# Popov-Type Criterion for Consensus in Nonlinearly Coupled Networks

**Article** *in* IEEE Transactions on Cybernetics · August 2015

DOI: 10.1109/TCYB.2014.2354519 · Source: PubMed



Anton V Proskurnikov
University of Groningen

# Popov-Type Criterion for Consensus in Nonlinearly Coupled Networks

Anton Proskurnikov, *Member, IEEE,* and Alexey Matveev

***Abstract*—This paper addresses consensus problems in nonlinearly coupled networks of dynamic agents described by a common and arbitrary linear model. Interagent interaction rules are uncertain but satisfy the standard sector condition with known sector bounds; both the agent's model and interaction topology are time-invariant. A novel frequency-domain criterion for consensus is offered that is similar to and extends the classical Popov's absolute stability criterion.**

***Index Terms*—Complex networks, consensus, multiagent systems, nonlinear network analysis, synchronization.**

## I. Introduction

THE captivating phenomenon of coordination of events in a multiagent system, when very local interactions between its components ultimately enforce the entire complex and large network to operate in unison, has attracted a long standing interest from various research communities. This does not come as a surprise since such coordination lies in the heart of many natural phenomena, ranging from physics to biology and social sciences, as well as engineering designs of networked systems. Examples include, but are not limited to, surprisingly regular and "smart," behaviors of large groups of animals [1], self-synchronization of coupled oscillators [2], coordination in multiagent robotics [3], sensor [4], and neural [5] networks.

Being mainly motivated by many emerging networking applications, the recent burst of research activity in this area is featured by emphasis on effects from the interaction topology and focus on rigorous mathematical study. The backbone of the employed analytic armamentarium is constituted by tools from algebraic graph theory, averaging dynamics of Markov chains, matrix theory, and control theory. The consensus problem is one of the central topics: it is required to disclose conditions under which multiple agents asymptotically agree upon a common value via local interactions. This is exemplified by alignment of the headings of mobile vehicles, averaging of local observations in sensor networks, synchronization, e.g., adjustment of rhythms (phase locking and frequency entrainment) in populations of coupled oscillators, etc. More examples, along with extended surveys of the recent progress in the area and its motivation, can be found in [6]–[9]. A problem of synchronization in a complex network [10]–[13] may also be considered as a multiagent consensus problem, where the "agents" correspond to the nodes of the network.

Consensus via linear couplings is now well-studied [7]–[9], [11], [14], [15], up to availability of constructive and exhaustive, i.e., necessary and sufficient, criteria in some cases, especially for linear agents. Lyapunov techniques and facts on convergence of infinite matrix products prevail in analysis of first order [7], [16] and some second-order protocols [17]. Some classes of neutrally stable high-order agents may be treated by reduction to the first-order case via internal model observers [18], [19]. Another approach is passivity-based analysis [20], [21], which is suitable not only for linear but also for nonlinear passive or semi-passive agents. Consensus criteria for general linearly coupled networks are mainly known for the case of time-invariant agents and topology, stemming from stability theory for LTI systems. An effective tool for time-domain analysis of such networks is the Laplacian matrix decomposition [7], [9], [11], [22], which reduces consensus to simultaneous stabilization of several systems. Frequency-domain approach, well-presented in the recent monograph [14], typically relies on a closed-form expression of the transfer matrix of the overall networked system in terms of transfer functions of the agents and couplings, along with the network adjacency matrix. The outcome of this approach is typically similar in spirit to the Nyquist stability test.

Nonlinearly coupled networks combine the challenge of large scale with that of intricate nonlinear dynamics, and are far less comprehensively studied, even for linear agents. Yet, they are of essential interest in many areas. Classic examples are given by pulse or sinusoidally coupled oscillator networks and smart grids [2], neuronal coupling [23], and cellular neural networks [24]. Nonlinearity may be also injected into otherwise linear couplings by a limited sensing or communication range [25]–[27], quantization or other distortion of communicated data, and saturation of data-based actuation.

Manuscript received March 2, 2014; revised June 22, 2014 and August 23, 2014; accepted August 24, 2014. This work was supported in part by the European Research Council under Grant ERCStG- 307207 and in part by RFBR under Grant 12-01-00808, Grant 13-08-01014, and Grant 14-08-01015. Sections VI-A and VI-B are supported only by Russian Scientific Foundation (RSF) under Grant 14-21-00041. Section VI-C is supported only by RSF under Grant 14-29-00142. This paper was recommended by Associate Editor J. Cao.

A. Proskurnikov is with the Institute of Technology and Management, University of Groningen, Groningen 9747AG, The Netherlands, and also with the Faculty of Mathematics and Mechanics, Saint Petersburg State University and Institute for Problems in Mechanical Engineering RAS, Saint Petersburg 198504, Russia (e-mail: avp1982@gmail.com).

A. Matveev is with the Faculty of Mathematics and Mechanics, Saint Petersburg State University, Saint Petersburg 198504, Russia (e-mail: almat1712@yahoo.com).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TCYB.2014.2354519

Up to now, consensus in nonlinearly coupled networks was mainly studied for only special or low-order agents. The influential paper [16] dealt with first-order discrete-time integrators and set forth the fruitful idea to base Lyapunov analysis on the shrinking property of the convex hull of the agents driven by a consensus protocol. This idea was further developed in [26] and [27], [28] for first-order agents with respect to the rendezvous problem and continuous-time networks, respectively, and was extended in [29] in the form of paracontraction analysis. In [30], a more general class of protocols was examined. In general, they are not shrinking and may fail to establish consensus, yet render the agents equidistant from a prespecified set. A Lyapunov technique confined to passive and passively coupled agents and close in spirit to that from [20] was offered in [21]. By and large, passivity protractedly determined the boundary outside which no general method of global consensus analysis has been elaborated for nonlinearly coupled higher order agents. Exceptions were given by the results of [7], [31], and [32], which however, dealt with only second-order agents and specific nonlinearities, e.g., hyperbolic tangents. A linearization-based criterion for local synchronization was established in [33] via the master-stability function approach. Reference [12] presents extensions of these techniques that allow to prove synchronization of periodic orbits under time-varying topologies.

This contrast between the panoramas of "linear," and "nonlinear," realms is expectedly inherited from the general theory of stability: whereas for linear systems constructive and exhaustive criteria, like Nyquist or Routh–Hurwitz methods, are known, no such results are available for nonlinear plants. For them, Lyapunov's second method is ubiquitous in stability analysis. However, its effect is impeded by the lack of systematic procedures to construct proper Lyapunov functions. Somewhat general methods were offered only for particular classes of nonlinear systems and kinds of stability. An effective sample is given by the absolute stability theory (AST) [34], [35], with its two keystone results: the circle and Popov's criteria. AST combines a certain exhaustiveness [36], [37], systematic techniques, strong touch with the important issue of robustness, and effective final results. Networks were basically outside its scope; a recent extension [38] was confined to linearly coupled networks and analysis of traditional robust input-output stability. To fully fit to networked scenarios, AST techniques still await serious development, especially with respect to the role of the network topology, and adjustment to analysis of unusual for AST phenomena, like consensus.

In the part concerned with the circle criterion, initial advancements to this end were undertaken in [39] and [40]. Specifically, a general method for analysis of consensus among nonlinearly coupled linear high-order agents was elaborated that extends far outside the above "passivity," boundary. Prerequisites for its applicability are sector or similar constraints on possibly uncertain nonlinear couplings, like in the circle criterion, and their antisymmetry in the spirit of the Newton's Third Law. Like in the case of linearly coupled networks [6], [22], a quantitative account for the network topology given by the algebraic connectivity of the interaction graph plays the central role in the convergence properties of the protocol. The results of [39] and [40] cover the case of networks with switching topology.

In this paper, a similar program is carried out with respect to Popov's criterion [34], [35], [41]. It complements the classic circle criterion and reduces its conservatism in the important case of time-invariant nonlinearities. Correspondingly, this paper reduces the conservatism in the results from [39] and [40] in the case of time-invariant couplings and interaction topology, proposing a novel Popov's type criterion for consensus among nonlinearly coupled linear high-order agents, with the same prerequisites as above. The benefits from this criterion are demonstrated for second-order agents by significantly improving both the results from [39] and [40], which stem from the circle criterion, and the well-known Ren's criterion [31] for consensus under hyperbolic tangent couplings.

Like the classical Popov criterion, the offered consensus condition is in frequency domain. Unlike the frequency-domain criteria from [14], it deals with nonlinearly coupled networks and cannot be derived from the results of [14], which are essentially underlain by the total linearity of the network. Moreover, we show that our criterion cannot be directly inferred from the classical Popov criterion or its multivariable extensions, except for the case of only two agents.

The rest of the paper is organized as follows. Section II is devoted to preliminaries. Sections III and IV introduce the problem setup and main assumptions, respectively. Section V offers the main result, whose applications to second-order agents and proof are given in Sections VI and VII, respectively. Section VIII discusses relations of the main result of this paper with the classic AST. Section IX offers brief conclusions. Two appendixes extend the main result on nonstationary topologies and directed balanced graphs.

## II. Preliminaries

The set composed by the elements $a_1, \dots, a_k$ is denoted by $\{a_1, \dots, a_k\}$, whereas $\underline{N}$ is a shorthand for $\{1, 2, \dots, N\}$. The vectors $y \in \mathbb{C}^m$ are treated as columns with superscript numbering of the entries $y = (y^1, \dots, y^m)^T$; which rule extends on diagonal matrices $D = diag(d^1, \dots, d^N)$. The symbol $^*$ stands for the Hermitian transpose, and $\mathbf{1}_N := (1, 1, \dots, 1)^T$.

Any $N \times N$ matrix $\Gamma = (\gamma_{jk})$ with nonnegative entries $\gamma_{jk} \geq 0$ can be associated with a weighted graph $G(\Gamma) = [\underline{N}, E, \Gamma]$. Here, $\underline{N}$ is the set of nodes, and the arc from $j$ to $k$ belongs to the set of arcs $E = E(\Gamma)$ if and only if $\gamma_{jk} > 0$ in which case $\gamma_{jk}$ is interpreted as the arc's weight. If $\Gamma = \Gamma^T$, the graph $G(\Gamma)$ is undirected $(j, k) \in E \Leftrightarrow (k, j) \in E$. A sequence of nodes $v_1, v_2, \dots, v_k$ where $(v_j, v_{j-1}) \in E$ for $j = 2, \dots, N$ is called the route from $v_1$ to $v_k$; the graph is connected if a route between any two different nodes exists. The degree of node $j$ is the sum of weights of incoming arcs $D^j(\Gamma) := \sum_{k=1}^{N} \gamma_{jk}$, and the Laplacian matrix is given by

$$L(\Gamma) := \text{diag}\left[D^1(\Gamma), \dots, D^N(\Gamma)\right] - \Gamma. \qquad (1)$$

If $\Gamma$ is symmetric, $L(\Gamma) = L(\Gamma)^T \geq 0$, the smallest eigenvalue of $L(\Gamma)$ is zero, and the second smallest eigenvalue

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.



$\lambda_2(\Gamma)$ (called the algebraic connectivity) is positive if and only if $G(\Gamma)$ is connected [42]. By the Courant–Fischer theorem

$$\lambda_2(\Gamma) = N \min_{z\in\mathbb{R}^N, z\not\parallel \mathbf{1}_N} \frac{\sum\limits_{j,k=1}^N \gamma_{jk}\left(z^k - z^j\right)^2}{\sum\limits_{j,k=1}^N \left(z^k - z^j\right)^2}. \tag{2}$$

## III. Problem Statement

We consider a team of identical agents indexed 1 through $N \geq 2$ and described by the following linear state space model:

$$\dot{x}_j(t) = Ax_j(t) + Bu_j(t),\ y_j(t) = Cx_j(t), t \geq 0, j \in \underline{N}. \tag{3}$$

Here, $x_j \in \mathbb{R}^n$, $u_j \in \mathbb{R}^m$, $y_j \in \mathbb{R}^m$ stand for the state vector, input, and output of the $j$th agent, respectively.

The agents interact in accordance with the rule of the form

$$u_j^q(t) = \sum_{k=1}^N \gamma_{jk}\varphi_{jk}^q\left[y_k^q(t) - y_j^q(t)\right],\ j \in \underline{N}, q \in \underline{m} \tag{4}$$

where $^q$ refers to the respective scalar entry of the vector. This rule (also called protocol) involves a structured nonparametric model uncertainty concerned with the continuous nonlinearities $\varphi_{jk}^q : \mathbb{R} \to \mathbb{R}$, called couplings. They belong to a common and known sector and are antisymmetric $\varphi_{jk}^q(y) = -\varphi_{kj}^q(-y)$; no extra knowledge about them may be available. The constant, symmetric, and known matrix of coupling gains $\Gamma = (\gamma_{jk})$ with $\gamma_{jk} \geq 0$ characterizes nominal (estimated, averaged) intensities of interactions between the agents and the interaction topology: the $j$th agent is directly affected by the $k$th one if and only if $\gamma_{jk} > 0$. We set $\gamma_{jj} := 0$ for the sake of definiteness: $\gamma_{jj}$ do not affect the right-hand side of (4) since the companion multiplier $\varphi_{jj}^q[y_j^q(t) - y_j^q(t)] = \varphi_{jj}^q(0) = 0$ due to both the sector and antisymmetry conditions.

The objective of this paper is to disclose conditions sufficient for establishing consensus in the following sense.

*Definition 1:* The protocol (4) establishes consensus among the agents (3) if any solution of (3) and (4) is prolongable up to $+\infty$ and is featured by convergence of the agents to each other

$$x_j(t) - x_k(t) \to 0 \quad \forall j, k \in \underline{N} \quad \text{as } t \to +\infty. \tag{5}$$

For controllable and observable agents, consensus is equivalent to the formally weaker requirement of output consensus: $y_k(t) - y_j(t) \to 0$ as $t \to \infty$; see [39, Remark 2] for details.

*Remark 1:* For the system at hand, any agent (3) has an equal number of inputs and outputs and the protocol (4) is component-wise decoupled: the $q$th input $u_j^q$ of any agent depends only on the matching outputs $y_k^q$ of the companions. Modulo a simple reduction, this case encompasses a more general situation: the dimensions of the input $m := \dim u_j$ and output $l := \dim y_j$ are not necessarily identical and the $q$th input may depend on all outputs in the following fashion:

$$u_j^s(t) = \sum_{k=1}^N \gamma_{jk} \sum_{r=1}^l \psi_{jk}^{s,r}\left[y_k^r(t) - y_j^r(t)\right],\ j \in \underline{N}, s \in \underline{m}. \tag{6}$$

The reduction is accomplished via introducing the augmented control and output $U_j, Y_j \in \mathbb{R}^{ml}$, where $Y_j^{l(s-1)+r} := y_j^r$ and $U_j^{l(s-1)+r} := \sum_k \gamma_{jk}\psi_{jk}^{s,r}(y_k^r(t) - y_j^r(t))$. Then $u_j^s = U_j^s + U_j^{s+m} + \ldots + U_j^{s+(l-1)m}$ and the system can be rewritten in the form (3) with equal number $ml$ of inputs and outputs

$$\dot{x}_j(t) = Ax_j(t) + \hat{B}U_j(t),\ Y_j(t) = \hat{C}x_j(t)$$

whereas (6) takes the form of (4)

$$U_j^q(t) := \sum_k \gamma_{jk}\Phi_{jk}^q\left[Y_k^q(t) - Y_j^q(t)\right], \quad \Phi_{jk}^q := \psi_{jk}^{s,r}.$$

Here, $s-1$ and $r$ are the quotient and remainder ($r \in \underline{l}$) of $q$ divided by $l$. This reduction will be employed in Section VI-B.

## IV. Main Assumptions

*Assumption 1:* The agents (3) are controllable and observable.

*Assumption 2:* The graph $G(\Gamma)$ determined by the matrix $\Gamma = (\gamma_{jk})$ of coupling gains is connected.

The focus of this paper is on the phenomenon of consensus caused by local interactions between the agents. In this context, Assumption 2 is necessary: its violation means disintegration of the network into a set of noninteracting clusters for which the above phenomenon is clearly impossible.

*Assumption 3:* The couplings are antisymmetric: $\varphi_{jk}^q(y) = -\varphi_{kj}^q(-y)\ \forall y \in \mathbb{R}, j, k, q$.

This is close in spirit to the Newton's Third Law and usually holds if the nodes are coupled via a physical interaction, like in oscillator networks, power grids, etc. Under Assumption 3

$$\sum_j u_j(t) \equiv 0. \tag{7}$$

So consensus implies that $x_j(t) - e^{tA}\tilde{x}_0 \to 0$ as $t \to \infty$, where $\tilde{x}_0 = \frac{1}{N}\sum_{j=1}^N x_j(0)$. For $A = 0$, this shapes into $x_j(t) \xrightarrow{t\to\infty} \tilde{x}_0\ \forall j$ and is referred to as the average consensus.

To describe uncertainty in the couplings, we introduce $-\infty \leq \alpha < \beta \leq \infty$ and define $S[\alpha; \beta]$ to be the set of all continuous mappings $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(0) = 0$ and

$$\alpha < \frac{\varphi(y)}{y} < \beta \qquad \forall y \neq 0 \tag{8}$$

$$\lim_{|y|\to\infty} |\varphi(y) - \alpha y| = \infty, \qquad \lim_{|y|\to\infty} |\beta y - \varphi(y)| = \infty. \tag{9}$$

Here, any relation automatically holds if the involved element $\alpha$ or $\beta$ is infinite. Geometrically (8) and (9) mean that the graph of $\varphi$ lies strictly inside the sector with the slopes $\alpha$ and $\beta$ and does not approach the sector boundaries as $|y| \to \infty$.

*Assumption 4:* There exist $-\infty \leq \alpha < \beta \leq \infty$ such that $\varphi_{jk}^q \in S[\alpha; \beta]$ for all $j, k, q$.

What is more, $\alpha$ and $\beta$ are known, whereas no extra data about the couplings may be available. So consensus criterion should be given in terms of $\alpha, \beta, A, B, C, \Gamma$, not the couplings themselves. Such a criterion in fact ensures consensus that is robust against the class of uncertainties from $S[\alpha; \beta]$ obeying the analog of the Newton's Third Law, i.e., Assumption 3.

This robustness involves that consensus is achieved for any linear couplings from that class $\varphi_{jk}^q(y) = \kappa y, \kappa \in (\alpha; \beta)$. For

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.



given $\kappa$, this consensus further implies [11, Th. 1] that whenever $\lambda$ is a nonzero eigenvalue of the Laplacian matrix (1), the feedback $u_j = -\kappa\lambda y_j$ stabilizes the agent (3)

$$\text{the matrix } A - \lambda\kappa BC \text{ is Hurwitz.} \tag{10}$$

So the following precondition for robust consensus is unavoidable. It is similar to standard assumptions of the circle and Popov's criteria and their extensions that somehow stipulate closed-loop stability for linear feedbacks from the sector.

*Assumption 5:* For any $\kappa \in (\alpha; \beta)$ and any eigenvalue $\lambda = \lambda_j(\Gamma) \neq 0$ of the Laplacian matrix $L(\Gamma)$, (10) holds.

This is valid if e.g., the matrix $A - \mu BC$ is Hurwitz whenever $\mu \in (\alpha\lambda_2(\Gamma); \beta\lambda_N(\Gamma))$, where the eigenvalues $\lambda_j(\Gamma)$ are enumerated in the ascending order.

*Remark 2:* A closer analysis of the proofs in Section VII shows that in fact we need (10) (for each $\lambda = \lambda_2(\Gamma), \dots, \lambda_N(\Gamma)$) only if $\kappa \approx \alpha$ or $\kappa \approx \beta$.

Our main interest is in agents with unstable open-loop dynamics: $A$ is not Hurwitz. Due to (10), robust consensus is impossible for them if $\kappa = 0 \in (\alpha, \beta)$. So we exclude such $\alpha, \beta$ from consideration by assuming that either $0 \leq \alpha < \beta$ or $\alpha < \beta \leq 0$. In fact, we focus on the first case

$$\alpha \geq 0 \tag{11}$$

since the second of them $\alpha < \beta \leq 0$ reduces to the first one by the substitution $(\alpha, \beta, \varphi^q_{jk}, B) \mapsto (-\beta, -\alpha, -\varphi^q_{jk}, -B)$: this does not alter the closed-loop equations [that result from insertion of (4) into (3)], keeps Assumptions 1–5 valid (for the altered data), and in the sector condition (8), makes the bound on the left nonnegative.

## V. Main Results

In this section, we suppose that Assumptions 1–5 and (11) hold, and use the following notations: $W(z) := C(zI - A)^{-1}B$—the transfer matrix of the agent (3); $\lambda_2$—the algebraic connectivity of the interaction graph; $D_{\max} := \max_j D^j(\Gamma)$—the maximal weighted degree of its nodes:

$$\delta := \frac{\alpha}{1 + \alpha(\beta)^{-1}}, \quad \rho := \frac{1}{2(\alpha + \beta)}. \tag{12}$$

Now, we are in a position to state the main result of the paper.

*Theorem 1:* Suppose that nonnegative diagonal matrices $\Psi := diag(\psi^1, \dots, \psi^m)$ and $\Theta := diag(\theta^1, \dots, \theta^m)$ exist such that the following frequency domain inequality holds:

$$\begin{aligned}\Pi_{\Psi,\Theta}(\omega) := {} & \rho D^{-1}_{\max}\Psi + \lambda_2\delta W(\imath\omega)^* \Psi W(\imath\omega) \\ & + (\Psi + \imath\omega\Theta) W(\imath\omega) + W(\imath\omega)^* (\Psi - \imath\omega\Theta) \geq 0, \\ & \forall\omega \in \mathbb{R} : \det(\imath\omega I_n - A) \neq 0.\end{aligned} \tag{13}$$

Then the protocol (4) establishes consensus provided that

$$\Psi > 0 \quad \text{and either} \quad \Theta > 0 \text{ or } \Theta = 0. \tag{14}$$

If (14) is violated, the following statements are of interest.

1) If $\Theta > 0$, then any solution of (3) and (4) is prolongable up to $+\infty$ and the deviation between the agents is finite

$$\sup_{t \geq 0} \left| x_j(t) - x_k(t) \right| < \infty \ \forall j, k \in \underline{N}. \tag{15}$$

2) If not necessarily $\Theta > 0$ but $\theta^q > 0$ for a particular $q \in \underline{m}$, then for any solution and any two agents, the deviation of the respective outputs stays bounded as time progresses: $\sup_{t\in\Delta} |y^q_j(t) - y^q_k(t)| < \infty$, where $\Delta$ is the maximal interval of existence of the solution at hand.
3) If (14) does not hold but $\psi^q > 0$ for a particular $q \in \underline{m}$, then for any solution that is prolongable up to $+\infty$, either the respective outputs of any two agents converge to each other or the time derivative of the output discrepancy $\dot{y}^q_j(t) - \dot{y}^q_k(t)$ is unbounded for some two agents $j \neq k$. The last option does not hold if $\Theta > 0$.

For $\beta < \infty$, all statements of the theorem remain valid even if the inequalities $\Theta \geq 0$, $\Theta > 0$ and $\theta^q > 0$ are everywhere replaced by $\Theta \leq 0$, $\Theta < 0$, and $\theta^q < 0$, respectively.

The proof of this theorem is given in Section VII.

*Remark 3:* If $\Theta = 0$, the first conclusion of Theorem 1, as well as 1) remain valid in the case of time-varying interaction topology and coupling gains; see Appendix A for details.

*Remark 4:* The matrix inequality (13) means that

$$\begin{aligned} Re\left[\tilde{u}^* (\Psi + \imath\omega\Theta)\, \tilde{y}\right] + \lambda_2\delta\tilde{y}^*\Psi\tilde{y} + \rho D^{-1}_{\max}\tilde{u}^*\Psi\tilde{u} \geq 0, \\ \forall\tilde{u} \in \mathbb{C}^m \end{aligned} \tag{16}$$

where $\tilde{y} := W(\imath\omega)\tilde{u}$.

*Remark 5:* In the criterion (13), the topology of interactions is concerned only through the algebraic connectivity $\lambda_2 > 0$ of the respective graph. Though its direct computation is an intricate problem for many graphs, various easily computable lower bounds on $\lambda_2$ are available [42]. Putting such a bound in place of $\lambda_2$ in (13) results in a new sufficient, though less conservative, condition since $\delta \geq 0$ and $\Psi \geq 0$.

### A. Consensus for SISO Agents and the Popov's Criterion

The criterion from Theorem 1 noteworthily simplifies in the case of SISO agents. This is demonstrated by the following.

*Theorem 2:* Let $u_j, y_j \in \mathbb{R}$ in (3). If $\theta \in \mathbb{R}$ exist such that

$$\pi_\theta(\omega) := Re[(1 + \imath\omega\theta)\, W(\imath\omega)] + \lambda_2\delta|W(\imath\omega)|^2 + \rho D^{-1}_{\max} \geq 0$$

whenever $\omega \in \mathbb{R}$ and $\det(\imath\omega I - A) \neq 0$, then the protocol (4) establishes consensus provided that either $\theta \geq 0$ or $\beta < \infty$.

Even if the premises of the above claim do not hold but $Re[\imath\omega W(\imath\omega)] \geq 0$ for any $\omega$ such that $\det(\imath\omega I - A) \neq 0$, then any solution of (3) and (4) is prolongable up to $+\infty$ and the deviation between the agents is finite, i.e., (15) holds.

*Proof:* The first claim is immediate from the main conclusion and the last statement of Theorem 1, with the both applied to $\Psi := 1$, $\Theta := \theta$. The second claim follows from 1) in Theorem 1 applied to $\Psi := 0$, $\Theta := 1$. ■

The criterion resulting from Theorem 2 by putting $\theta := 0$ can be extended to time-variant topology (see Appendix A) and essentially coincides with the criterion from [39]. As is shown in [39, Appendix A], the latter can be viewed as a counterpart of the circle criterion [34], [35] for networked systems.

Taken as a whole (i.e., with not necessarily $\theta = 0$), Theorem 2 extends the celebrated Popov's criterion [34], [35], [41]. Specifically, this criterion can be portrayed as that from this theorem in the particular case of only two agents $N = 2$, as discussed below in Section VIII.

## VI. Examples: Consensus of Second-Order Agents

In this section, we illustrate the potential of Theorems 1 and 2 by applying them to second-order agents. Consensus among such agents is a topic of extensive recent research [7], [8], [31], [32], [43], [44], basically motivated by concerns of multiagent mobile robotics. Unlike first-order agents, that research mainly dealt with linear couplings only. To the best of knowledge of the authors, rare exceptions are represented by [7], [31], and [32], where couplings were given by the hyperbolic tangents, and [39] and [40], where analogs of the circle criterion were obtained.

Now, we extend the results from [7], [31], and [32] on a much wider class on nonlinear couplings, including uncertain ones. We also significantly reduce the conservatism of the criteria from [39] and [40] in the case of the fixed interaction topology.

The bulk of recent research on consensus among second-order agents is mainly distributed over double integrators

$$\ddot{z}_j(t) = v_j(t) \in \mathbb{R}, \qquad j \in \underline{N} \tag{17}$$

and second-order oscillators. For the sake of definiteness, we focus on double integrators. The respective protocols available in the literature can be broadly classified into three groups. They are formed by the protocols with access, respectively, to:

1) the absolute velocity of every agent, see [44];
2) only relative velocities of the agents [31];
3) no velocity measurements [45].

These groups will be concerned in Sections VI-A, VI-B, and VI-C, respectively.

### A. Protocols With Access to the Absolute Velocity

We consider the team of agents (17) driven by the protocol

$$v_j(t) = -\mu \dot{z}_j + \sum_{k=1}^{N} \gamma_{jk}\varphi_{jk}\left[z_k(t) - z_j(t)\right] \tag{18}$$

where $\mu$ is a constant. This protocol relies on access to the absolute velocity $\dot{z}_j$ of every agent $j$, whereas only relative positions $z_k - z_j$ of the agents with respect to each other are measured. This situation is exemplified by some marine applications where autonomous surface vessels or underwater vehicles are equipped with relative position sensors (like radars, hydrolocators, etc.) and ground logs, measuring the speed over ground, but have no GPS sensors.

*Theorem 3:* Let Assumptions 2 and 3 hold and $\varphi_{jk} \in S[0; \infty]$. Then for any $\mu > 0$, the protocol (18) establishes consensus and, moreover, $\dot{z}_j(t) \to 0, z_j(t) \to z_* \ \forall j$ as $t \to \infty$, where the limit $z_*$ depends on the initial data only.

*Proof:* By introducing the outputs $y_j(t) := z_j(t)$ and new controls $u_j(t) := v_j(t) + \mu\dot{z}_j$, the system (17), (18) can be viewed as the team of linear agents (3)

$$\ddot{z}_j(t) + \mu\dot{z}_j(t) = u_j(t), \qquad y_j(t) = z_j(t) \tag{19}$$

with the respective states $x_j := (z_j, \dot{z}_j)^T$. They are coupled via the protocol (4) with $m = 1$ and $\varphi^1_{jk} := \varphi_{jk}$ and clearly satisfy Assumption 1. Assumptions 2–4 with $\alpha := 0, \beta := \infty$ are directly stipulated; Assumption 5 holds since the feedback $u_j = -\kappa z_j$ stabilizes the system (19) for any $\kappa > 0$. In Theorem 2, $\rho = \delta = 0$ by (12), $W(\imath\omega) = [\imath\omega(\imath\omega + \mu)]^{-1}$, and so $\pi_\theta(\omega) = [\theta\mu - 1]/(\omega^2 + \mu^2) \geq 0$ for any $\theta \geq \mu^{-1}$. So this theorem guarantees consensus: $z_j - z_k \to 0$ and $\dot{z}_j - \dot{z}_k \to 0$ as $t \to \infty$. Then (18) and (19) sequentially imply that $u_j \to 0$ and $\dot{z}_j \to 0$ as $t \to \infty$. Finally, by summing up (19) over $j$ with regard to (7), we see that $\bar{z} := \sum_j \dot{z}_j + \mu \sum_j z_j \equiv \bar{z}(0)$. Hence, $z_j \to z_* := \bar{z}(0)/(\mu N) \ \forall j$ as $t \to \infty$. ■

Theorem 3 enhances the results [39], [40] applied to networks with a fixed interaction topology. Specifically, it extends them to an infinite sector and arbitrary damping $\mu > 0$.

### B. Protocols With Access to Only Relative Velocities

If the agents can measure only their relative velocities $\dot{z}_k - \dot{z}_j$, the focus is shifted to another consensus algorithm

$$v_j(t) = \sum_{k=1}^{N} \gamma_{jk}\varphi_{jk}\left\{z_k(t) - z_j(t) + \mu\left[\dot{z}_k(t) - \dot{z}_j(t)\right]\right\}. \tag{20}$$

Surprisingly, this shift entails almost no change in the consensus conditions, as is shown by the following.

*Theorem 4:* Let Assumptions 2 and 3 hold, $\varphi_{jk} \in S[0; \infty]$, and

$$\varliminf_{|y| \to 0} \varphi_{jk}(y)/y > 0. \tag{21}$$

Then for any $\mu > 0$, the protocol (20) establishes consensus. Moreover, $\dot{z}_j(t) \to \upsilon_*$ and $z_j(t) - z_* - t\upsilon_* \to 0$ as $t \to \infty$ for all $j$, where $z_*, \upsilon_*$ depend only on the initial data.

*Proof:* By putting $u_j(t) := v_j(t), y_j(t) := z_j + \mu\dot{z}_j$, the system (17), (20) can be viewed as a group of linear agents (3)

$$\ddot{z}_j(t) = u_j(t), \ y_j(t) = z_j + \mu\dot{z}_j \tag{22}$$

with the common transfer function $W_y(\imath\omega) = (\imath\omega)^{-2} + \mu(\imath\omega)^{-1}$. Assumptions 1–5 are checked like in the proof of Theorem 3. Consensus will be justified via two steps.

We first additionally assume that $\varphi_{jk} \in S[\varepsilon; \infty] \ \forall j, k$ for some $\varepsilon > 0$. Then $\alpha = \varepsilon, \beta = \infty$ in (12) and so consensus is justified by Theorem 2 since $\delta = \varepsilon, \rho = 0$ and

$$\pi_\theta(\omega) = \theta\mu - \frac{1}{\omega^2} + \varepsilon\lambda_2\frac{1 + \mu^2\omega^2}{\omega^4} \geq 0 \, \forall\omega \neq 0$$

whenever $\theta \geq |1 - \varepsilon\lambda_2\mu^2|^2/(4\mu\varepsilon\lambda_2)$.

Now, we proceed to the general case $\varphi_{jk} \in S[0; \infty]$ and reduce it to the previous one. To this end, we consider a particular solution $z^0_j, y^0_j, u^0_j$ of (22) and (20). Since $Re[\imath\omega W_y(\imath\omega)] = \theta\mu > 0$ for any $\theta > 0$, Theorem 2 guarantees that this solution is prolongable up to $\infty$ and $M := \sup_{t \geq 0, j, k} |y^0_j(t) - y^0_k(t)| < \infty$. Noting that $\varepsilon := \min_{j,k} \inf_{|y| \leq M} \varphi_{jk}(y)/y > 0$ by (8) (where $\alpha = 0$) and (21), we alter the nonlinearities in (20)

$$\tilde{\varphi}_{jk}(y) := \begin{cases} y \max\left\{\varphi_{jk}(y)/y, \varepsilon_0\right\} & \text{if } y \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

This does not violate (20) for the solution at hand by the choice of $M$ and $\varepsilon$; Assumption 3 is also kept valid: $\tilde{\varphi}_{kj}(-y) =$

$-y\max\{-\varphi_{kj}(-y)/y,\varepsilon\} = -y\max\{\varphi_{jk}(y)/y,\varepsilon\} = -\tilde{\varphi}_{jk}(y)$. Finally, we note that since $\tilde{\varphi}_{jk} \in S[\varepsilon;\infty]$, consensus holds by the foregoing: $z_j^0 - z_k^0 \to 0$, $\dot{z}_j^0 - \dot{z}_k^0 \to 0$ as $t \to \infty$. The proof is completed like for Theorem 3: (7) $\Rightarrow \sum_j \ddot{z}_j = 0$, and so consensus entails that the limits $\upsilon_* = \lim_{t\to\infty} \dot{z}_j(t)$ and $z_* = \lim_{t\to\infty}(z_j(t) - \upsilon_* t)$ exist and do not depend on $j$. ■

It should be noted that (21) is implied by $\varphi'_{jk}(0) > 0$.

The protocols (20) were addressed in [39] and [40] for more general networks, whose interaction topology varies over time, but under a much stronger assumption about the couplings: $\varphi_{jk} \in S[\alpha;\infty]$, where $\alpha$ should be not only positive but also large enough by exceeding a certain topology-dependent bound. Theorem 4 sweeps away the last two requirements, though is concerned only with a fixed topology.

Consensus protocols with nonlinearities of another type

$$v_j(t) = \sum_{k=1}^{N} \gamma_{jk}\left\{\varphi_{jk}^1\left[z_k(t) - z_j(t)\right] + \varphi_{jk}^2\left[\dot{z}_k(t) - \dot{z}_j(t)\right]\right\} \quad (23)$$

were examined in [31] for $\varphi_{jk}^1(y) = \varphi_{jk}^2(y) = \tanh y$. This choice was motivated in [31] by the need to produce bounded controls $v_j$, which however, is equally achieved by any bounded couplings $\varphi_{jk}$. Protocols (23) are of interest, e.g., whenever the relative positions and velocities are separately accessed via nonlinearly distorted measurements or/and are fed to different saturated actuators. Our next result extends the criterion from [31] to a much wider class of nonlinearities.

*Theorem 5:* Suppose Assumptions 2 and 3 hold, $\varphi_{jk}^1, \varphi_{jk}^2 \in S[0;\infty]$, the functions $\varphi_{jk}^2$ are bounded on $\mathbb{R}$, and

$$0 < \varliminf_{|y|\to 0} \varphi_{jk}^q(y)/y \le \varlimsup_{|y|\to 0} \varphi_{jk}^q(y)/y < \infty \quad q = 1, 2. \quad (24)$$

Then the protocol (20) establishes consensus. Moreover, $\dot{z}_j(t) \to \upsilon_*$ and $z_j(t) - z_* - t\upsilon_* \to 0$ as $t \to \infty$ for all $j$, where $z_*, \upsilon_*$ depend only on the initial data.

*Proof:* Consider the system (17), (23) as a team of agents

$$\ddot{z}_j(t) = u_j^1(t) + u_j^2(t),\ y_j(t) = x_j(t) := \left[z_j(t), \dot{z}_j(t)\right]^T \quad (25)$$

where the $j$th agent has the state $x_j$, output $y_j$, control $u_j = [u_j^1, u_j^2]^T$, and the transfer matrix $W(\lambda) = (\lambda^{-2};\lambda^{-1})^T(1;1)$. The agents are coupled via the protocol (4) with $m = 2$. Assumptions 1–4 are checked like in the proof of Theorem 3; Assumption 5 holds since the feedback $u_j^q(t) = -\kappa y_j^q$ $(q = 1, 2)$ clearly stabilizes the agent whenever $\kappa > 0$. Now, $\alpha = 0, \beta = \infty$ in Assumption 4 and so $\delta = \rho = 0$ by (12).

For $\psi^1 := 0$, $\theta^1 := \psi^2 = 1$, $\theta^2 := 0$, the frequency-domain inequality (16) takes the form

$$Re\left[\imath\omega\frac{\tilde{u}_1^*(\tilde{u}_1 + \tilde{u}_2)}{(\imath\omega)^2} + \frac{\tilde{u}_2^*(\tilde{u}_1 + \tilde{u}_2)}{(\imath\omega)}\right] \ge 0$$

and so is true since its left-hand side is identically zero. So 2) in Theorem 1 ensures that for any solution of (17) and (23), the discrepancy $z_j(t) - z_k(t) = y_j^1(t) - y_k^1(t)$ is bounded on the maximal interval where the solution exists. Since the function $\varphi_{jk}^2$ is bounded, (17) and (23) imply that $\ddot{z}_j(t)$ also is bounded on that interval. This in turns entails that this interval extends to $+\infty$. Then by 3) in Theorem 1

$$\dot{z}_j - \dot{z}_k \to 0 \quad \text{as} \quad t \to \infty \qquad \forall j, k. \quad (26)$$

Like in the proof of Theorem 4, now we focus on a particular solution and alter the nonlinearities $\varphi_{jk}^q \mapsto \tilde{\varphi}_{jk}^q$ so that $\tilde{\varphi}_{jk}^1, \tilde{\varphi}_{jk}^2 \in S[\alpha;\beta]$ with some $\alpha > 0, \beta < \infty$, Assumption 3 remains true, and equations (4) are not violated

$$u_j^q(t) := \sum_{k=1}^{N} \gamma_{jk}\tilde{\varphi}_{jk}^q\left[y_k^q(t) - y_j^q(t)\right], \quad q = 1, 2. \quad (27)$$

It is easy to see that this can be achieved by putting $M := \max_{j,k,q}\sup_{t\ge 0}(|y_k^q(t) - y_j^q(t)|) < \infty$ and taking

$$0 < \alpha < \alpha_0 := \min_{q=1,2}\min_{j,k}\inf_{|y|\le M}\varphi_{jk}^q(y)/y \overset{(24)}{>} 0$$

$$\beta > \beta_0, \text{ where } \beta_0 := \max_{q=1,2}\max_{j,k}\sup_{|y|\le M}\varphi_{jk}^q(y)/y \overset{(24)}{<} \infty$$

$$\tilde{\varphi}_{jk}^q(y) := y\min\left\{\max\left[\varphi_{jk}^q(y)/y;\alpha_0\right];\beta_0\right\}.$$

For the altered protocol (27) and $\psi^1 = 1$, $\psi^2 = \theta^1 = \xi$, $\theta^2 = 0$, the frequency-domain inequality (16) shapes into

$$\begin{aligned}0 \le{}& \psi^1\rho|\tilde{u}_1|^2 + \psi^2\rho|\tilde{u}^2|^2 + \left(\frac{\psi^1}{\omega^4} + \frac{\psi^2}{\omega^2}\right)\delta\lambda_2|\tilde{u}^1 + \tilde{u}^2|^2\\ &+ Re\left[(\tilde{u}^1)^*\left(\tilde{u}^1 + \tilde{u}^2\right)\frac{\psi^1 + \theta^1\imath\omega}{(\imath\omega)^2} + (\tilde{u}^2)^*\left(\tilde{u}^1 + \tilde{u}^2\right)\frac{\psi^2 + \theta^2\imath\omega}{\imath\omega}\right]\\ \overset{(a)}{=}{}& \rho|\tilde{u}^1|^2 + \xi\rho|\tilde{u}^2|^2 + \left(\frac{1}{\omega^4} + \frac{\xi}{\omega^2}\right)\lambda_2\delta|\tilde{u}^1 + \tilde{u}^2|^2\\ &- \frac{1}{\omega^2}Re\left[(\tilde{u}^1)^*\left(\tilde{u}^1 + \tilde{u}^2\right)\right] \overset{(a)}{=} \rho|\tilde{u}_1|^2 + \rho\xi|\tilde{u}_2|^2\\ &+ \frac{1}{\omega^2}Re\left[\tilde{u}_2^*(\tilde{u}_1 + \tilde{u}_2)\right] + \frac{1}{\omega^4}\lambda_2\delta|\tilde{u}_1 + \tilde{u}_2|^2 + \frac{\xi\lambda_2\delta - 1}{\omega^2}|\tilde{u}_1 + \tilde{u}_2|^2\end{aligned}$$

where (a) holds since $\tilde{u}_1^*(\tilde{u}_1 + \tilde{u}_2) = |\tilde{u}_1 + \tilde{u}_2|^2 - \tilde{u}_2^*(\tilde{u}_1 + \tilde{u}_2)$ and so $Re[\tilde{u}_1^*(\tilde{u}_1 + \tilde{u}_2)/(\imath\omega) + \tilde{u}_2^*(\tilde{u}_1 + \tilde{u}_2)/(\imath\omega)] = 0$. It is easy to see that the resultant Hermitian form in $\tilde{u}_1, \tilde{u}_2$ is nonnegative for all $\omega \ne 0$ if $\xi\lambda_2\delta - 1 \ge 0$ and $\rho\xi\lambda_2\delta > 1/4$: in this case, we have $\rho|\tilde{u}_1|^2 \ge 0$, $(\xi\lambda_2\delta - 1)|\tilde{u}_1 + \tilde{u}_2|^2 \ge 0$ and $\rho\xi|\tilde{u}_2|^2 + \lambda_2\delta|\tilde{u}_1 + \tilde{u}_2|^2/\omega^4 \ge |\tilde{u}_2||(\tilde{u}_1 + \tilde{u}_2)|/\omega^2 \ge Re[\tilde{u}_2^*(\tilde{u}_1 + \tilde{u}_2)]/\omega^2$. By taking such $\xi$ and invoking (26), we see that for the chosen solution of the system (17), (23), $z_j(t) - z_k(t) = y_j^1(t) - y_k^1(t) \to 0$ as $t \to +\infty$ thanks to 3) in Theorem 1. This and (26) mean consensus by Definition 1. The proof is completed like for Theorems 3 and 4. ■

### C. Protocols With No Velocity Measurements

In the case where neither absolute nor relative velocities of the agents are measured, [45] proposed to use the following protocol based on positional measurements only:

$$\dot{w}_j = -\nu_1 w_j + \xi_j,\ \xi_j = \sum_{k=1}^{N}\gamma_{jk}(z_k - z_j) \quad (28)$$

$$v_j(t) = \nu_2\dot{w}_j + \xi_j = -\nu_2\nu_1 w_j + (\nu_2 + 1)\xi_j,\ j \in \underline{N} \quad (29)$$

where $\nu_1, \nu_2 > 0$ are constants. The intuition behind it stems from the conventional interpretation of the first equation as that of a low-pass differentiator $\dot{w}_j \approx \dot{\xi}_j$, which estimates the combination of the unobserved velocities $\sum_{k=1}^{N}\gamma_{jk}(\dot{z}_k - \dot{z}_j)$. By formally substituting it for $\dot{w}_j$ in (29), we see that (28), (29) can be viewed as an approximation of the protocol (20) with the

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.



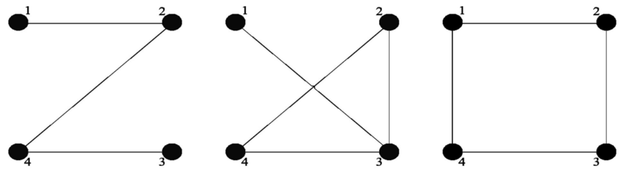


Fig. 1. Undirected graphs $G_1, G_2, G_3$.

identity couplings $\varphi_{jk}(y) = y$, which protocol is not directly implementable under the circumstances.

Now, we extend the protocol (28), (29) and related results [40] on the case of nonlinear coupling

$$\dot{w}_j(t) = -\nu_1 w_j(t) + \xi_j(t),\ v_j(t) = \nu_2 \dot{w}_j(t) + \xi_j(t)$$
$$\xi_j(t) = \sum_{k=1}^{N} \gamma_{jk}\varphi_{jk}\left[z_k(t) - z_j(t)\right]. \tag{30}$$

*Theorem 6:* Under the assumptions of Theorem 4, the protocol (30) establishes consensus for any $\nu_1, \nu_2 > 0$.

*Proof:* Let us consider (30) as a team of agents (3)

$$\ddot{z}_j = \nu_2 \dot{w}_j + u_j,\ \dot{w}_j = -\nu_1 w_j + u_j,\ y_j(t) = z_j(t) \tag{31}$$

coupled by the protocol (4), where $m = 1$. Here, the state of the $j$th agent $x_j = (z_j, \dot{z}_j, w_j)^T$. Assumptions 1–4 are checked like in the proof of Theorem 3; Assumption 5 holds since the feedback $u_j(t) = -\kappa y_j$ clearly stabilizes the agent whenever $\kappa > 0$. Similarly to the proof of Theorem 4, it suffices to examine the case where $\varphi_{jk} \in S[\alpha; \infty]$ with some $\alpha > 0$.

For the transfer function $W$ of the agent (31), we have

$$W(\lambda) = \frac{(1+\nu_2)\lambda + \nu_1}{\lambda^2(\lambda + \nu_1)}, \quad Re\,W(\imath\omega) = -\frac{1}{\omega^2} - \frac{\nu_2}{\nu_1^2 + \omega^2}$$
$$Re\,[\imath\omega W(\imath\omega)] = \frac{\nu_2\nu_1}{\nu_1^2 + \omega^2}, \quad |W(\imath\omega)|^2 = \frac{(1+\nu_2)^2\omega^2 + \nu_1^2}{\omega^4\left(\nu_1^2 + \omega^2\right)}.$$

So the frequency-domain condition in Theorem 2 shapes into

$$0 \le -\frac{(\nu_2+1)\omega^2 + \nu_1^2}{\omega^2(\nu_1^2+\omega^2)} + \frac{\theta\nu_1\nu_2}{\nu_1^2+\omega^2} + \delta\lambda_2\frac{(1+\nu_2)^2\omega^2+\nu_1^2}{\omega^4(\nu_1^2+\omega^2)}$$

or equivalently $(\theta\nu_1\nu_2 - \nu_2 - 1)\omega^4 + a\omega^2 + \delta\lambda_2\nu_1^2 \ge 0\ \forall\omega$. It remains to note that this inequality really holds for sufficiently large $\theta$, and to apply Theorem 2. ■

### D. Numerical Examples

To confirm the results of this section, we consider a team of $N = 4$ double integrator agents (17), coupled via three different protocols, corresponding respectively to the conditions of Theorems 3, 4, and 6. We assume that $\gamma_{jk} \in \{0; 1\}$ and the graph $G(\Gamma)$ coincides with one of the graph $G_1, G_2, G_3$ depicted on Fig. 1. All of these graphs obviously satisfy Assumption 2. Initial conditions for the agents are respectively $z_1(0) = -5, \dot{z}_1(0) = 1,\ z_2(0) = 1, \dot{z}_2(0) = -3,$ $z_3(0) = 5, \dot{z}_3(0) = 2,$ and $z_4(0) = -4, \dot{z}_4(0) = -2$.

We start with the protocol (18) with $\mu = 2.5$, $G(\Gamma) = G_1$ and $\varphi_{jk}(y) = y|y|/(1+y^2)$. This coupling map is odd and satisfies the assumptions of Theorem 3. Since this theorem, unlike Theorem 4, does not require a positive gain at zero (21), the consensus is reached. However, a fast decay of $\varphi_{jk}(\cdot)$ at zero implies rather slow convergence, as can be seen in Fig. 2.

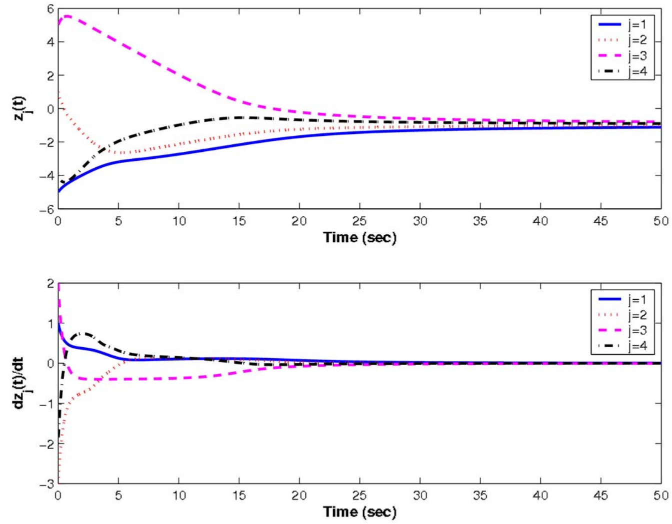


Fig. 2. Protocol (18), $\varphi_{jk}(y) = y|y|/(1+y^2)$, $G(t) = G_1$.

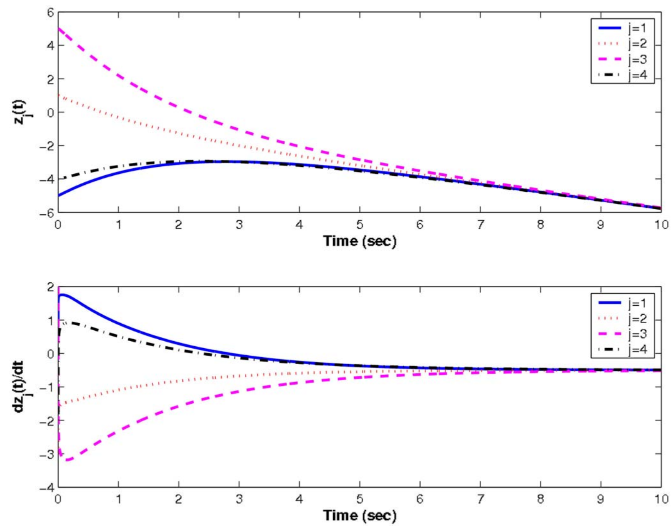


Fig. 3. Protocol (20), $\varphi_{jk}(y) = y + y^3$, $G(t) = G_2$.

The next test deals with $G(\Gamma) = G_2$ and the protocol (20), where $\mu = 2$, $\varphi_{jk}(y) = y + y^3$. This coupling satisfies (21) though is unbounded, unlike the previous example. The result of the test is displayed in Fig. 3 and confirms Theorem 4.

Our last test addresses the protocol (30), where $w_j(0) = 0$, $G(\Gamma) = G_3$, $\nu_i = 1, \varphi_{jk}(y) = y^2 \tanh y$. This coupling combines features from the previous examples: it is $O(y^2)$ as $y \to 0$ and is unbounded as $y \to \infty$. The achievement of consensus, justified by Theorem 6, is illustrated by Fig. 4.

## VII. Proof of Theorem 1

In this section, all assumptions of Theorem 1, including (13), are supposed to hold with one reservation. We assume that either $\Theta \ge 0$, as is required at the beginning of the theorem, or $\Theta \le 0$ and $\beta < \infty$, as is permitted at its end.

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.



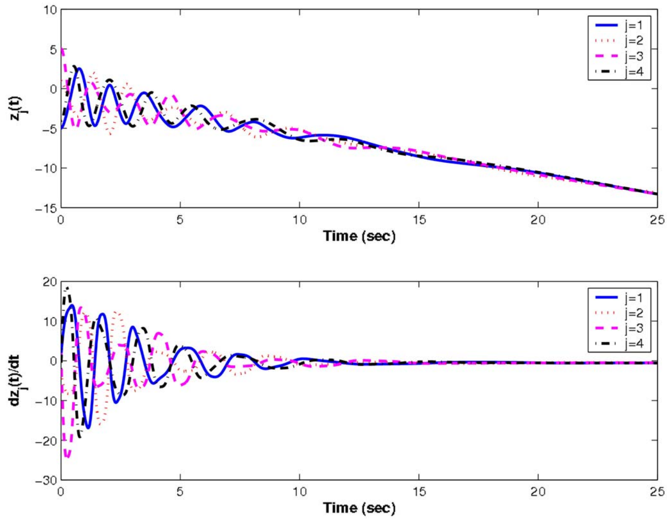


Fig. 4. Protocol (30), $\varphi_{jk}(y) = y^2 \tanh y$, $G(t) = G_3$.

To prove a Popov's type criterion, we follow [34] by employing the Kalman–Yakubovich–Popov (KYP) lemma and a Lyapunov function in the Lur'e–Postnikov form (a quadratic form plus the integral of nonlinearity). Its introduction is prefaced by auxiliary technical constructions.

By invoking $\delta, \rho, \lambda_2, D_{\max}$ from the beginning of Section V, we introduce the Hermitian forms

$$F(\sigma,\xi) := -Re(\bar{\sigma}\xi) - \lambda_2\delta|\sigma|^2 - \rho D_{\max}^{-1}|\xi|^2,\ \sigma,\xi \in \mathbb{C}$$

$$\mathcal{F}_q(\mathbf{u},\mathbf{y}) := \sum_{j,k=1}^{N} F\left(u_k^q - u_j^q, y_k^q - y_j^q\right)$$

$$\mathcal{F}(\mathbf{u},\mathbf{y}) := \sum_{q=1}^{m} \psi^q \mathcal{F}_q(\mathbf{u},\mathbf{y}). \tag{32}$$

Here and throughout, $\overline{\sigma}$ is the conjugate of $\sigma$, whereas $\mathbf{u} := (u_1^T,\ldots,u_N^T)^T$, $\mathbf{y} := (y_1^T,\ldots,y_N^T)^T$, and $u_j, y_j \in \mathbb{C}^m$. We also introduce the following functions:

$$\sigma \in \mathbb{R} \mapsto \Phi_{jk}^q(\sigma) := \int_0^\sigma \left[\varphi_{jk}^q(s) - \alpha s\right] ds$$

$$\sigma \in \mathbb{R} \mapsto \tilde{\Phi}_{jk}^q(\sigma) := \int_0^\sigma \left[\beta s - \varphi_{jk}^q(s)\right] ds \ (\equiv +\infty \text{ if } \beta = \infty),$$

$$\mathfrak{S}(\mathbf{y}) := N \sum_{q=1}^{m}\sum_{j,k=1}^{N} \theta^q \gamma_{jk} \Phi_{jk}^q\left(y_k^q - y_j^q\right)$$

$$\tilde{\mathfrak{S}}(\mathbf{y}) := -N \sum_{q=1}^{m}\sum_{j,k=1}^{N} \theta^q \gamma_{jk} \tilde{\Phi}_{jk}^q\left(y_k^q - y_j^q\right). \tag{33}$$

From the interaction topology and individual properties (8), (9) of the couplings, the first lemma derives an integrated scalar quadratic inequality on the overall solution of the networked system. This lemma extends [39, Lemma 1] to the case of arbitrary $\gamma_{jk} \geq 0$ (not necessarily 0 or 1).

*Lemma 1:* For any $q \in \underline{m}$ and any solution of (3) and (4), $\mathcal{F}_q[\mathbf{y}(t),\mathbf{u}(t)] \geq 0\ \forall t$. Moreover given $\nu > 0$, there exists $\varkappa = \varkappa(\nu) > 0$, such that $\mathcal{F}_q[\mathbf{y}(t),\mathbf{u}(t)] > \varkappa$ whenever

$$\max_{1\leq j,k\leq N} \left|y_k^q(t) - y_j^q(t)\right| > \nu. \tag{34}$$

*Proof:* By (8), $[\varphi_{jk}^q(\sigma) - \alpha\sigma][\beta^{-1}\sigma - \varphi_{jk}^q(\sigma)] \geq 0 \overset{(12)}{\Longrightarrow} f_{jk}^q(\sigma) := \sigma\varphi_{jk}^q(\sigma) - \delta|\sigma|^2 - 2\rho|\varphi_{jk}^q(\sigma)|^2 \geq 0$. Moreover, (8), (9) $\Rightarrow \inf_{|\sigma|\geq\varepsilon} f_{jk}^q(\sigma) > 0\ \forall\varepsilon > 0$. Substituting $\sigma := y_k^q(t) - y_j^q(t)$, multiplying by $\gamma_{jk}$, denoting $\widetilde{\varphi}_{jk}^q(t) := \varphi_{jk}^q[y_k^q(t) - y_j^q(t)]$ and $\xi_{jk}^q(t) := \gamma_{jk}\widetilde{\varphi}_{jk}^q(t)$, and summing up yields

$$\underbrace{\sum_{j,k=1}^{N} \xi_{jk}^q(t)\left[y_k^q(t) - y_j^q(t)\right]}_{=a} - \delta \underbrace{\sum_{j,k=1}^{N} \gamma_{jk}\left|y_k^q(t) - y_j^q(t)\right|^2}_{=b}$$
$$- 2\rho \underbrace{\sum_{j,k=1}^{N} \gamma_{jk}\left|\widetilde{\varphi}_{jk}^q(t)\right|^2}_{=c} \geq \sum_{j,k=1}^{N} \gamma_{jk} f_{jk}^q\left[y_k^q(t) - y_j^q(t)\right] \geq 0.$$

Since $u_j^q(t) = \sum_k \xi_{jk}^q(t)$ by (4) and $\xi_{jk}^q(t) = -\xi_{kj}^q(t)$ by Assumption 3, we have with regard to (7) $a = -\sum_{j,k=1}^N \xi_{jk}^q(t) y_j^q(t) - \sum_{j,k=1}^N \xi_{kj}^q(t) y_k^q(t) = -2\sum_j u_j^q y_j^q = -N^{-1}\sum_{j,k=1}(u_k^q - u_j^q)(y_k^q - y_j^q)$, whereas $b \geq N^{-1}\lambda_2 \sum_{j,k=1} |y_k^q - y_j^q|^2$ by (2). Meanwhile, due to the Cauchy–Schwartz inequality, $|u_j^q(t)|^2 = |\sum_k \gamma_{jk}\widetilde{\varphi}_{jk}^q(t)|^2 \leq \sum_k \gamma_{jk} \sum_k \gamma_{jk} |\widetilde{\varphi}_{jk}^q(t)|^2$, where $\sum_k \gamma_{jk} \leq D_{\max}$. Hence, $c \geq D_{\max}^{-1} \sum_j |u_j^q(t)|^2 \overset{(7)}{=\!=} (2ND_{\max})^{-1} \sum_{j,k} |u_j^q(t) - u_k^q(t)|^2$. By substituting these bounds for $a, b, c$ into the displayed formula and multiplying the result by $N$, we arrive at the first claim of the lemma $\mathcal{F}_q[\mathbf{y}(t),\mathbf{u}(t)] \geq 0\ \forall t$.

Now let (34) be true. Then $|y_k^q(t) - y_j^q(t)| > \varepsilon := \nu/(N-1)$ and $\gamma_{jk} > 0$ for some $(j,k)$ by Assumption 2. Due to (9) and the strict inequalities in (8), not only $f_{jk}^q(\sigma) \geq 0\ \forall\sigma$ not also $\inf_{|\sigma|\geq\varepsilon} f_{jk}^q(\sigma) > 0$. With this observation at hand, the second claim results from merely retracing the above arguments. ■

*Corollary 1:* The quadratic form (32) is nonnegative, and $\mathcal{F}[\mathbf{y}(t),\mathbf{u}(t)] \geq \psi^q\varkappa(\nu)$ whenever (34) holds.

Now, we address properties of the functions from (33).

*Lemma 2:* The following claims hold:

1) $\Phi_{jk}^q(\sigma), \tilde{\Phi}_{jk}^q(\sigma) \geq 0$ and $\Phi_{jk}^q(\sigma), \tilde{\Phi}_{jk}^q(\sigma) \xrightarrow{|\sigma|\to\infty} \infty$;
2) $\mathfrak{S}(\mathbf{y}) \geq 0\ \forall \mathbf{y} = (y_1,\ldots,y_N)^T \in \mathbb{R}^{mN}$ if $\Theta \geq 0$;
3) $\tilde{\mathfrak{S}}(\mathbf{y}) \geq 0\ \forall \mathbf{y}$ if $\Theta \leq 0$ and $\beta < \infty$;
4) let for some solution of (3), (4) and $q \in \underline{m}$, either $\sup_t \mathfrak{S}[\mathbf{y}(t)] < \infty, \Theta \geq 0, \theta^q > 0$ or $\sup_t \tilde{\mathfrak{S}}[\mathbf{y}(t)] < \infty, \Theta \leq 0, \beta < \infty, \theta^q < 0$. Then the discrepancy $y_j^q(t) - y_k^q(t)$ is bounded over time for all $k, j$.

*Proof:* The first claim is immediate from (8) and (9) and implies the other claims due to (33). ■

Now based on the KYP lemma, we convert the frequency-domain inequality (16) into feasibility of a special LMI, equivalently expressed in terms of a quadratic inequality.

*Lemma 3:* There exists a real $n \times n$ matrix $H_0 = H_0^T$ such that the following inequality holds along the solutions of (3) and (4) with $V_0(\mathbf{x}) := \sum_{j,k=1}^N (x_k - x_j)^T H_0 (x_k - x_j)$:

$$\frac{dV_0(\mathbf{x})}{dt} + \mathcal{F}(\mathbf{y},\mathbf{u}) \leq \sum_{j,k=1}^{N} (u_k - u_j)^T \Theta\left(\dot{y}_k - \dot{y}_j\right)$$

$$\overset{(7)}{=\!=} 2N \sum_{j=1}^{N} u_j^T \Theta \dot{y}_j, \quad \text{where } \mathbf{x} = \left(x_1^T,\ldots,x_N^T\right)^T. \tag{35}$$

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.



*Proof:* It is easy to check that (35) is implied by

$$2x^T H_0(Ax+Bu) - u^T\Psi y - \lambda_2\delta y^T\Psi y - \rho u^T\Psi u - u^T\Theta\dot{y} \le 0$$

where $y := Cx, \dot{y} := C(Ax+Bu)$ and $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ are arbitrary. By the KYP lemma [34], [46], the last inequality has a solution $H_0 = H_0^T \in \mathbb{R}^{n\times n}$ if and only if for any $\tilde{u} \in \mathbb{C}^m, \tilde{y} := W(\imath\omega)\tilde{u}, \omega \in \mathbb{R} : \det(\imath\omega I - A) \ne 0$, we have

$$-Re\left[\tilde{u}^*\tilde{y} + \imath\omega\tilde{u}^*\Theta\tilde{y}\right] - \delta\lambda_2\tilde{y}^*\Psi\tilde{y} - \rho D_{\max}^{-1}\tilde{u}^*\Psi\tilde{u} \le 0.$$

It remains to note that this inequality is equivalent to (16). ∎

Now, we introduce the Lyapunov candidate function

$$\begin{aligned} V(\mathbf{x}) &:= V_0(\mathbf{x}) + N\sum_{q=1}^{m}\sum_{j,k=1}^{N}\theta^q\gamma_{jk}\int_0^{\left(y_k^q - y_j^q\right)}\varphi_{jk}^q(s)\,ds \\ &= \begin{cases} V_\alpha(\mathbf{x}) + \mathfrak{S}(\mathbf{y}) & \text{if } \Theta \ge 0 \\ V_\beta(\mathbf{x}) + \tilde{\mathfrak{S}}(\mathbf{y}) & \text{if } \Theta \le 0, \beta < \infty. \end{cases} \end{aligned} \tag{36}$$

Here, $y_j := Cx_j$, $\mathfrak{S}, \tilde{\mathfrak{S}}$ are given by (33), and

$$V_\alpha(\mathbf{x}) := V_0(\mathbf{x}) + \alpha\mathcal{V}(\mathbf{x}), \quad V_\beta(\mathbf{x}) := V_0(\mathbf{x}) - \beta\mathcal{V}(\mathbf{x})$$

$$\mathcal{V}(\mathbf{x}) := \frac{N}{2}\sum_{q=1}^{m}\theta^q\sum_{j,k=1}^{N}\gamma_{jk}|y_k^q - y_j^q|^2.$$

Now we disclose Lyapunov-like properties of $V$.

*Lemma 4:* The function $V$ decays along the trajectories of (3) and (4) $\frac{d}{dt}V[\mathbf{x}(t)] \le 0$, and $V(\mathbf{x}) \ge 0$. Furthermore, $V_\alpha(\mathbf{x}) \ge 0\ \forall\mathbf{x}$ if $\Theta \ge 0$, and $V_\beta(\mathbf{x}) \ge 0\ \forall\mathbf{x}$ if $\Theta \le 0$ and $\beta < \infty$. If $\Theta = 0$ and $\Psi > 0$, then $V = V_\alpha = V_0$, and $V_0(\mathbf{x}) = 0$ if and only if $x_1 = \cdots = x_N$, i.e., $H_0 > 0$.

*Proof:* The first claim is proved via direct calculation

$$\begin{aligned} \frac{d}{dt}V\left[\mathbf{x}(t)\right] &= \frac{d}{dt}V_0\left[\mathbf{x}(t)\right] \\ &\quad + N\sum_{q=1}^{m}\sum_{j,k=1}^{N}\theta^q\xi_{jk}^q(t)\left(\dot{y}_k^q(t) - \dot{y}_j^q(t)\right) \overset{(4)}{=} \frac{d}{dt}V_0\left[\mathbf{x}(t)\right] \\ &\quad - 2N\sum_{j=1}^{N}u_j(t)^T\Theta\dot{y}_j(t) \overset{(35)}{\le} -\mathcal{F}\left[\mathbf{y}(t),\mathbf{u}(t)\right] \overset{\text{Corollary 1}}{\le} 0. \end{aligned} \tag{37}$$

This holds for any $\varphi_{jk}^q$ satisfying Assumptions 3 and 4 (with given $\alpha, \beta$). Among such $\varphi_{jk}^q$'s, let us temporarily focus on $\varphi_{jk}^q(y) = \kappa y$ with $\kappa \in (\alpha;\beta)$. Let $V^{[\kappa]}$ be the respective function (36). For any $\mathbf{x}(0) = \mathbf{x}_0$, (5) holds by [11, Th. 1] and so $V^{[\kappa]}[\mathbf{x}(t)] \to 0$ as $t \to \infty$. This and (37) imply that $V^{[\kappa]}[\mathbf{x}_0] \ge 0\ \forall\mathbf{x}_0$. So $V[\mathbf{x}] \ge 0$ if $\Theta = 0$ since $V \equiv V^{[\kappa]}$.

Now let $\Theta \ge 0$ and let $\mathbf{x}^{[\kappa]}, \mathbf{y}^{[\kappa]}, \mathbf{u}^{[\kappa]}$ be the solution for (3) and (4) with $\varphi_{jk}^q(y) = \kappa y$ and $\mathbf{x}^{[\kappa]}(0) = \mathbf{x_0}$. For the respective function $\mathfrak{S}^{[\kappa]}$ from (33), $\mathfrak{S}^{[\kappa]}[\mathbf{y}^{[\kappa]}(t)] = (\kappa - \alpha)\mathcal{W}[\mathbf{x}^{[\kappa]}(t)]$, where $\mathcal{W}(\mathbf{x}) \ge 0$ is a quadratic form independent of $\kappa$. Hence, $V_\alpha(\mathbf{x_0}) = V^{[\kappa]}(\mathbf{x_0}) - (\kappa-\alpha)\mathcal{W}(\mathbf{x_0}) \ge -(\kappa-\alpha)\mathcal{W}(\mathbf{x_0})$. Letting $\kappa \to \alpha$ yields the third claim $V_\alpha(\mathbf{x_0}) \ge 0$ of the lemma. The fourth claim is established likewise. The second claim $V \ge 0$ follows from (36) and Lemma 2.

Let $\Theta = 0$, $\Psi > 0$ and $V(\mathbf{x}_0) = 0$. Putting $\mathbf{x}^{[\kappa]}, \mathbf{y}^{[\kappa]}, \mathbf{u}^{[\kappa]}$ into (37) and integrating yields $\int_0^\infty \mathcal{F}(\mathbf{y}^{[\kappa]}(t), \mathbf{u}^{[\kappa]}(t))dt = 0$. Hence, $y_1^{[\kappa]}(t) \equiv \ldots \equiv y_N^{[\kappa]}(t)$ and $u_1^{[\kappa]}(t) \equiv \ldots \equiv u_N^{[\kappa]}(t) \equiv 0$ by Lemma 1. It remains to note that then $x_1^{[\kappa]}(t) \equiv \ldots \equiv x_N^{[\kappa]}(t)\ \forall t$ due to observability. ∎

*Proof of Theorem 1:* For any solution of (3) and (4), $V[\mathbf{x}(t)] \le M := V\ [\mathbf{x}(0)]$ for $t \ge 0$ due to (37). So by (36) and Lemma 4, $\mathfrak{S}[\mathbf{y}(t)] \le M$ if $\Theta \ge 0$, and $\tilde{\mathfrak{S}}[\mathbf{y}(t)] \le M$ if $\Theta \le 0$ and $\beta < \infty$. This and iv) in Lemma 2 yield 2) from Theorem 1. In the context of 1), all discrepancies $y_j(t) - y_k(t)$ are bounded by 2), so are $u_j(t)$ by (4). The last claim and (3) entail that every $x_j(t)$ is prolongable up to $+\infty$. Furthermore, we see that for the auxiliary linear plant with the state $\Delta_{jk} = x_j - x_k$

$$\dot{\Delta}_{jk} = A\Delta_{jk} + Bu_{jk}, \qquad u_{jk} = u_j - u_k \tag{38}$$

the output $C\Delta_{jk} = y_j(t) - y_k(t)$ and control are also bounded. This implies 1) in Theorem 1 thanks to Assumption 1.

To prove 3), we examine a solution of (3) and (4) that is prolongable up to $+\infty$, and assume that $\psi^q > 0$. Due to (37), $\mathcal{F}[\mathbf{y}(\cdot),\mathbf{u}(\cdot)] \in L^1[0;\infty]$ and so $\mathcal{F}_q[\mathbf{y}(\cdot),\mathbf{u}(\cdot)] \in L^1[0;\infty]$ by (32) and Lemma 1. If all discrepancies $\dot{y}_j^q(t) - \dot{y}_k^q(t)$ are bounded, then $\mathcal{F}_q[\mathbf{y}(\cdot),\mathbf{u}(\cdot)]$ is uniformly continuous and so $\mathcal{F}_q[\mathbf{y}(t),\mathbf{u}(t)] \to 0$ as $t \to \infty$ by the Barbalat lemma [35]. Thanks to Lemma 1, this proves the claim given by the first sentence in 3) from Theorem 1. To complete the proof of 3), suppose that $\Theta > 0$. Then (15) holds by 1). Hence, $y_j(t) - y_k(t) \overset{(3)}{=} C[x_j(t) - x_k(t)]$ is bounded; so is $u_j(t)$ due to (4). It remains to note that then $\dot{y}_j(t) - \dot{y}_k(t) = CA[x_j(t) - x_k(t)] + Bu_j(t) - Bu_k(t)$ is bounded as well.

To prove the first claim from Theorem 1, we suppose that (14) holds, and show first that all solutions are prolongable up to $+\infty$. If $\Theta > 0$, this is true by 1). If $\Theta = 0$, then $H_0 > 0$ by Lemma 4, and (37) implies that the functions $x_j(t) - x_k(t)$ are bounded on the maximal interval $I$ of existence. So are $y_j(t) - y_k(t)$ and $u_j(t)$ due to (3) and (4), respectively. The last claim, along with (3), entails the required property $I = [0,\infty)$.

By 3), $y_j(t) - y_k(t) \to 0$ as $t \to \infty$. Then $u_j(t) \to 0$ due to (4) since $\varphi_{jk}(0) = 0$ by Assumption 3. Thus, the output and control of the plant (38) go 0 as $t \to \infty$. So (5) holds due to Assumption 1. Definition 1 completes the proof. ∎

## VIII. Theorem 1 and AST

In this section, we discuss relationships between consensus and absolute stability, as well as similarities and differences between our consensus criterion and the Popov criterion.

By interpreting some of the agents as a virtual "leader," the consensus may be viewed as stability of the deviations of the "leader" from its flock-mates, which is nothing but absolute stability of a certain reduced-order networked system. This hint shows that in the case of a trivial network of $N = 2$ scalar agents, the criterion from Theorem 2 is in fact a reformulation of the classic Popov criterion. However, the latter criterion is not applicable to $N \ge 3$ agents since the reduced-order system contains multiple nonlinearities. This motivates application of multivariable extensions of the Popov criterion [47], [48]. Now we show that this application entails overly conservative consensus conditions, as compared with the results of this paper, even for the case of $N = 3$ scalar agents.

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.



Given a solution of (3) and (4), we put $\xi_j = u_j - u_N$, $\eta_j = x_j - x_N$, and $\sigma_j = y_j - y_N$, where $j \in \underline{N-1}$. Since the agents are identical, (3) shapes into

$$\dot{\eta}_j = A\eta_j + B\xi_j,\ \sigma_j = C\eta_j,\ j \in \underline{N-1}. \quad (39)$$

These may be interpreted as equations of a reduced-order system. Similarly by (4), $u_N(t) = \sum_{k=1}^{N-1} \gamma_{Nk}\varphi_{Nk}(\sigma_k(t))$ and $u_j(t) = \sum_{k=1}^{N-1} \gamma_{jk}\varphi_{jk}(\sigma_k(t) - \sigma_j(t)) + \gamma_{jN}\varphi_{jN}(-\sigma_j(t))$. So

$$\xi_j^q(t) = \sum_{k=1}^{N-1} \gamma_{jk}\varphi_{jk}^q\left(\sigma_k^q(t) - \sigma_j^q(t)\right) - \gamma_{Nj}\varphi_{Nj}\left(\sigma_j^q(t)\right) - \sum_{k=1}^{N-1} \gamma_{Nk}\varphi_{Nk}\left(\sigma_k^q(t)\right), \quad j \in \underline{N-1}, q \in \underline{m}. \quad (40)$$

By the following lemma, consensus may be viewed as stability of the system (39), (40). This system is said to be stable if any its solution is prolongable up to $+\infty$, and $\eta_j(t) \to 0$ (and so $\xi_j(t), \sigma_j(t) \to 0$) as $t \to +\infty$.

*Lemma 5:* The protocol (4) establishes consensus if and only if the auxiliary system (39), (40) is stable.

*Proof:* We start with the "if" part. Any solution $(x_j, u_j, y_j)$ of (3) and (4) is associated with a solution $(\eta_j, \xi_j, \sigma_j) := (x_j - x_N, u_j - u_N, y_j - y_N)$ of (39) and (40). If the latter system is stable, the functions $(\eta_j, \xi_j, \sigma_j)$ remain bounded on any compact interval. Then this property is evidently valid for $u_N(t)$ and hence for $x_N, y_N$. Thus, the functions $x_j, u_j, y_j$ cannot blow up in finite time, i.e., the solution is infinitely prolongable. Meanwhile, the consensus (5) obviously holds since $\xi_j(t) \xrightarrow[t\to+\infty]{} 0$. To prove the "only if" part, we consider an arbitrary solution $(\eta_j, \xi_j, \sigma_j)$ of (39) and (40). Let $u_N(t) = \sum_{k=1}^{N-1} \gamma_{Nk}\varphi_{Nk}(\sigma_k(t))$ and $x_N, y_N$ be some solution of (3) (where $j = N$). By putting $x_j := \xi_j + x_N, y_j := \sigma_j + y_N, u_j := \xi_j + u_N\ \forall j = 1, \dots, N-1$, we obtain a solution of (3) and (4). So this solution is infinitely prolongable and (5) holds. It follows that $(\eta_j(t), \xi_j(t), \sigma_j(t))$ is infinitely prolongable and converges to 0 as $t \to +\infty$. ■

*Remark 6:* The "partial" consensus considered in Theorem 1 ($x_j^q - x_k^q \to 0$ for some $q \in \underline{m}$) may be viewed as stability of the system (39), (40) with respect to the output $(\eta_1^q, \dots, \eta_{N-1}^q)$: $\eta_j^q(t) \to 0$ as $t \to +\infty$ for any $j = 1, \dots, N-1$ and any initial data.

The focus of this paper is on situations where the consensus is robust, i.e., holds for any nonlinearities $\varphi_{jk}^q \in S[\alpha; \beta]$ satisfying Assumption 3 and any graph satisfying Assumption 6. Lemma 5 transforms the problem of robust consensus into that of absolute stability: it is required to disclose conditions under which stability of (39) and (40) simultaneously holds for all afore-mentioned nonlinearities $\varphi_{jk}$. So consensus criteria can be derived from well-elaborated absolute stability criteria. Now, we show that this method gives the result of this paper only in the simplest case of $N = 2$ scalar agents, whereas its outcome is much weaker in the other cases.

We start with two scalar agents: $N = 2$, $m := \dim u_j = \dim y_j = 1$, and $\alpha = 0$. Then (39), (40) is a Lurie system. It is constituted by the linear plant (39) (where the index $j = 1$ will be dropped) in the feedback interconnection $\xi_1(t) = \Phi(\sigma_1(t))$ with an uncertain "nonlinearity" $\Phi(\sigma) := -2\gamma_{12}\varphi_{12}^1(\sigma)$ that comes from the "protocol" equations and satisfies the sector condition $\Phi \in S[0; \mu]$ with $\mu := 2\gamma_{12}\beta$. These are exactly the context of the classic Popov's criterion [34], [35]. It states that absolute stability holds whenever the plant is stable in open loop (i.e., $A$ is Hurwitz) and for some $\theta \in \mathbb{R}$

$$Re\,[(1 + \imath\omega\theta)\, W(\imath\omega)] + \mu^{-1} > 0\ \forall\omega \in \mathbb{R}:\ \det(\imath\omega I - A) \neq 0. \quad (41)$$

It is easy to see by inspection that this result basically coincides with that from Theorem 2 applied to the case at hand (in which case, $D_{\max} = \lambda_2 = 2\gamma_{12}$ and $\delta = 0$ in Theorem 2), except for two mismatches. Firstly, the assumption of open-loop stability from the classic criterion is replaced with Assumption 5. In fact, Assumption 5 is weaker since in the face of the frequency domain inequality (41), this assumption is implied by the open-loop stability [34]. At the same time, Assumption 5 does not imply open-loop stability; it guarantees only that the system is neutrally stable in open loop. Secondly, for nonlinearities that lie strictly inside the sector $S[0; \mu]$, our Theorem 2 in fact relaxes inequality (41) employed by the classic criterion by substituting the nonstrict inequality $\geq$ in place of $>$ (this can be shown by retracing the relevant arguments from [34, Sec. 2.1.6]). Thus, Theorem 2 modifies the classic Popov criterion, providing stability for nonlinearities from the open sector and neutrally stable linear part.

Now, we show that even in a bit more complex case of $N = 3$ scalar agents, direct application of the multivariable Popov criterion gives much more conservative consensus conditions than ours. We proceed from a recent extension of the Popov criterion [47], [48] and carry out comparison in the simplified context of a complete graph $\gamma_{12} = \gamma_{23} = \gamma_{13} = 1$ and $\alpha = 0$.

Let $\Phi_1 := \varphi_{31}^1$, $\Phi_2 := \varphi_{32}^1$, $\Phi_3 := \varphi_{12}^1$, and $\sigma_3 := \sigma_2 - \sigma_1 = y_1 - y_2$. Then the system (39), (40) shapes into

$$\dot{\bar{\eta}} = \mathcal{A}\bar{\eta} + \mathcal{B}\Phi(\bar{\sigma}),\ \bar{\sigma} = \mathcal{C}\bar{\eta}. \quad (42)$$

Here, $\bar{\eta} := (\eta_1^T, \eta_2^T)^T$, $\bar{\sigma} := (\sigma_1, \sigma_2, \sigma_3)^T$, $\Phi(\bar{\sigma}) := (\Phi_1(\sigma_1), \Phi_2(\sigma_2), \Phi_3(\sigma_3))$, and

$$\mathcal{A} := I_2 \otimes A,\ \mathcal{B} = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \end{bmatrix} \otimes B,\ \mathcal{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \otimes C$$

where $\otimes$ stands for the Kronecker product of two matrices. For the system (42), the transfer function $G(s)$ from $\Phi$ to $\eta$ is given by $G(s) = -G_0 W(s)$, where $W(s)$ is the transfer function of (3) (which is assumed to be scalar) and $G_0 = \left[\begin{smallmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{smallmatrix}\right]$.

The criterion from [48, Sec. 3] applied to $K = \beta I_3$ guarantees stability under the frequency-domain condition[1]

$$He\left[\beta^{-1} I + (1 + \imath\omega D)\, G_0 W(\imath\omega)\right] \geq 0. \quad (43)$$

[1]In fact, some modification of this criterion is needed. The criterion from [48] deals with stability in the closed sector that includes both $\Phi \equiv 0$ and $\Phi(\sigma) = \beta\sigma$, thus assuming the open-loop stability. A simple extra analysis of the proofs shows that for the open sector, this requirement may be replaced by stability under any feedback $\Phi(\sigma) = \mu\sigma$ with $\mu \in (0; \beta)$; and the strict inequality $>$ in (43) may be relaxed into the nonstrict counterpart $\geq$ along the lines discussed in the foregoing with respect to the classic Popov's criterion.

Here, $He(M) := M + M^*$ and $D$ is some diagonal matrix. Unfortunately, this criterion is not applicable to the agents (22) since (43) fails to be true for $\omega \approx 0$. (Indeed, $W(\imath\omega) = \mu(\imath\omega)^{-1} - \omega^{-2}$ and so the left-hand side of (43) is asymptotically equivalent to $-\omega^{-2}G_0 < 0$ as $\omega \to 0$.) In other words, the MIMO Popov criterion does not guarantee stability for nonlinearities $\varphi^1_{jk} \in S[0;\beta]$, whereas Theorem 1 does so. Meanwhile the "two-sided" Popov's criterion from [47] ensures stability for $\varphi^1_{jk} \in S[\alpha;\beta]$ if

$$He\left[(\beta-\alpha)^{-1}I + (1+\imath\omega D)\left[I+\alpha G_0 W(\imath\omega)\right]^{-1} G_0 W(\imath\omega)\right] \geq 0.$$

This inequality holds only if $\alpha > 0$ is sufficiently large and $\beta < \infty$. So the result of Theorem 4 about consensus in the infinite sector ($\alpha = 0, \beta = \infty$) and even more general Theorem 2 are not deducible from the Popov's criterion.

Finally, as $N$ grows, the worst-case number of nonlinearities in the system (39), (40) has order of $O(N^2)$. So straightforward application of classical absolute stability criteria is computationally expensive, up to looking as hardly tractable for large-scale networks $N \approx \infty$. Conversely, the complexity of the proposed consensus criterion, whose main part is constituted by the frequency-domain inequality (13), does not grow as $N \to \infty$ and depends only on the number of inputs $m$. So this criterion is better fitted to deal with large-scale systems. For instance, Theorem 2 gives easily verifiable conditions irrespectively of the system's scale and multiple nonlinearities.

## IX. Conclusion

We considered networks of identical linear agents of arbitrary order with time-invariant and undirected interaction topologies and uncertain nonlinear couplings. They satisfy the sector condition with known slopes and are featured by a symmetry in the spirit of the Newton's Third Law. A novel Popov's type criterion for robust consensus was obtained. Its potential was illustrated by a series of corollaries concerned with double integrators, including extension of a recent result from [31] on a much wider class of nonlinearities and essential reduction of the conservatism in recent criteria from [39] and [40].

## Appendix A
## Networks With Switching Interaction Topology

If $\Theta = 0$ in (14), the first claim of Theorem 1 remains true for switching topology, i.e., if the gains in (4) depend on time

$$u^q_j(t) = \sum_{k=1}^{N} \gamma_{jk}(t)\varphi^q_{jk}\left[y_k(t) - y_j(t)\right]. \tag{44}$$

Here, $\gamma_{jk} : [0;\infty) \to \mathbb{R}$ are Lebesgue measurable functions such that $\Gamma(t) := (\gamma_{jk}(t))$ always belongs to a compact set $\boldsymbol{\Gamma}$ of symmetric $N \times N$-matrices with nonnegatives entries.

The following analogs of Assumptions 2 and 5 are adopted.

*Assumption 6:* The graph $G(\Gamma)$ is connected for any $\Gamma \in \boldsymbol{\Gamma}$.

*Assumption 7:* There exists $\kappa \in (\alpha;\beta)$ and $\Gamma_* \in \boldsymbol{\Gamma}$ such that the matrix $A - \kappa\lambda BC$ is Hurwitz whenever $\lambda \neq 0$ is an eigenvalue of the Laplacian $L(\Gamma_*)$.

Due to [49], this is equivalent to the fact that the protocol (44) with $\Gamma(t) \equiv \Gamma_*$, $\varphi^q_{jk}(y) = \kappa y$ establishes consensus.

*Theorem 7:* Suppose that Assumptions 1, 3, 4, 6, and 7 hold, and there exists a diagonal matrix $\Psi := \mathrm{diag}(\psi^1,\ldots,\psi^m) > 0$ such that the frequency domain inequality (13) is true with $\Theta = 0$, $\delta$ and $\rho$ given by (12), and

$$\lambda_2 := \min_{\Gamma\in\boldsymbol{\Gamma}} \lambda_2(\Gamma), \quad D_{\max} = \max_{j\in\underline{N}} \max_{\Gamma\in\boldsymbol{\Gamma}} D_j(\Gamma).$$

Then the protocol (44) establishes consensus.

*Proof:* It is easy to see by inspection that Lemma 1 remains valid due to compactness of $\boldsymbol{\Gamma}$. Similarly, Lemma 3 also works, and the first and last claims of Lemma 4 remain valid. Based on these, merely retracing the relevant arguments from the proof of Theorem 1 assures that 3) in this theorem is also true. The proof is completed like in the last two paragraphs from the proof of Theorem 1. ■

Theorem 7 extends the result of [39] from SISO agents to MIMO agents and from two valued gains $\gamma_{jk}(t) = 0, 1$ to arbitrary real valued ones $\gamma_{jk}(t) \geq 0$.

## Appendix B
## Balanced Protocols Over Directed Graphs

According to [39], the protocol (4) is said to be balanced if for any $j$ and $y_1,\ldots,y_N \in \mathbb{R}$, the following relation holds:

$$\sum_{k=1}^{N} \gamma_{jk} w^q_{jk}(y_k - y_j) = \sum_{k=1}^{N} \gamma_{kj} w^q_{kj}(y_j - y_k). \tag{45}$$

Here, the "gain" $w^q_{jk}$ is given by $w^q_{jk}(y) := \varphi^q_{jk}(y)/y$ if $y \neq 0$ and $w^q_{jk}(0) := 0$. For undirected graphs ($\gamma_{jk} = \gamma_{kj}$), (45) is true under Assumption 3 since then $\gamma_{jk}w^q_{jk}(y_k - y_j) = \gamma_{kj}w^q_{kj}(y_j - y_k)$. For directed graphs, (45) is typically fulfilled only if the gains $w^q_{jk}(y) \equiv w^q_{jk}$ are constant.

*Theorem 8:* Theorem 1 remains valid for directed graphs if in its statement, Assumption 3 and condition $\Gamma = \Gamma^T$ are replaced by the requirement that the protocol is balanced, and connectivity in Assumption 2 is understood in the strong sense.

*Proof:* By retracing the arguments from [39, Appendix C], it is easy to see that (7) and Lemma 1 remain valid in the current context. So Lemma 3 also works. The proof is completed like the proof of Theorem 1. ■

## References


[1] C. Reynolds, "Flocks, herds, and schools: A distributed behavioral model," *Comput. Graph.*, vol. 21, no. 4, pp. 25–34, 1987.

[2] S. Strogatz, "From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators," *Physica*, vol. 143, nos. 1–4, pp. 1–20, Sep. 2000.

[3] Q. Lu, S. Liu, X. Xie, and J. Wang, "Decision making and finite-time motion control for a group of robots," *IEEE Trans. Cybern.*, vol. 43, no. 2, pp. 738–750, Apr. 2013.

[4] S. Zhu, C. Chen, W. Li, B. Yang, and X. Guan, "Distributed optimal consensus filter for target tracking in heterogeneous sensor networks," *IEEE Trans. Cybern.*, vol. 43, no. 6, pp. 1963–1976, Dec. 2013.

[5] J. Wang and H. Wu, "Synchronization and adaptive control of an array of linearly coupled reaction-diffusion neural networks with hybrid coupling," *IEEE Trans. Cybern.*, vol. 44, no. 8, pp. 1350–1361, Aug. 2014.

[6] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton, NJ, USA: Princeton Univ. Press, 2010.

[7] W. Ren and R. Beard, *Distributed Consensus in Multi-Vehicle Cooperative Control: Theory and Applications*. New York, NY, USA: Springer, 2008.

[8] W. Ren and Y. Cao, *Distributed Coordination of Multi-Agent Networks*. New York, NY, USA: Springer, 2011.
[9] R. Olfati-Saber, J. Fax, and R. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.
[10] S. Strogatz, "Exploring complex networks," *Nature*, vol. 410, pp. 268–276, Mar. 2001.
[11] Z. Li, Z. Duan, G. Chen, and L. Huang, "Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 57, no. 1, pp. 213–224, Jan. 2010.
[12] J. Lü and G. Chen, "A time-varying complex dynamical network model and its controlled synchronization criteria," *IEEE Trans. Autom. Control*, vol. 50, no. 6, pp. 841–846, Jun. 2005.
[13] Z. Meng, Z. Li, A. Vasilakos, and S. Chen, "Delay-induced synchronization of identical linear multiagent systems," *IEEE Trans. Cybern.*, vol. 43, no. 2, pp. 476–489, Apr. 2013.
[14] Y.-P. Tian, *Frequency-Domain Analysis and Design of Distributed Control Systems*. Hoboken, NJ, USA: Wiley, 2012.
[15] D. Meng and K. Moore, "Studies on resilient control through multiagent consensus networks subject to disturbances," *IEEE Trans. Cybern.*, to be published.
[16] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 169–182, Feb. 2005.
[17] Y. Chen, J. Lü, X. Yu, and Z. Lin, "Consensus of discrete-time second-order multiagent systems based on infinite products of general stochastic matrices," *SIAM J. Control Optim.*, vol. 51, no. 4, pp. 3274–3301, 2013.
[18] P. Wieland, R. Sepulchre, and F. Allgöwer, "An internal model principle is necessary and sufficient for linear output synchronization," *Automatica*, vol. 47, no. 5, pp. 1068–1074, 2011.
[19] Y. Su and J. Huang, "Cooperative output regulation with application to multi-agent consensus under switching network," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 42, no. 3, pp. 864–875, Jun. 2012.
[20] A. Pogromsky and H. Nijmeijer, "Cooperative oscillatory behavior of mutually coupled dynamical systems," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 48, no. 2, pp. 152–162, Feb. 2001.
[21] N. Chopra and M. Spong, "Passivity-based control of multi-agent systems," in *Advances in Robot Control*, S. Kawamura and M. Svinin, Eds. Berlin, Germany: Springer-Verlag, 2006.
[22] C. Wu, *Synchronization in Complex Networks of Nonlinear Dynamical Systems*. Singapore: World Scientific, 2007.
[23] K. Friston, "The labile brain: I. Neuronal transients and nonlinear coupling," *Philos. Trans. Roy. Soc. B, Biol. Sci.*, vol. 355, no. 1394, pp. 215–236, Feb. 2000.
[24] R. Dogaru, *Systematic Design for Emergence in Cellular Nonlinear Networks: With Applications in Natural Computing and Signal Processing* (Studies in Computational Intelligence), vol. 95. Berlin, Germany: Springer, 2008.
[25] H. Tanner, A. Jadbabaie, and G. Pappas, "Flocking in fixed and switching networks," *IEEE Trans. Autom. Control*, vol. 52, no. 5, pp. 863–868, May 2007.
[26] J. Lin, A. Morse, and B. Anderson, "The multi-agent rendezvous problem (2 parts)," *SIAM J. Control Optim.*, vol. 46, no. 6, pp. 2096–2147, 2007.
[27] Z. Lin, B. Francis, and M. Maggiore, "State agreement for continuous-time coupled nonlinear systems," *SIAM J. Control Optim.*, vol. 46, no. 1, pp. 288–307, 2007.
[28] A. Ajorlou, A. Momeni, and A. Aghdam, "Sufficient conditions for the convergence of a class of nonlinear consensus algorithms," *Automatica*, vol. 47, no. 3, pp. 625–629, 2011.
[29] L. Fang and P. Antsaklis, "Asynchronous consensus protocols using nonlinear paracontractions theory," *IEEE Trans. Automat. Control*, vol. 53, no. 10, pp. 2351–2355, Nov. 2008.
[30] Y. Chen, J. Lü, and Z. Lin, "Consensus of discrete-time multi-agent systems with transmission nonlinearity," *Automatica*, vol. 49, no. 6, pp. 1768–1775, 2013.
[31] W. Ren, "On consensus algorithms for double-integrator dynamics," *IEEE Trans. Autom. Control*, vol. 53, no. 6, pp. 1503–1509, Jul. 2008.
[32] A. Abdessameud and A. Tayebi, "On consensus algorithms for double-integrator dynamics without velocity measurements and with input constraints," *Syst. Control Lett.*, vol. 59, no. 12, pp. 812–821, Dec. 2010.
[33] L. Pecora and T. Carroll, "Master stability functions for synchronized coupled systems," *Phys. Rev. Lett.*, vol. 80, no. 10, pp. 2109–2112, 1998.
[34] A. Gelig, G. Leonov, and V. Yakubovich, *Stability of Stationary Sets in Control Systems With Discontinuous Nonlinearities*. Singapore: World Scientific, 2004.
[35] H. Khalil, *Nonlinear Systems*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1996.
[36] V. Yakubovich, "Necessity in quadratic criterion for absolute stability," *Int. J. Robust Nonlinear Control*, vol. 10, nos. 11–12, pp. 889–907, 2000.
[37] A. Megretski and A. Rantzer, "System analysis via integral quadratic constraints," *IEEE Trans. Autom. Control*, vol. 42, no. 6, pp. 819–830, Jun. 1997.
[38] U. Jönsson, C. Kao, and H. Fujioka, "A Popov criterion for networked systems," *Syst. Control Lett.*, vol. 56, nos. 9–10, pp. 603–610, 2007.
[39] A. Proskurnikov, "Consensus in switching networks with sectorial nonlinear couplings: Absolute stability approach," *Automatica*, vol. 49, no. 2, pp. 488–495, 2013.
[40] A. Proskurnikov, "Nonlinear consensus algorithms with uncertain couplings," *Asian J. Control*, vol. 16, no. 5, pp. 1277–1288, 2014.
[41] V. Popov, *Hyperstability of Control Systems*. Berlin, Germany: Springer, 1973.
[42] R. Chung, *Spectral Graph Theory*. Providence, RI, USA: American Mathematical Society, 1997.
[43] W. Yu, G. Chen, M. Cao, and J. Kurths, "Second-order consensus for multiagent systems with directed topologies and nonlinear dynamics," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 40, no. 3, pp. 881–891, Jun. 2010.
[44] J. Qin, W. Zheng, and H. Gao, "Coordination of multiple agents with double-integrator dynamics under generalized interaction topologies," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 42, no. 1, pp. 44–57, Feb. 2012.
[45] Y. Zheng and L. Wang, "Consensus of heterogeneous multi-agent systems without velocity measurements," *Int. J. Control*, vol. 85, no. 7, pp. 906–914, 2012.
[46] A. Rantzer, "On the Kalman–Yakubovich–Popov lemma," *Syst. Control Lett.*, vol. 28, no. 1, pp. 7–10, 1996.
[47] D. Bernstein, W. Haddad, and A. Sparks, "A Popov criterion for uncertain linear multivariable systems," *Automatica*, vol. 31, no. 7, pp. 1061–1064, 1995.
[48] W. Heath and G. Li, "Lyapunov functions for the multivariable Popov criterion with indefinite multipliers," *Automatica*, vol. 45, no. 12, pp. 2977–2981, 2009.
[49] W. Yu, G. Chen, W. Ren, J. Kurths, and W. Zheng, "Distributed higher order consensus protocols in multiagent dynamical systems," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 58, no. 8, pp. 1924–1932, Aug. 2011.


**Anton Proskurnikov** (M'13) was born in 1982. He received the M.Sc. (Specialist) and the Ph.D. (Candidate of Science) degrees, both from Saint Petersburg State University, Saint Petersburg, Russia (supervised by Prof. V. A. Yakubovich), in 2003 and 2005, respectively.

He is currently a Post-Doctoral Researcher with the Research Institute of Technology and Management of the University of Groningen, Groningen, The Netherlands. He is also a part-time Researcher with Saint Petersburg State University and Institute for Problems of Mechanical Engineering RAS, Saint Petersburg. His current research interests include dynamics and control of complex networks, multiagent and decentralized control nonlinear control, robust control, optimal control, and optimization.

Dr. Proskurnikov was the recipient of the Russian Academy of Sciences Medal for prominent young researchers in 2009.

**Alexey Matveev** was born in Leningrad, Russia, in 1954. He received the M.S. and the Ph.D. degrees, both from the Leningrad University, Saint Petersburg, Russia, in 1976 and 1980, respectively.

He is currently a Professor with the Department of Mathematics and Mechanics, Saint Petersburg University, Saint Petersburg. His current research interests include estimation and control over communication networks, hybrid dynamical systems, and navigation and control of mobile robots.