y \neq 0$.*

The proof of Theorem 3.1 for protocols (5) under Assumption 3.2 retraces that from the previous section, taking $\phi_{jk}(t) := \varphi_{jk}(y_k(t) - y_j(t))$.

4 Proof of the Main Result

The proof of Theorem 3.1 relies on the fundamental results from control theory known as the Kalman-Szegö (or discrete-time Kalman-Yakubovich) lemma [12].

Lemma 4.1 *Consider a control system obeying the equations as follows*

$$x(t+1) = Px(t) + Qu(t) \in \mathbb{R}^d,\, y(t) = Rx(t) + Su(t) \in \mathbb{R}^s,\, u(t) \in \mathbb{R}^m. \tag{6}$$

Here P,Q,R,S are constant matrices. Assume the system to be controllable and observable [12] and let $W(\lambda) := S + R(\lambda I - P)^{-1}Q$ be its transfer function. Let $F(y,u)$ ($y \in \mathbb{C}^s, u \in \mathbb{C}^m$) be a Hermitian form. Then two claims are equivalent:

(i) *there exists a Hermitian form $V(x) = x^* Hx$ ($x \in \mathbb{C}^d$) such that*

$$V(x(t+1)) - V(x(t)) + F(y(t), u(t)) \le 0 \iff \tag{7}$$

$$(Px + Qu)^* H(Px + Qu) - x^* Hx + F(Rx + Su, u) \le 0 \forall x, u. \tag{8}$$

(ii) $F(W(\lambda)u, u) \le 0$ *for any* $u \in \mathbb{C}^m$ *and* $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$ *and* $\det(\lambda I - P) \ne 0$.

The equation of the delayed agent (1) may be easily rewritten in the state-space form (6) with $S_j = 0$ and appropriate P,Q,R by putting $x(t) = x_j(t) := (y_j(t), u_j(t-1), \ldots, u_j(t-\tau_j))$, $y(t) = y_j(t)$ and $u(t) = u_j(t)$ so that $x_j(t+1) = (y_j(t) + u_j(t-\tau_j), u_j(t), u_j(t-1), \ldots, u_j(t-\tau_j+1)) = Px_j(t) + Qu_j(t)$ and $y_j(t) = Rx_j(t)$. The system (6) obtained in this way is easily shown to be controllable and observable with transfer function $W_j(\lambda) = \lambda^{-\tau_j}/(\lambda - 1)$. Given a Hermitian form $F_j(y,u) := -2Re(\bar{y}u) - \alpha_j |u|^2$, where $y, u \in \mathbb{C}$, it is easily shown that $F_j(W_j(\lambda)u, u) = -|u|^2(\alpha_j + 2Re W_j(\lambda)) \le 0$ if and only if $\alpha_j + 2Re W_j(\lambda) \ge 0$ whenever $\lambda = e^{i\theta}$ with $\theta \in \mathbb{R}$ and $\theta \ne 0$. A straightforward computation gives

$$2Re W_j(\lambda) = \frac{2\cos\theta(\tau_j - 1) - 2\cos\theta\tau_j}{|\cos\theta - 1|^2} = -\frac{\sin\left(\frac{(2\tau_j+1)\theta}{2}\right)}{\sin\frac{\theta}{2}} \ge -(1 + 2\tau_j),$$

whence we get that condition (ii) holds if (and in fact, only if) $\alpha_j \ge 1 + 2\tau_j$. As a consequence of Lemma 4.1, we get the *Willems dissipativity* [13] for agents (1).

Corollary 4.2 *Let $\alpha_j \ge 1 + 2\tau_j$. Then a form $V_j(x) = x^* H_j x \ge 0$ ($x \in \mathbb{C}^{1+\tau_j}$) exists such that for solutions of* (1) *and $x_j(t) = (y_j(t), u_j(t-1), \ldots, u_j(t-\tau_j))$ one has*

$$V_j(x_j(t+1)) - V_j(x_j(t)) \le y_j(t)u_j(t) + \alpha_j |u_j(t)|^2 \quad \forall t = 0, 1, \ldots \tag{9}$$

Proof. The existence of $V_j(x)$ satisfying (9) is immediate from Lemma 4.1. To prove its positivity, consider a solution of (1), along which $u_j(t) = -\varepsilon y_j(t)$, where $\varepsilon \in (0;1)$ is so small that $-\varepsilon + \varepsilon^2 \alpha_j < 0$. Then $x_j(t) = (1-\varepsilon)^t x_j(0) \to 0$ as $t \to \infty$. Summing inequalities (9) up over all $t \geq 0$ yields that $\lim\limits_{t\to\infty} V_j(x_j(t)) - V_j(x_j(0)) = -V_j(x_j(0)) \leq 0$. Since $x_j(0)$ is arbitrary, then the form V_j is non-negative definite.□

Proof of Theorem 3.1. Let assumptions of this Theorem hold. Putting $\phi_{jk}(t) := y_k(t) - y_j(t)$, $\eta_{jk}(t) := a_{jk}(t)|\phi_{jk}(t)|^2$, equation (2) entails that

$$u_j(t) = \sum_{k=1}^{N} a_{jk}(t)\phi_{jk}(t),\ |u_j(t)|^2 = \left|\sum_{k=1}^{N} a_{jk}^{1/2}(t) a_{jk}^{1/2}(t)\phi_{jk}(t)\right|^2 \leq \bar{d}_j \sum_k \eta_{jk}(t) \quad (10)$$

along any solution of (1),(2). By definition of ϕ_{jk} the inequality holds

$$\phi_{jk}(t)(y_k(t) - y_j(t)) - \beta|\phi_{jk}(t)|^2 - (1-\beta)|\phi_{jk}(t)|^2 \geq 0, \quad (11)$$

here $1 > \beta > \max_j[\bar{d}_j(1+2\tau_j)]$. Multiplying (11) by $a_{jk}(t) \geq 0$, summing up over all j,k, noticing that $\phi_{jk} = -\phi_{kj}$ and using (10), one arrives at the inequality

$$-\sum_{j=1}^{N}\left[2y_j(t)u_j(t) + \beta(\bar{d}_j)^{-1}|u_j(t)|^2 + (1-\beta)\sum_{k=1}^{N}\eta_{jk}\right] \geq 0. \quad (12)$$

Taking $\alpha_j := \beta(\bar{d}_j)^{-1} > 1 + 2\tau_j$ and V_j from Corollary 4.2, (12) implies that

$$\sum_{j=1}^{N}\left[V_j(x_j(t)) - V_j(x_j(t+1)) - (1-\beta)\sum_{k=1}^{N}\eta_{jk}(t)\right] \geq 0 \quad \forall t \geq 0,$$

and hence we get that $\sum_{t=0}^{\infty}\eta_{jk}(t) < \infty$ for any j,k and thus $\eta_{jk}(t) \to 0$ and $u_j(t) \to 0$ as $t \to \infty$. To prove consensus, suppose on the contrary the existence of $j,k \in \underline{N}$, $\delta > 0$ and a sequence $t_n \to \infty$ with $|y_j(t_n) - y_k(t_n)|^2 > 2\delta$. For large n this implies that $|y_j(t) - y_k(t)|^2 > \delta$ whenever $t \in [t_n; t_n + T]$, where T is a period from Assumption 2.1. Thus $t \in [t_n; t_n + T]$ and j', k' exist such that $a_{j'k'}(t) \geq \varepsilon$ and $|y_{j'}(t) - y_{k'}(t)| > \delta/N$, and hence $\sum_{j,k=1}^{N}\sum_{t=t_n}^{t_n+T}\eta_{jk}(t) \geq \varepsilon\delta/N$ for large $t \geq 0$. One arrives at the contradiction since $\eta_{jk}(t) \to 0 \forall j,k \in \underline{N}$, which finishes the proof.□

5 Conclusion

We examine how robust is multi-agent consensus against heterogeneous "inner" delays in the agents. The agents obey the integrator model, network topology is undirected and satisfies the assumption of repeated connectivity. A criterion for consensus under sufficiently small delays is obtained, which may be used e.g. for analysis of microscopic traffic flow models. This criterion may be extended to nonlinear symmetric networks. Its extension on more general networks, including those with directed topology and higher-order agents is subject of ongoing research.

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