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The Yakubovich quadratic criterion, F-stability and multi-agent consensus. ⋆

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Abstract: Generalizing numerous results on absolute stability of nonlinear Lurie systems, Vladimir A. Yakubovich established a fundamental abstract criterion of absolute stability under uncertain nonlinearities. His Quadratic Criterion offers elegant conditions for the uniform output L^2-stability in a wide class of uncertain nonlinearities, obeying anytime ("local") or integral quadratic constraints; these conditions are based on either quadratic Lyapunov functions whose existence is proved via the KYP lemma, or the method of integral quadratic constraints dating back to the works by V.M. Popov. In the present paper, we extend the Yakubovich quadratic criterion, replacing the output L^2-stability by boundedness of some quadratic integral performance index, defined by a quadratic form F and referred to as F-stability. We demonstrate that our F-stability criteria enables one to obtain easily many stability criteria for the "critical" case, where the frequency-domain inequality is non-strict and usual L^2-stability of the solution cannot be proved, e.g., we derive the circle and Popov's stability criteria in this degenerate case. All of these criteria are especially effective for scalar nonlinearities, nevertheless, we demonstrate that our approach is also applicable to a class of large scale nonlinear multi-agent networks, and derive the "networked" circle criterion for such networks as a consequence of our results.

Keywords: Networks, synchronization, consensus, nonlinear systems, absolute stability.

1. INTRODUCTION

The theory of absolute stability, taking its origin from the pioneering works of A.I. Lur'e (Lur'e [1957]), is now a commonly used and efficient tool in nonlinear and robust control. Its main results and historical milestones may be found in recent surveys and monographs Gelig et al. [2004], Khalil [1996], Leonov et al. [1996], Yakubovich [2002], Liberzon [2006] and references therein.

Two basic methods of the absolute stability theory are the KYP lemma Gelig et al. [2004], Rantzer [1996], giving an efficient criterion for the existence of a quadratic Lyapunov function, and the method of integral quadratic constraints Megretski and Rantzer [1997], Yakubovich [2002], dating back the the Popov's method of "a priori integral indices" Popov [1973]. Using one of this approach, one is able to get a condition for L^2-stability of a nonlinear system, obtained via feedback superposition of some *known* linear block and generally *uncertain* nonlinearity, the only available information about which comes to a set of anytime ("local") or integral *quadratic constraints*. Elegant general criteria, ensuring the *absolute stability* under such constraints (which term emphasizes that stability is guaranteed for any uncertain nonlinearity and, moreover, a uniform bound for the solution's L^2-norm is available), has been recently elaborated in Yakubovich [2002], Megretski and Rantzer [1997], Yakubovich [2000]. The sufficient (and necessary under some assumptions) condition for stability employs the *frequency-domain inequality* for the linear part and some technical conditions of "minimal stability".

In practice, however, the L^2-stability of the full solution often cannot be established; in this case one requires criteria establishing some "partial stability". In the papers Yakubovich [1998, 2000] V.A. Yakubovich proposed a more general version of the absolute stability criterion (which he called a *Quadratic Criterion*) that guarantees L^2-stability of some output. In this paper, we propose more general concept of "partial stability", which we call $\mathcal{F}$-stability. This property requires ultimate boundedness of some quadratic integral index, which is defined by a Hermitian form $\mathcal{F}$, along the trajectory. We present a criterion for such a stability, extending the result from Yakubovich [1998], and show how our $\mathcal{F}$-stability criterion may be used for deriving conditions for stability in critical cases, where the frequency-domain inequalities are non-strict. A new application which also discussed below is *consensus* and stability of multi-agent networks, which has been uncovered by the classical absolute stability theory.

2. AN ENHANCED QUADRATIC CRITERION FOR ABSOLUTE STABILITY

In this section, we summarize the main ideas, lying in the heart of the quadratic criterion for the absolute stability from Yakubovich [1998], and give its extension, which will

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be used later to get criteria for stability and consensus. To simplify matters, we confine ourselves to systems of ordinary differential equations, whose processes are nonlinearly constrained.

Consider a class of nonlinear systems, obtained via feedback interconnection of a *linear part* as follows

$$\dot{x}(t) = Ax(t) + Bu(t) \in \mathbb{R}^n, y(t) = Cx(t) \in \mathbb{R}^k, \qquad (1)$$

and a nonlinear constraint of the form

$$z(\cdot) = [x(\cdot), u(\cdot)] \in \mathcal{N}. \qquad (2)$$

Here the matrices A, B, C are known, (A, B) is controllable and (A, C) is observable; $\mathcal{N} = \{z(\cdot)\}$ stands for some set of locally L^2-summable functions $z : [0; +\infty) \to \mathbb{R}^n \times \mathbb{R}^m$.

Basically, the class of nonlinearities $\mathcal{N}$ is defined by the relation $u(t) = \Phi[x(t)]$, where Φ is a nonlinear function that can be uncertain but belongs to some known class, being e.g. a sector-restricted or slope-restricted nonlinearity.

Definition 1. A solution $z(\cdot) = [x(\cdot), \xi(\cdot)]$ of (1) is referred to as a *process*. We say the process is *stable* (respectively, *output stable*) if $z(t)$ is defined for all $t \geq 0$ and $\int_0^\infty |z(t)|^2\, dt < \infty$ (respectively, $\int_0^\infty |y(t)|^2\, dt < \infty$).

The quadratic criterion obtained in Yakubovich [1998, 2000, 2002] offers and effective criterion for the *absolute* output L^2-stability, assuming that the class $\mathcal{N}$ is described by a set of integral quadratic constraints (IQC). This criterion guarantees that $\|y\|^2_{L^2[0;\infty]} \leq C(|x(0)|^2 + \Gamma)$ for any process belonging to $\mathcal{N}$, where C and Γ depend on the system of IQC but not on the process itself. A natural extension of the output L^2-stability, considered by V.A. Yakubovich, is boundedness of some general integral quadratic performance index, which we refer to as the *$\mathcal{F}$-stability*, where $\mathcal{F}$ is some Hermitian form. As will be shown subsequently, the criteria for $\mathcal{F}$-stability obtained in this Section not only extend the Yakubovich criteria of absolute stability, but also allow to derive easily many criteria of absolute stability in the "critical cases", where the frequency-domain inequality is non-strict.

We start with a definition of the $\mathcal{F}$-stability property.

Definition 2. Let $\mathcal{F}(z) = z^* P z$ be a Hermitian form (here $P = P^*$ and $z \in \mathbb{C}^{n+m}$). The process $z(\cdot)$ is *$\mathcal{F}$-stable* if $z(t)$ exist for all $t \geq 0$ and $I_{\mathcal{F}}[z(\cdot)] := \varlimsup_{T\to\infty} \int_0^T \mathcal{F}(z(t))dt < \infty$. The system (1),(2) is *absolutely $\mathcal{F}$-stable* if all its processes are $\mathcal{F}$-stable (and hence are infinitely prolongable).

In the applications discussed below, we are primarily interested in the $\mathcal{F}$-stability of the processes under assumption that $\mathcal{F}$ is non-negatively definite or at least $\mathcal{F}[z(t)] \geq 0$, so $I_{\mathcal{F}}[z(\cdot)] = \int_0^\infty \mathcal{F}[z(t)]dt$; the finiteness of this quadratic performance index may be considered as "partial stability". It is obvious that a stable process is always $\mathcal{F}$-stable; the inverse is also true only when $P > 0$. For a Hermitian form $F_y(x, u) = |Cx|^2$ the F_y-stability is nothing else than the output stability and $I_{F_y}[z(\cdot)] = \|y(\cdot)\|^2_{L^2}$. Absolute output stability in the sense of Yakubovich [1998] implies absolute F_y-stability (with some explicit bound on I_{F_y}).

Suppose that any process from $\mathcal{N}$ satisfies the following:

$$\int_0^T F_j[x(t), u(t)]\, dt + \gamma_j \geq 0 \quad \forall T \geq 0,\ j = 1, \ldots, M. \ (3)$$

Here constants γ_j may depend on the solution $[x(\cdot), u(\cdot)]$ and F_j are some Hermitian forms on $\mathbb{C}^{n+m}$.

Sufficient condition of the absolute $\mathcal{F}$-stability for the system (1),(22) is usually obtained via the trick called S-procedure (see e.g. Gelig et al. [2004]). For a given Hermitian form $\mathcal{F}$ and $\theta \in \mathbb{R}^M$, let $\tilde{\mathcal{F}}_\theta := \mathcal{F} + \sum_{j=1}^M \theta^j F_j$.

Lemma 3. Suppose the process obeying (22) is $\tilde{\mathcal{F}}_\theta$-stable for some non-negative vector θ. Then it is also $\mathcal{F}$-stable. If $\mathcal{F}[x(t), u(t)] \geq 0\, \forall t \geq 0$, the constraints (22) are sufficient to hold only for $T = T_n$, where $T_n \uparrow +\infty$ is some sequence.

Proof is obvious from the following inequality

$$\int_0^T \mathcal{F}[x(t), u(t)]\, dt \leq \int_0^T \tilde{\mathcal{F}}_\theta[x(t), u(t)]\, dt + \sum_{j=1}^M \theta_j \gamma_j, \ (4)$$

which implies that $I_{\mathcal{F}}[x, u] \leq I_{\tilde{\mathcal{F}}_\theta}[x, u] + \sum_j \theta_j \gamma_j$. In the case where $\mathcal{F}(x(t), u(t)) \geq 0$, one has $I_{\mathcal{F}}[x, u] = \int_0^\infty \mathcal{F}[x(t), u(t)]dt$ and the latter inequality is valid even if (4) holds only for $T = T_n$, where $T_n \uparrow +\infty$. □

Lemma 3 shows that if one is able to prove the $\mathcal{F}_\theta$-stability for *any* process, this automatically implies the $\mathcal{F}$-stability under constraints (22). So we now proceed with conditions of "global" $\mathcal{F}$-stability, sufficient conditions for which easily follow from teh KYP lemma and related theory of singular LQR problems.

Lemma 4. (Willems [1971]) Let $\mathcal{F}$ satisfy the *frequency-domain* inequality condition whenever $\tilde{u} \in \mathbb{C}^m$:

$$\mathcal{F}((\imath\omega I - A)^{-1}B\tilde{u}, \tilde{u}) \leq 0\ \forall \omega \in \mathbb{R} : \det(\imath\omega I - A) \neq 0. \ (5)$$

Then $V(x_0) := \sup\{\int_0^\infty \mathcal{F}[x(t), u(t)]dt : x(0) = x_0\} < \infty$ for any $x_0 \in \mathbb{R}^n$. Here the sup is taken over the set of all *stable* processes with $x(0) = x_0$ so the integral is well defined. Moreover, V is a quadratic form and

$$\dot{V}(x, u) + \mathcal{F}[x, u] = \frac{\partial V}{\partial x}(Ax + Bu) + \mathcal{F}(x, u) \leq 0. \quad (6)$$

As a corollary, we obtain the following $\mathcal{F}$-stability criteria for solutions of (1).

Lemma 5. Under condition (5) any process $[x(\cdot), u(\cdot)]$ defined for all $t \geq 0$ and having $x(\cdot)$ *bounded*, is $\mathcal{F}$-stable. The boundedness is not needed, assuming the following:

(a) for any $\varepsilon > 0, x_0 \in \mathbb{R}^n$ a *stable* process $[x(\cdot), u(\cdot)]$ exists with $x(0) = x_0$ and $\int_0^\infty \mathcal{F}[x(t), u(t)]\, dt > -\varepsilon$.

Proof. In accordance with Lemma 4, (5) implies that (7) holds; integrating this inequality one has

$$\int_0^T \mathcal{F}[x(t), u(t)]dt \leq V(x(0)) - V(x(T)) \quad \forall T. \quad (7)$$

If the process is bounded one has $\sup_{T>0} |V(x(T))| < \infty$, and thus it is $\mathcal{F}$-stable. Under supposition (a), one has $V(x_0) \geq 0$ for any x_0 and hence (7) implies the $\mathcal{F}$-stability without additional assumptions. □

Remark 6. Given a *stable* process, (5) implies, due to (7), that $\int_0^\infty F[x(\cdot), u(\cdot)]dt \leq V(x(0))$ since $x(t) \xrightarrow[t\to\infty]{} 0$.

It may be noticed that the "dissipation inequality" (7) with $V \geq 0$, which follows from (a) and (5), is nothing else than the *Willems dissipativity* with the storage function V

and supply rate $(-\mathcal{F})$ (Willems [1972]). While its validity for *some* quadratic form V is equivalent to (5) due to the KYP lemma, for the dissipativity no easily verifiable conditions are known except for special forms $\mathcal{F}$ (e.g. passivity conditions Khalil [1996], Fradkov [2003]).

It should be emphasized that Lemma 5 says nothing about solutions, defined on finite intervals. However, assuming that $\mathcal{F} \geq 0$ along such a process and condition (a) holds in some stronger version, it is possible to guarantee that the state vector $x(t)$ remains bounded (wherever defined).

Lemma 7. Let (5) holds and a process $[x(t), u(t)]$, defined for $t \in [0; T)$, satisfies an inequality $\mathcal{F}[x(t), u(t)] \geq 0\,\forall t \in [0; T)$. Suppose that also the following stronger modification of condition (a) is valid:

(a') for any $x_0 \neq 0$ a *stable* process $[x(\cdot), u(\cdot)]$ exists with $x(0) = x_0$ such that $\int_0^\infty \mathcal{F}[x(t), u(t)]\,dt > 0$.

Then a constant $c > 0$ exists (depending only on $A, B, \mathcal{F}$) such that $|x(t)| \leq c|x(0)|$ for any $t \in [0; T)$.

Proof. Condition (a') implies that $V(x_0) > 0$ for any $x_0 \neq 0$. Since $\mathcal{F}[x(t), u(t)] \geq 0$, (7) implies that $\lambda_1 |x(t)|^2 \leq V(x(t)) \leq V(x(0)) \leq \lambda_n |x(0)|^2$ for any $t \in [0; T)$ where λ_1, λ_N are the minimal and the maximal eigenvalues of V; hence $|x(t)| \leq c|x(0)|$ with $c = \sqrt{\lambda_N/\lambda_1}$.

We now return to the problem of absolute $\mathcal{F}$-stability of nonlinear uncertain systems (1),(2), assuming that nonlinearities from Φ obey the quadratic constraints (22). Combining Lemmas 3 and 5, one arrives at the following.

Theorem 8. Given a set of Hermitian forms $F, F_1, \ldots, F_M : \mathbb{C}^{n+m} \to \mathbb{R}$, let a non-negative vector $\theta \in \mathbb{R}^M$ exist such that $\mathcal{F} := F + \sum_j \theta_j F_j$ satisfies the frequency-domain inequality (5). Then any process $z(t) = [x(t), u(t)]$, defined for all $t \geq 0$, satisfying (22) and having *bounded* state $x(\cdot)$, is F-stable. Furthermore, if $F[z(t)] \geq 0\,\forall t \geq 0$, it is sufficient that (22) hold only for $T = T_1, T_2, \ldots, T_n, \ldots$, where $T_n \uparrow \infty$ is some sequence. The boundedness assumption may be discarded, provided that condition (a) holds.

Like many "abstract" criteria in the absolute stability theory, Theorem 8 establishes a *dichotomy* property: under the frequency-domain condition, any process obeying the quadratic constraint is either "stable" in some sense (in our case, F-stable) or *unbounded*. To proof the "stability" for unbounded processes as well, additional assumptions are needed, referred to as the *minimal stability* conditions Yakubovich [1998, 2000, 2002], Barabanov [2000]. Condition (a) is one of the possible conditions of this type; it holds for $\mathcal{F} := F + \sum_j \theta_j F_j$, for instance, if any of $F, F_1, \ldots, F_M$ satisfies (a). More convenient conditions were proposed in Yakubovich [1998, 2000, 2002] and are based on the idea of stable prolongations. Given $F, F_1, \ldots, F_M$, we introduce the following condition:

(b) for any process $z(t) = [x(t), u(t)]$, $t \geq 0$, satisfying (22), and any $T \geq 0$ there exists a *stable* process $z_T(t)$ such that $z_T(t) = z(t)$ for $t \in [0; T]$, the IQCs hold

$$\int_0^\infty F_j[z_T(t)]\,dt + \gamma_j \geq 0 \quad \forall T \geq 0\,\forall j, \tag{8}$$

and, finally, $J := \inf_{T>0} \int_T^\infty F[z(t)]\,dt > -\infty$.

Theorem 9. Let Assumptions of Theorem 8 and (b) hold. Then any process $z(t) = [x(t), u(t)]$, defined for $t \geq 0$ and satisfying (22), is F-stable with $I_F[z(\cdot)] \leq c|x(0)|^2 + \sum_j \theta_j \gamma_j - J$, where c depends on A, B, F, F_j, θ_j. If $F[z(t)] \geq 0\,\forall t \geq 0$, the validity of (22) and condition (b) for only $T = T_n$, where $T_n \uparrow \infty$, is sufficient.

Proof Fixing $T > 0$, condition (b) and Remark 6 entail

$$\int_0^T F[z(t)]dt = \int_0^\infty F[z_T(t)]dt - J \leq \int_0^\infty \mathcal{F}[z_T(t)]dt + \\ + \sum_j \theta_j \gamma_j - J \leq V(x(0)) + \sum_j \theta_j \gamma_j - J. \tag{9}$$

Here V is the quadratic form from Lemma 4. Passing to the limit as $T \to \infty$, one obtains that $I_F[z(\cdot)] \leq V(x(0)) + \sum_j \theta_j \gamma_j - J$. which imply the F-stability of the process $z(\cdot)$. The second claim is proved similarly, replacing $T > 0$ with $T = T_n$ and passing to the limit as $n \to \infty$.□

In the case where $F[x, u] = F_y[x, u] := |Cx|^2$ the last assumption in (b) is obvious (one may take J=0) and condition (b) coincides with the minimal stability assumption from Yakubovich [1998]. Hence, our result extends the stability criterion from Yakubovich [1998] to the case of general F-stablity under quadratic constraints (22).

3. EXAMPLES

In this section we illustrate the potential of general criteria, obtained in Section 2, by applying them to special classes of quadratic constraints. To simplify matters, we assume hereinafter that nonlinear uncertain constraint (2) corresponds to the output-input relation

$$u(t) = \varphi(t, y(t)), \quad \varphi(\cdot) \in \Phi. \tag{10}$$

Here $\Phi = \{\varphi(\cdot)\}$ stands for some class of maps, assumed to satisfy Caratheodory conditions, i.e. $\varphi(t, \cdot)$ is continuous for almost all $t \geq 0$, $\varphi(\cdot, x)$ is measurable for any x, and $\max_{|x| \leq \delta} |\varphi(t, x)|$ is locally summable for any $\delta > 0$. We also assume that $\varphi(t, 0) \equiv 0\,\forall \varphi \in \Phi$. Then for any initial conditions $x(0) = x_0$ a solution of (1),(2) is either infinitely prolongable or grows unbounded in finite time.

We say the system (1),(10) is *minimally stable* if for any initial condition $x(0) = x_0$ there exists at least one map $\varphi_0 \in \Phi$ such that the process $[x(t), u(t)]$ with $x(0) = x_0$ and $u(t) = \varphi_0(t, Cx(t))$ is stable. We call the system *strictly minimally stable* if $x_0 \neq 0$ implies that $F[y(t), \varphi_0(t, y(t))] > 0$ for some $t \geq 0$ (due to observability, it is sufficient that $F[y(t), \varphi_0(t, y(t))] \equiv 0 \Longrightarrow y(t) \equiv 0$).

3.1 Stability under local constraints and the circle criterion

The simplest type of quadratic constraint is anytime or *local* constraint Gelig et al. [2004], Yakubovich [2002]

$$F(y(t), u(t)) \geq 0, \tag{11}$$

where F is a Hermitian form on $\mathbb{C}^k \times \mathbb{C}^m$. A typical example is the sector constraint: $\alpha y(t)^2 \leq u(t)y(t) \leq \beta y(t)^2$ or $(u(t) - \alpha y(t))(\beta y(t) - u(t)) \geq 0$ (here $y, u \in \mathbb{R}$).

Theorem 10. Let any nonlinearity $\varphi \in \Phi$ satisfies constraint (11) and the Hermitian form $\mathcal{F}(x, u) = F(Cx, u)$

satisfy (5). Then any process $[x(t), u(t)]$ of the system (1),(10), where $t \in [0; \infty)$ and $x(\cdot)$ is bounded, is $\mathcal{F}$-stable:

$$I_F := \int_0^\infty F[y(t), u(t)]dt < \infty, \quad y(t) = Cx(t). \qquad (12)$$

If the system (1),(10) is minimally stable, (12) holds for *any* infinitely prolongable process and $I_F \leq V(x(0))$ where V is defined in Lemma 4. If it is *strictly* minimally stable, any process is infinitely prolongable with bounded state.

Proof is obvious from Lemmas 5 and 7 since minimal stability implies condition (a), whereas strict minimal condition is sufficient for (a'). □

Corollary 11. Under assumptions of Theorem 10, let all mappings $\varphi \in \Phi$ satisfy the *strict* quadratic constraint:

$$\inf_{t\geq 0} \inf_{c_1 \leq |y| \leq c_2} F(y, \varphi(t, y)) > 0 \quad \forall c_2 > c_1 > 0, \qquad (13)$$

and, furthermore, $\sup_{t\geq 0} \sup_{|y|\leq c} |\varphi(t, y)| < \infty \, \forall c > 0$. Assume that the system (1),(10) is minimally stable. Then any process is infinitely prolongable, has bounded state, and is output stable in the sense that $y(t) \to 0$ as $t \to \infty$.

Proof. Indeed, minimal stability and (11) imply *strict* minimal stability, which implies that any process is bounded and prolongable up to ∞. To prove that $y(t) \to 0$, assume on the contrary that $|y(t_n)| \geq 2\delta > 0$ along some sequence $t_n \uparrow \infty$; passing to its subsequence, one may assume without loss of generality that $t_{n+1} - t_n > 1$. Due to (1), $\dot{y}(t)$ is bounded and hence for $\varepsilon > 0$ sufficiently small one has $|y(t_n)| \geq \delta > 0$ whenever $t \in \Delta_n := (t_n - \varepsilon; t_n + \varepsilon)$. This obviously contradicts to (12) due to (13): there exists some $\mu > 0$ such that $F(y(t), u(t)) > \mu$ for $t \in \Delta_n$. □

Theorem 10 and Corollary 11 may be further refined in the case, where Φ satisfies a system of local constraints:

$$F_1(y(t), u(t)) \geq 0, \ldots, F_M(y(t), u(t)) \geq 0.$$

Choosing a non-negative vector $\theta \in \mathbb{R}^M$, the claims of Theorem 10 and Corollary 11 remain valid, taking $F(y, u) = \sum_j \theta_j F_j(y, u)$. By playing with a vector θ, one often may provide the frequency-domain condition (5) for the Hermitian form F, although it fails to be valid for F_j.

As a special case of Corollary 11, we derive the circle criterion for scalar nonlinearities. Assume that $y(t), u(t) \in \mathbb{R}$. For $-\infty \leq \alpha < \beta \leq \infty$ we introduce the class $S[\alpha; \beta]$ of nonlinear maps $\varphi : [0; \infty) \times \mathbb{R} \to \mathbb{R}$ such that

$$\alpha < \inf_{t\geq 0} \inf_{c_1 \leq |y| \leq c_2} \frac{\varphi(t, y)}{y} < \sup_{t\geq 0} \sup_{c_1 \leq |y| \leq c_2} \frac{\varphi(t, y)}{y} < \beta \quad (14)$$

$\varphi(t, 0) \equiv 0$ and $\sup_{t\geq 0} \sup_{|y|\leq c} |\varphi(t, y)| < \infty$ for any $c, c_1, c_2 > 0$.

As usual, $\varphi(\cdot)$ is assumed to be a Caratheodory map.

Let $W_y(\lambda) = C(\lambda I - A)^{-1}B$ be the transfer function of the linear block (1). Define the set $D[\alpha; \beta] \subset \mathbb{C}$ as the set of such $\lambda \in \mathbb{C}$, that $Re\, \rho_{\alpha,\beta}(\lambda) \geq 0$, where

$$\rho_{\alpha,\beta}(\lambda) = \begin{cases} (1 - \alpha\lambda)^*(1 - \beta\lambda), & \alpha, \beta \in \mathbb{R}; \\ -(1 - \alpha\lambda)^*\lambda, & \alpha \in \mathbb{R}, \beta = +\infty; \\ (1 - \beta\lambda)^*\lambda, & \alpha = -\infty, \beta \in \mathbb{R}. \end{cases} \qquad (15)$$

For $\alpha, \beta \neq 0$ and $\alpha\beta < 0$ (respectively, $\alpha\beta > 0$) this set is the closed disc, based on the line segment $[\alpha^{-1}, \beta^{-1}]$ as a diameter (or, respectively, the closure of its exterior). When $\alpha = 0$ or $\beta = 0$, $D[\alpha; \beta]$ reduces to a half-plane.

The following theorem is the celebrated circle criterion for the case of *non-strict* frequency-domain inequality ("critical case"), which guarantees stability for any nonlinearity, whose graph lies *strictly* inside the sector in the sense of (14). Traditionally, the circle criterion is formulated for the *strict* frequency-domain inequalities Gelig et al. [2004], Yakubovich [2000], Khalil [1996], in this case it is applicable to non-strict sector condition, and, moreover, guarantees *exponential* stability.

Theorem 12. Suppose that the feedback $u = \varkappa y$ stabilizes the plant (1) for some $\varkappa \in (\alpha; \beta)$ (hence, $A - \varkappa BC$ is a Hurwitz matrix) and the Nyquist curve $\{W_y(\imath\omega)\}$ is contained by the set $D[\alpha; \beta]$, i.e. $Re\, \rho_{\alpha;\beta}(W_y(\imath\omega)) \geq 0$. Under nonlinear feedback $u(t) = \varphi(t, y(t))$, $\varphi \in S[\alpha; \beta]$, the solutions of the closed-loop system are bounded, prolongable up to ∞ and output stable: $y(t) \to 0$ as $t \to \infty$.

Proof. Let $F_{\alpha,\beta}(\tilde{y}, \tilde{u}) = -Re[(\tilde{u} - \alpha\tilde{y})^*(\tilde{u} - \beta\tilde{y})]$ for $\alpha, \beta \in \mathbb{R}$, $F_{\alpha,\beta}(\tilde{y}, \tilde{u}) = Re[(\tilde{u} - \alpha\tilde{y})^*\tilde{y}]$ if $\beta = +\infty$ and $F_{\alpha,\beta}(\tilde{y}, \tilde{u}) = -Re[\tilde{y}^*(\tilde{u} - \beta\tilde{y})]$ when $\alpha = -\infty$. A straightforward computation shows that (14) implies the strict local constraint (13) with $F = F_{\alpha;\beta}$. Our claim is now immediate from Corollary 11 since the frequency-domain condition (5) boils down to $W_y(\imath\omega) \in D[\alpha; \beta]$. □

3.2 The Popov criterion for critical case

Whereas the circle criterion gives simple and convenient condition for stability under sector constraints, it is known to be overly conservative in the case of stationary sector nonlinearities. This motivated further refinements of this criterion, among which the Popov stability criterion (Popov [1973], Gelig et al. [2004], Khalil [1996]) is the simplest. In its classical formulation, the Popov criterion addresses nonlinear system with *stable* linear part (the matrix A in (1) is Hurwitz) and scalar *static* and continuous sectorial nonlinearity in the sector $0 \leq \varphi(y)y \leq \beta y^2$. Under these conditions, the Popov criterion ensures stability provided that for some $\theta \in \mathbb{R}$ one has

$$-Re[W(\imath\omega)(1 + \theta\imath\omega)] + \beta^{-1} > 0 \quad \forall \omega \in \mathbb{R}, \qquad (16)$$

which inequality reduces to the circle criterion (with strict inequalities) in the special case $\theta = 0$. Many applications, however, lead to the "critical case" where the matrix A is not Hurwitz but has purely imaginary eigenvalues, in which case the sector should be "half-open" $0 < \varphi(y)y \leq \beta y^2$ (Gelig et al. [2004],Yakubovich [1964]), however, the frequency-domain inequality (16) remains strict (with some modifications, caused by imaginary poles). Below we derive the version of the Popov criterion with non-strict inequalities, dealing with nonlinearities in an open sector.

Given $-\infty \leq \alpha < \beta \leq \infty$, let $S_0[\alpha; \beta] \subset S[\alpha; \beta]$ stands for the class of continuous maps $y \mapsto \varphi(y)$, such that

$$\alpha < \frac{\varphi(y)}{y} < \beta \quad \forall y \neq 0, \quad \varphi(0) = 0, \qquad (17)$$

and, additionally $\psi_1(y) := \int_0^y (\varphi(\sigma) - \alpha\sigma)d\sigma \geq 0$, $\psi_2(y) := \int_0^y (\beta\sigma - \varphi(\sigma))d\sigma \geq 0$ satisfy the condition $\psi_1(y), \psi_2(y) \to +\infty$ as $|y| \to \infty$. Notice that if $\alpha = -\infty$ (respectively, $\beta = +\infty$), one has $\psi_1 \equiv \infty$ (respectively, $\psi_2 \equiv \infty$).

Theorem 13. Suppose that the feedback $u = \varkappa y$ stabilizes the plant (1) for *any* $\varkappa \in (\alpha; \beta)$. Assume there exists $\theta \in \mathbb{R}$ such that the following frequency-domain inequality holds:

$$Re[\rho_{\alpha;\beta}(W_y(\imath\omega)) + \theta\imath\omega W_y(\imath\omega)] \geq 0, \tag{18}$$

and, additionally, $\theta \leq 0$ if $\alpha = -\infty$ and $\theta \geq 0$ if $\beta = \infty$. Then any nonlinear feedback $u(t) = \varphi(y(t))$ asymptotically stabilizes the system: $x(t) \to 0$ as $t \to \infty$.

Proof. In the case where $\theta = 0$ the claim follows from the circle criterion (Theorem 12), so we may assume that $\theta \neq 0$. Let $F = F_{\alpha,\beta}$ be defined as in the proof of Theorem 12. Suppose that $\theta > 0$ and hence $\alpha > -\infty$; define $F_1(\tilde{x}, \tilde{u}) = Re[(\tilde{u} - \alpha\tilde{y})^*\dot{\tilde{y}}]$, here, by definition, $\tilde{y} = C\tilde{x}$ and $\dot{\tilde{y}} = C(A\tilde{x} + B\tilde{u})$. Given a solution of (1), one has[1] $F_1[x(t), u(t)] = (u(t) - \alpha y(t))\dot{y}(t) = \frac{d}{dt}\psi_1(y(t))$. This implies the integral quadratic constraint as follows:

$$\int_0^T F_1[x(t), u(t)]dt = \psi_1(y(T)) - \psi_1(y(0)), \tag{19}$$

which is a special case of (22) with $M = 1$.

It is sufficient to prove that any process of the closed-loop system is F-stable and bounded. If this is the case, one can obviously retrace arguments from Theorem 12. To do thus, we use Lemma 5 and Theorem 8.

It is easily noticed that (18) is equivalent to the frequency-domain condition for the Hermitian form $\mathcal{F} = F + \theta F_1$, that also satisfies condition (a). Indeed, for any initial condition $x(0) = x_0$ consider a *stable* process $[x^\varkappa, u^\varkappa]$, corresponding to the feedback $u = \varkappa y$. For this linear feedback the correspondent map ψ_1 is $\psi_1^\varkappa(y) = (\varkappa - \alpha)y^2/2$ and hence $\psi_1^\varkappa(x_0) \to 0$ as $\varkappa \to \alpha$, hence $\int_0^\infty \mathcal{F}[x^\varkappa(t), u^\varkappa(t)]dt \geq -\psi_1^\varkappa(x_0) \to 0$ as $\varkappa \to \alpha + 0$. As was discussed in the proof of Lemma 5, (7) holds with $V \geq 0$ and hence $\int_0^T \mathcal{F}[x(t), u(t)]dt \leq V(x(0))$ for any T such that the process $x(t), u(t)$ exist on $t \in [0; T)$. Applying this to processes of the closed-loop system, one has $F[y(t), u(t)] \geq 0$, which implies thanks to (19) that $\theta\psi_1(y(T)) \leq V(x(0)) + \theta\psi_1(y(0))$. Therefore, $y(t)$ and $u(t)$ remain bounded, which proves that the process is prolongable up to ∞ and bounded (due to the observability). Applying Theorem 8, one shows that it is F-stable.

The case where $\theta < 0$ and hence $\beta < \infty$ is proved in the same way, taking $F_1(\tilde{x}, \tilde{u}) = Re[(\beta\tilde{y} - \tilde{u})^*\dot{\tilde{y}}]$. □

4. MULTI-AGENT CONSENSUS AND INCREMENTAL $\mathcal{F}$-STABILITY

In this section we demonstrate how our criteria may be applied to examining of synchronization in multi-agent diffusively coupled networks, consisting of identical agents that are coupled via some distributed protocols.

Consider now an ensemble of $N > 1$ identical systems (1), indexed 1 through N and standing for independent *agents*:

$$\dot{x}_j(t) = Ax_j(t) + Bu_j(t) \in \mathbb{R}^n, y_j(t) = Cx_j(t) \in \mathbb{R}^k. \tag{20}$$

Here $j = 1, 2, \ldots, N$, $x_j \in \mathbb{R}^n, u_j \in \mathbb{R}^m, y_j \in \mathbb{R}^k$ stand, respectively, for the state, control, and output of the jth agent. The agents are coupled via some protocol, generally nonlinear and time-varying, which may e.g. be as follows:

$$u_j(t) = U_j(t, y_1(t) - y_j(t), \ldots, y_N(t) - y_j(t)),$$

where $U_j(t, \xi_1, \xi_2, \ldots, \xi_N)$ state for some maps, describing couplings between the agents. Many practical examples of such systems may be found in recent monographs Ren and Beard [2008], Ren and Cao [2011], Mesbahi and Egerstedt [2010] and references therein. Although most results, concerning multi-agent networks, assume that the maps U_j are *linear*, many applications in fact require nonlinear couplings, as discussed in Ren and Beard [2008], Ren and Cao [2011], Proskurnikov and Matveev [2015].

A natural question arises how effective are absolute stability criteria for the large-scale multi-agent networks with nonlinear coupling maps U_j. For the first glance, the absolute stability approach is not very promising because the nonlinearity's dimension depends on N, and thus the corresponding frequency-domain conditions (neither LMI solvability) cannot be verified in practice. Nevertheless for some important classes of networks one can derive quadratic constraints of special structure which we call *incremental*. Although such a constraint involves the solutions of N systems (20), the verification of the frequency-domain condition reduces to the case of single agent.

Given a sequence of vectors $\xi_1, \ldots, \xi_N$, we denote their stack column vector with $\bar{\xi} = (\xi_1^\top, \ldots, \xi_N^\top)^\top$. Given a Hermitian form $\mathcal{F} : \mathbb{C}^{n+m} \to \mathbb{R}$ and the vectors $x_j \in \mathbb{R}^n$, $u_j \in \mathbb{R}^m$, $j \in 1:N$, let $\bar{\mathcal{F}}(\bar{x}, \bar{u}) := \sum_{i,j=1}^{N} \mathcal{F}(x_j - x_i, u_j - u_i)$.

The team of agents (20) may be considered as one block

$$\dot{\bar{x}}(t) = (I_N \otimes A)\bar{x}(t) + (I_N \otimes B)\bar{u}(t), \bar{y}(t) = (I_N \otimes C)\bar{x}(t). \tag{21}$$

We say the process $[\bar{x}(t), \bar{u}(t)]$ is *incrementally $\mathcal{F}$-stable* if it is $\bar{\mathcal{F}}$-stable and *incrementally stable* if $|x_j - x_k| + |u_j - u_k| \in L^2[0; \infty]$ for any j, k. It may be noticed that the incremental stability implies that $x_i(t) - x_j(t) \to 0$ as $t \to \infty$ for any i, j, which relation is referred to as the *consensus* or *synchronization* among the agents. The problem of consensus reaching is a central for the multi-agent control, since many other types of cooperative behavior may be reduced to synchronization of the agents state vectors or their outputs Ren and Beard [2008], Olfati-Saber et al. [2007]. The incremental $\mathcal{F}$-stability of the solution may be considered as a "consensus-like" behavior.

Retracing the theory, presented in Section 2, one may establish conditions for the incremental $\mathcal{F}$-stability of all the processes $[\bar{x}(t), \bar{u}(t)]$ or those satisfying *incremental quadratic constraints*

$$\int_0^T \bar{F}_l[\bar{x}(t), \bar{u}(t)]\, dt + \gamma_l \geq 0 \quad \forall T \geq 0,\ l = 1, \ldots, M. \tag{22}$$

It should be noticed that the frequency-domain inequality (5) for the form $\bar{\mathcal{F}}$ and the linear block (21) is equivalent to the original (5), moreover, if $V(x_0) = \sup \int_0^\infty \mathcal{F}(x, u)dt$ is a function from Lemma 4; the corresponding supremum for $\bar{\mathcal{F}}$ is nothing else than $\bar{V}(\bar{x}_0)$. Hence analogues of Lemmas 5 and 7 and of the subsequent results ensure the *incremental* $\mathcal{F}$-stability under the frequency-domain condition, whose complexity is independent of N!

4.1 The circle criterion for undirected multi-agent networks

In this subsection we discuss F-stability criteria for multi-agent networks, deriving a "networked" circle criterion.

[1] such a relation is often referred to as the *differential constraint* Barabanov [1982]

Consider the team of agents (20) (where $\dim y_j = \dim u_j = 1$), coupled via the following control protocol

$$u_i(t) = \sum_{j=1}^{N} \gamma_{ij}\varphi_{ij}(t, y_j(t) - y_i(t)). \tag{23}$$

Here the matrix $\Gamma = (\gamma_{ij})$ describes the (weighted) interaction graph; we always assume that $\gamma_{ij} \geq 0$ and $\gamma_{ii} = 0$. The maps $\varphi_{ij} : \mathbb{R} \to \mathbb{R}$ are called *couplings* and describe how the agents interact. Let $d_i := \sum_j \gamma_{ij}$, $D = diag(d_1, \ldots, d_N)$ and $L := D - \Gamma$. We confine ourselves to the case of *undirected* topology, i.e. $\Gamma = \Gamma^\top$ and hence $L = L^\top$. Due to the Gershgorin disc theorem Ren and Beard [2008], Mesbahi and Egerstedt [2010] we have $L \geq 0$. The second eigenvalue in the ascending order $0 = \lambda_1(L) \leq \lambda_2(L) \leq \ldots$ is called the *algebraic connectivity* Fiedler [1973] of the network, $\lambda_2(L) > 0$ if and only if the L corresponds to the connected graph.

Lemma 14. Suppose the couplings are *anti-symmetric* $\varphi_{ij}(t, -y) = -\varphi_{ji}(t, y)$ and belong to $S[\alpha; \beta]$, where $0 \leq \alpha < \beta \leq \infty$. Define a Hermitian form $F(y, u)$ as follows

$$F(\tilde{y}, \tilde{u}) = -\,Re\,\tilde{y}^*\tilde{u} - \delta|\tilde{y}|^2 - \upsilon|\tilde{u}|^2,\ \tilde{y}, \tilde{u} \in \mathbb{C},$$

where the parameters $\delta, \upsilon \geq 0$ are defined by

$$\delta := \frac{\alpha\beta^{-1}\lambda_2(\Gamma)}{1 + \alpha\beta^{-1}}, \quad \upsilon := \frac{1}{2(\alpha + \beta)\max_j D_j}.$$

Then the solutions of system (20),(23) satisfy an *incremental* quadratic constraint $\bar{F}[\bar{y}(t), \bar{u}(t)] \geq 0$. Moreover, if $\lambda_2(L) > 0$, then this constraint is strict in the following sense: for any $c_1, c_2 > 0$ there exist $\varepsilon > 0$ so that $\bar{F}[\bar{y}(t), \bar{u}(t)] \geq \varepsilon$ whenever $c_1 \leq \max_{i,j} |y_j(t) - y_i(t)| \leq c_2$.

The proof of Lemma 14 may be found in Proskurnikov [2013, 2014] (in the case where $\gamma_{ij} \in \{0; 1\}$) and also in Proskurnikov and Matveev [2015] and is omitted here due to space limitations. This lemma may be easily extended to time-varying topology (Proskurnikov and Matveev [2015]).

An incremental analogue of the *minimal stability* (see Subsect. 3.1) is guaranteed by the following assumption.

Assumption 15. The protocol (23) establishes consensus if $\varphi_{ij}(t, y) = \mu y$ with some $\mu \in (\alpha; \beta)$. Equivalently Olfati-Saber et al. [2007], for $\lambda \neq 0$ being an eigenvalue of $L(\Gamma)$, the matrices $A - \mu\lambda BC$ are Hurwitz.

Retracing the proof of circle criterion in Subsect. 3.1, one obtains the following "multi-agent" circle criterion.

Theorem 16. Suppose that $\lambda_2(L) > 0$, Assumption 15 and suppositions of Lemma 14 be valid, and the condition holds

$$Re\,W_y(\imath\omega) + \delta|W_y(\imath\omega)|^2 + \upsilon \geq 0 \quad \forall \omega \in \mathbb{R}.$$

Then the protocol (23) establishes *output consensus*: any solution is infinitely prolongable with $y_j(t) - y_i(t) \xrightarrow[t\to\infty]{} 0$.

By introducing "differential" constraints like in the proof of Popov's criterion, one may also get its multi-agent analogue Proskurnikov and Matveev [2015].

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