UNITING OF GLOBAL AND LOCAL CONTROLLERS UNDER ACTING DISTURBANCES

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Abstract: A switching control is presented, which unites global stabilising control law and local optimal one. In the presence of disturbances this switching control ensures boundedness of all trajectories of the system and additionally provides output attractiveness property, if disturbance is absent.

Keywords: uniting control, hybrid control, switching, disturbances.

1. INTRODUCTION

The problem of uniting of global and local controllers is rather not new. Frequently in practical applications a situation is appeared, then for some plant it is possible to design a global stabilizing control law, which ensures global boundedness of the system solution in the presence of disturbances without desired performance, and local control law, that guarantees to solution optimality in some sense for absent disturbances. It is necessary to design unite controller, which inherits properties of both controllers. Really, the most of well known techniques, such as backstepping, forwarding, feedback linearization and passivation (Fomin, *et al.*, 1981; Isdori, 2000; Krstić, *et al.*, 1995; Sepulchre, *et al.*, 1997), which solve task of robust system stabilization, usually do not address the problem of system performance even in some small neighborhood of the origin. In these approaches it is supposed, that the main goal consists in disturbance rejecting or compensating. But in real applications the quality of transient processes is very important and a compromise should be found. Uniting controller can be considered as a such compromise, if it equals to the optimal controller near the origin and provide boundedness of system solution, then disturbances presence is detected.

The problem of uniting control was solved in (Teel, and Kapoor, 1997), where a dynamic time invariant controller was proposed, which converges to local optimal controller near the origin only under some special conditions. In paper (Morin, *et al.*, 1998) a static time invariant controller was presented under hard verified condition of existing continuous path between global and local controllers. In works (Prieur, 2001; Prieur, and Praly, 1999) an example was found, which does not allow any continuous or even discontinuous time invariant controllers. Additionally, in these works several solutions of uniting control problem were proposed (continuous, discontinuous, hybrid and time-varying) for case of external disturbances absence. A kind of uniting controls for chained nonholonomic systems was developed in (Prieur, and Astolfi, 2002), where robust properties of such controller with respect to some sufficiently small disturbances were analyzed. Here we present a hybrid uniting control for systems with not measured bounded external disturbances. In Section 2 all definitions and notations can be found, in Section 3 main result formulation and proofs are summarized. Conclusion in Section 4 finishes the paper.

2. DEFINITIONS AND STATEMENTS

Let the model of dynamical system under consideration can be presented as follows:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u},\mathbf{d}),\ \mathbf{y} = \mathbf{h}(\mathbf{x}), \qquad (1)$$

where $\mathbf{x} \in R^n$ is state space vector of the system; $\mathbf{y} \in R^m$ is output vector; function $\mathbf{f}$ is locally Lipschitz continuous, $\mathbf{f}(0,0,0) = 0$; function $\mathbf{h}$ is continuous and $\mathbf{h}(0) = 0$; $\mathbf{u} \in R^p$ and $\mathbf{d} \in R^q$ are control and disturbance inputs respectively, it is supposed that they are Lebesgue measurable and essentially bounded functions of time:

$$\|\mathbf{u}\|_{[0,t)} = \operatorname*{ess\,sup}_{\tau\in[0,t)}\{|\mathbf{u}(\tau)|\} < +\infty,$$

$$\|\mathbf{d}\|_{[0,t)} = \operatorname*{ess\,sup}_{\tau\in[0,t)}\{|\mathbf{d}(\tau)|\} < +\infty,$$

classes of such functions for $\mathbf{u}$ and $\mathbf{d}$ we denote as $\mathcal{M}_p$ and $\mathcal{M}_q$ respectively (here $|\cdot|$ denotes usual Euclidean norm, $\|\mathbf{u}\|_{[0,+\infty)} = \|\mathbf{u}\|$). Thus, for any initial conditions $\mathbf{x}_0 \in R^n$ and any $\mathbf{u} \in \mathcal{M}_p$ and $\mathbf{d} \in \mathcal{M}_q$, solution of (1) $\mathbf{x}(t,\mathbf{x}_0,\mathbf{u},\mathbf{d})$ is well defined at the least locally on some time interval $[0,T)$ (further we will omit dependence of $\mathbf{x}_0$ and $\mathbf{u},\mathbf{d}$ if it is clear from the context and will simply write $\mathbf{x}(t)$ or for systems without controls $\mathbf{x}(t,\mathbf{x}_0,\mathbf{d})$; denote output solution as $\mathbf{y}(t,\mathbf{x}_0,\mathbf{u},\mathbf{d})$ or for system without controls $\mathbf{y}(t,\mathbf{x}_0,\mathbf{d})$ correspondingly). If $T = +\infty$ for all initial conditions and any $\mathbf{u} \in \mathcal{M}_p$ and $\mathbf{d} \in \mathcal{M}_q$, then system (1) is called forward complete.

As usually, it is said, that function $\rho: R_{\geq 0} \to R_{\geq 0}$ belongs to class $\mathcal{K}$, if it is strictly increasing and

$\rho(0)=0$; $\rho \in \mathcal{K}_\infty$ if $\rho \in \mathcal{K}$ and $\rho(s) \to \infty$ for $s \to \infty$ (radially unbounded). Function $\beta : R_{\geq 0} \times R_{\geq 0} \to R_{\geq 0}$ is from class $\mathcal{KL}$, if it belongs to class $\mathcal{K}$ for the first argument (for any fixed second) and strictly decreases to zero for the second (for any fixed first argument). The distance from given point $\zeta \in R^n$ to any non empty subset $\mathcal{A}$ from R^n is denoted as

$$|\zeta|_{\mathcal{A}} \overset{\Delta}{=} \text{dist}[\zeta, \mathcal{A}] = \inf_{\boldsymbol{\eta} \in \mathcal{A}} [\zeta, \boldsymbol{\eta}],$$

in this way $|\zeta|_0 = |\zeta|$. Now we are ready to introduce main properties of local $\mathbf{u}_l$ and global $\mathbf{u}_g$ controllers for system (1).

A s s u m p t i o n 1. *There exist a constant $K > 0$ and a continuous local control $\mathbf{u}_l : R^n \to R^p$, such, that system*

$$\dot{\mathbf{x}} = \mathbf{f}_l(\mathbf{x}, \mathbf{d}) = \mathbf{f}(\mathbf{x}, \mathbf{u}_l(\mathbf{x}), \mathbf{d}), \; \mathbf{y} = \mathbf{h}(\mathbf{x}), \quad (2)$$

admits the following properties:

A. *Forward completeness for any $\mathbf{d} \in \mathcal{M}_q$, i.e. there exist some functions $\chi_1, \chi_2, \chi_3, \chi_4 \in \mathcal{K}_\infty$ and positive constant χ, such, that (Angeli, and Sontag, 1999)*

$$|\mathbf{x}(t, \mathbf{x}(t_0), \mathbf{d})| \leq \chi_1(t - t_0) + \chi_2(|\mathbf{x}(t_0)|) + \\ + \chi_3\left(\int_{t_0}^{t} \chi_4(|\mathbf{d}(\tau)|) d\tau\right) + \chi.$$

B. *If $\mathbf{d}(t) \equiv 0$ for all $t \geq 0$, then solution of system (2) is locally stable:*

$$\mathbf{x}(t_0) \in \mathrm{X}_l, \; \mathrm{X}_l = \{\mathbf{x} \in R^n : |\mathbf{x}| \leq K\} \Rightarrow \\ \Rightarrow |\mathbf{x}(t, \mathbf{x}(t_0), 0)| \leq \rho(|\mathbf{x}(t_0)|), \; \rho \in \mathcal{K};$$

and locally output asymptotically stable, i.e. for all $t_0 \geq 0$ and $\mathbf{x}(t_0) \in \mathrm{X}_\rho$, $\mathrm{X}_\rho = \{\mathbf{x} \in R^n : |\mathbf{x}| \leq \rho(K)\}$ there exists some function $\beta \in \mathcal{KL}$, such, that

$$|\mathbf{y}(t, \mathbf{x}(t_0), 0)| \leq \beta(|\mathbf{x}(t_0)|, t - t_0) \text{ for all } t \geq t_0. \;\blacksquare$$

Necessary and sufficient conditions for forward completeness property can be found in (Angeli, and Sontag, 1999). Local stability property ensures, that solutions of system (2) $\mathbf{x}(t, \mathbf{x}(t_0), 0)$ starting with $\mathbf{x}(t_0)$ at set $\mathrm{X}_l \subseteq \mathrm{X}_\rho$ stay in set X_ρ for all $t \geq t_0$. Let us stress, that Assumption 1 says nothing about optimality of system (2) solution for case of disturbance $\mathbf{d}$ absence. In property B of Assumption 2 function β reflects character of output of system (2) decreasing to zero with initial conditions lying in set X_ρ. It is supposed, that such decreasing is optimal in some sense, but for our proofs such supposition is not important one and it is not introduced explicitly. For example, control $\mathbf{u}_l$ can be obtained as a solution of optimal control task for linearized model of system (1) with $\mathbf{d} = 0$ (Fradkov, *et al.*, 1999).

A s s u m p t i o n 2. *There exist a constant $k \geq 0$ and a continuous global control $\mathbf{u}_g : R^n \to R^p$, such, that system*

$$\dot{\mathbf{x}} = \mathbf{f}_g(\mathbf{x}, \mathbf{d}) = \mathbf{f}(\mathbf{x}, \mathbf{u}_g(\mathbf{x}), \mathbf{d}), \; \mathbf{y} = \mathbf{h}(\mathbf{x}), \quad (3)$$

possesses the following properties for any $t_0 \geq 0$, initial conditions $\mathbf{x}(t_0) \in R^n$ and $\mathbf{d} \in \mathcal{M}_q$:

A. *Practical global stability (GS), i.e. there exists some function $\sigma \in \mathcal{K}$, such, that*

$$|\mathbf{x}(t, \mathbf{x}(t_0), \mathbf{d})| \leq \sigma(|\mathbf{x}(t_0)|) + \sigma(\|\mathbf{d}\|) + k.$$

B. *Practical asymptotic gain (AG) with some function $\gamma \in \mathcal{K}_\infty$:*

$$\overline{\lim_{t \to +\infty}} |\mathbf{x}(t, \mathbf{x}(t_0), \mathbf{d})| \leq \gamma(\|\mathbf{d}\|) + k,$$

or equivalently

$$\forall \varepsilon > 0 \; \forall \kappa > 0 \;\; \exists T = T(\varepsilon, \kappa, \mathbf{d}) \geq t_0 \; : \; |\mathbf{x}(t_0)| \leq \kappa, \\ \sup_{t \geq T} |\mathbf{x}(t, \mathbf{x}(t_0), \mathbf{d})| \leq \gamma(\|\mathbf{d}\|) + k + \varepsilon, \; \mathbf{d} \in \mathcal{M}_q. \;\blacksquare$$

Thus, global control $\mathbf{u}_g$ provides boundedness of system solution (property A of Assumption 2) and attractiveness (property B of Assumption 2) of some neighborhood of the origin

$$\mathrm{X}_g = \{\mathbf{x} \in R^n : |\mathbf{x}| \leq k\}$$

uniformly with respect to initial conditions. Term "practical" used above in Assumption 2 formulation means that both properties A and B hold with some static error k.

R e m a r k 1. From another point of view, control $\mathbf{u}_g$ provides for system (3) input-to-state stability (ISS) property with respect to compact set X_g, if it is possible to reformulate Assumption 2 as GS and AG properties for set X_g:

$$|\mathbf{x}(t, \mathbf{x}(t_0), \mathbf{d})|_{\mathrm{X}_g} \leq \sigma(|\mathbf{x}(t_0)|_{\mathrm{X}_g}) + \sigma(\|\mathbf{d}\|), \\ \lim_{t \to +\infty} |\mathbf{x}(t, \mathbf{x}(t_0), \mathbf{d})|_{\mathrm{X}_g} \leq \gamma(\|\mathbf{d}\|).$$

In paper (Sontag, and Wang, 1996) equivalence of properties GS and AG to ISS property with respect to non empty closed compact set was proven. It is easy to see, that property B from Assumption 2 exactly coincides with AG property with respect to set X_g, but GS property does not follow from property A in Assumption 2. The main difference consists in dependence from initial conditions, while in Assumption 2 a boundedness by state space vector is supposed, GS property needs boundedness with respect to distance to the set. Note also, that if AG property with respect to set X_g introduced in Assumption 2 holds uniformly with respect to initial distances to the set and input disturbances:

$$\forall \varepsilon > 0 \; \forall \kappa > 0 \;\; \exists T = T(\varepsilon, \kappa) \geq t_0 \; : \; |\mathbf{x}(t_0)|_{\mathrm{X}_g} \leq \kappa, \\ \sup_{t \geq T} |\mathbf{x}(t, \mathbf{x}(t_0), \mathbf{d})|_{\mathrm{X}_g} \leq \gamma(\|\mathbf{d}\|) + \varepsilon, \; \forall \; \mathbf{d} \in \mathcal{M}_q,$$

then such property is called uniform asymptotic gain and it is equivalent to ISS property with respect to set X_g (Sontag, and Wang, 1996).

Above discussion deals with necessary link from Assumption 2 to ISS property with respect to some compact set, but it is clear that sufficient relation holds without any additional requirements. In other words, if some control ensures for system (1) ISS property with respect to set X_g, then it can be considered as a global control $\mathbf{u}_g$. Necessary and sufficient conditions for ISS Control Lyapunov function existence with respect to the origin for affine in controls nonlinear systems were presented in paper (Liberzon, *et al.*, 2001), where also continuous control law provided such property was developed. Sufficient conditions for input-to-output stabilization and controls were obtained in (Efimov, 2002), that result also can be used to calculate control $\mathbf{u}_g$. ■

The basic desired properties of uniting controller, that should be synthesized for system (1) starting from Assumptions 1 and 2, can be formulated as follows for any $t_0 \geq 0$, $\mathbf{x}(t_0) \in R^n$ and $\mathbf{d} \in \mathcal{M}_q$:

– practical GS;
– output global asymptotic stability (oGAS) for case $\mathbf{d}(t) \equiv 0$, $t \geq 0$;
– there exists a neighborhood Ξ of the origin, such, that uniting controller equals to control $\mathbf{u}_l$ on this set.

The first two goals fix requirements to stability properties of closed loop system, while the last goal ensures optimality property for system solution near the origin.

3. MAIN RESULTS

If we can directly measure external disturbance $\mathbf{d}$ or simply detect its presence in the system, the following simple switching or so-called supervision algorithm solves proposed uniting task for $k < K$: if disturbance presence is detected or $\mathbf{x}(t) \notin X_l$, then $\mathbf{u}(\mathbf{x}) = \mathbf{u}_g(\mathbf{x})$, and $\mathbf{u}(\mathbf{x}) = \mathbf{u}_l(\mathbf{x})$ otherwise. Indeed control $\mathbf{u}_g$ is connected to system (1) always, except to case of disturbances absence and then system trajectory enters a small neighborhood of the origin. In such situation property A of Assumption 2 provides GS property for the system and property B with $k < K$ ensures, that there exists a finite time instant, such, that solution reaches for set X_l if disturbance is absent. Local control $\mathbf{u}_l$ is linked with plant only on set X_l with $\mathbf{d}(t) \equiv 0$, $t \geq 0$, when it provides convergence of output $\mathbf{y}$ to zero. Combining this oLAS property with GS property and global attractiveness of set X_l received by $\mathbf{u}_g$ control, we obtain oGAS for case $\mathbf{d}(t) \equiv 0$, $t \geq 0$. It is clear, that in this situation $\Xi = X_l$.

But due to problem formulation, disturbance $\mathbf{d}$ is not measured, and such algorithm needs a modification, that deals with task of indirect disturbances detecting. Control $\mathbf{u}_g$, due to property B of Assumption 2, for case of disturbance $\mathbf{d}$ absence provides attractiveness of set X_g:

$$\forall \varepsilon > 0 \ \forall \kappa > 0 \ \ \exists T = T(\varepsilon, \kappa, 0) \geq t_0 \ :$$
$$|\mathbf{x}(t_0)| \leq \kappa \,, \ \sup_{t \geq T} |\mathbf{x}(t, \mathbf{x}(t_0), 0)| \leq k + \varepsilon \,.$$

If $k < K$, then there exists some $\varepsilon > 0$, such, that $k + \varepsilon = K$ and

$$\forall \kappa > 0 \ \ \exists T_1 = T_1(K, \kappa, 0) \geq t_0 \ :$$
$$|\mathbf{x}(t_0)| \leq \kappa \,, \ \sup_{t \geq T} |\mathbf{x}(t, \mathbf{x}(t_0), 0)| \leq K \,. \qquad (4)$$

Property (4) means that if disturbance $\mathbf{d}(t) \equiv 0$ for all $t \geq t_0$, then there exists a finite time instant $T_1 \geq t_0 \geq 0$ uniformly with respect to initial conditions on compact sets, such, that trajectory of system (3) is entering into set X_l and stays there. Further switching to control $\mathbf{u}_l$ is desirable. Such property can be utilized for indirect disturbance $\mathbf{d}$ presence detecting. Of course it is possible a situation, then trajectory reaches for set X_l, but disturbance $\mathbf{d}$ is acting on the system and switching to control $\mathbf{u}_l$ causes instability of system solution. According to Assumption 1 control $\mathbf{u}_l$ does not guarantee any boundedness of system solution for system (1) with disturbance, only forward completeness can be obtained. Such critical situation will be analyzed later, during proof of our main result. Before it we should make some remarks about solutions existence in switching systems with disturbances.

Frequently, in switching systems under acting of disturbances a strange behaviour can arise, which is called chattering regime. Such chattering regime originates from fast switching that can take place in the system due to disturbances presence. Classical definitions of differential equations or differential inclusions solution do not suit well for dynamical system in chattering regime, hence, some special methods are used in logic-based switching control theory to prevent such regime arising. In this work we will firstly use so-called dwell time technique (Morse, 1995), which is traditionally used only in linear switching systems due to finite time escape phenomena in nonlinear systems.

So, supervision algorithm can be described as follows:

$$i(t) = \begin{cases} i(t_j) \text{ if } \tau < \tau_D \,; \\ \begin{cases} l, & \text{if } \mathbf{x}(t) \in X_l \,; \\ g, & \text{if } \mathbf{x}(t) \notin X_\rho \,; \\ i(t_j), & \text{otherwise;} \end{cases} \text{ if } \tau \geq \tau_D \,; \end{cases} \qquad (5)$$
$$\dot{\tau} = 1, \ \tau(t_j) = 0,$$

where auxiliary variable τ represents internal supervisor timer dynamics, $\tau_D > 0$ is dwell time constant and t_j, $j = 0, 1, 2, \ldots$ are moments of switching (moments then signal $i(t)$ changes its value), j is number of the last switching. The operating of algorithm (5) can be explaining in the following way: after each switching internal timer τ is initialised to zero, while $\tau < \tau_D$ signal $i(t)$ can not change its value. Dwell time presence in algorithm (5) help us to prevent fast switching arising in the system. Then dwell time is over, signal $i(t)$ can be set up to *l*, if a trajectory reaches for set X_l; signal $i(t)$ would be set up to *g*, if variable $\mathbf{x}$ leaves set X_ρ. From Assumption 1 solutions of system (2) stay in set X_ρ starting in X_l for $\mathbf{d}(t) \equiv 0$, $t \geq t_0$, thus, if control $\mathbf{u}_l$ can not provide forward invariant property of set X_ρ, it means that disturbances are acting on the system and control $\mathbf{u}_g$ have to be switched on. There are two situations with $i(t) = l$ or $i(t) = g$ and $\mathbf{x}(t) \in \mathcal{N}$, where $\mathcal{N} = X_\rho / X_l$, then switching is not needed and signal $i(t)$ saves its current value. Indeed, situation with $i(t) = l$ and $\mathbf{x}(t) \in X_\rho$ is admissible for local controller for case of disturbance absence. Then $i(t) = g$ and $\mathbf{x}(t) \in \mathcal{N}$ we should wait to switch on local controller until trajectory crosses set $\mathcal{N}$ due to workability property of local controller was claimed in Assumption 1 only for trajectories with initial conditions belonged to set X_l. The main properties of such uniting controller are substantiated in the following theorem.

T h e o r e m 1 . *Let Assumptions 1 and 2 be true and* $k < K$. *Then system*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}_i(\mathbf{x}), \mathbf{d}), \ \mathbf{y} = \mathbf{h}(\mathbf{x}), \ i \in \{l, g\} \qquad (6)$$

with supervision algorithm (5) possesses the following properties for all $\mathbf{x}(t_0) \in R^n$ *and* $t_0 \geq 0$:

1. *Global stability with respect to set* X_g *for any* $\mathbf{d} \in \mathcal{M}_q$:

$$|\mathbf{x}(t)|_{X_g} \leq \sigma(R(\mathbf{x}(t_0), \mathbf{d})) + \sigma(\|\mathbf{d}\|),$$

$$R(\mathbf{x}(t_0), \mathbf{d}) = \max\left\{ \begin{array}{l} \chi_1(\tau_D) + \chi_2(\rho(K)) + \\ + \chi_3(\tau_D \, \chi_4(\|\mathbf{d}\|)) + \chi, \ \mathbf{x}(t_0) \end{array} \right\}.$$

2. *Global output attractiveness for* $\mathbf{d}(t) \equiv 0$, $t \geq t_0$:
$\forall \, \varepsilon > 0 \ \exists \, T_y = T_y(\mathbf{x}(t_0)) : |\mathbf{y}(t)| \leq \varepsilon$ *for all* $t \geq T_y$.

P r o o f . Firstly, from definition of supervision algorithm (5), on any time interval $[T_s, T_e)$ with $T_s \geq t_0 \geq 0$, $T_e > T_s$, a finite number of switches $N[T_s, T_e)$ is possible and the following upper estimate holds:

$$N[T_s, T_e) \leq 1 + \frac{T_e - T_s}{\tau_D}.$$

Between switches uniting controller is continuous and equals to local $\mathbf{u}_l$ or global $\mathbf{u}_g$ controllers, which are continuous functions for all $\mathbf{x} \in R^n$. Thus resulting control is piecewise continuous function of time and solution of system (5), (6) is continuous and defined at the least locally $[t_0, T)$. Secondly, let us investigate stability properties of local and global controllers separately.

C l a i m 1 . *Let* $t \in \Omega_1$, $\Omega_1 = \{t \in [t_0, T) : i(t) = l\}$. *Then all trajectories are bounded:*

$$|\mathbf{x}(t)| \leq \chi_1(\tau_D) + \chi_2(\rho(K)) + \chi_3(\tau_D \, \chi_4(\|\mathbf{d}\|)) + \chi.$$

P r o o f . If $t \in \Omega_1$ then initial conditions always belong to set X_l and according to Assumption 1 system is forward complete. By definition of switching algorithm (5), $t \in \Omega_1$ while solution stays in set X_ρ. If a trajectory leaves set X_ρ, then after time interval with maximum length τ_D supervisor (5) switches on global controller. Hence proposed upper estimate should hold for the system in this case.

C l a i m 2 . *Let* $t \in \Omega_2$, $\Omega_2 = \{t \in [t_0, T) : i(t) = g\}$. *Then all trajectories are bounded*

$$|\mathbf{x}(t)|_{X_g} \leq \sigma(R(\mathbf{x}(t_0), \mathbf{d})) + \sigma(\|\mathbf{d}\|).$$

P r o o f . According to property A of Assumption 2 solution in this case is bounded in the following manner:

$$|\mathbf{x}(t)| \leq \sigma(|\mathbf{x}(t_j)|) + \sigma(\|\mathbf{d}\|) + k,$$

where t_j, $j = 0, 1, 2, ...$ are instants of time, then global controller was plugged to the plant (1). For $j > 0$, $\mathbf{x}(t_j)$ by continuity property admits estimate from Claim 1, thus the last estimate can be rewritten as follows:

$$|\mathbf{x}(t)| \leq \sigma(R(\mathbf{x}(t_0), \mathbf{d})) + \sigma(\|\mathbf{d}\|) + k,$$

from that claimed inequality can be obtained.

It is obvious, that $[t_0, T) = \Omega_1 \cup \Omega_2$, but solution of switching system is bounded for all Ω_z, $z = 1, 2$, thus, $T = +\infty$ and the following inequality is satisfied for all $t \geq 0$:

$$|\mathbf{x}(t)|_{X_g} \leq \sigma(R(\mathbf{x}(t_0), \mathbf{d})) + \sigma(\|\mathbf{d}\|).$$

Indeed, if estimate from Claim 2 holds, then estimate from Claim 1 is also satisfied, due to property $\sigma(s) \geq s$.

The last thing, that we should base is output attractiveness for system (5), (6) for case of disturbance $\mathbf{d}$ absence. Suppose, that there exists some $T_d \geq t_0$, such, that $\mathbf{d}(t) \equiv 0$ for all $t \geq T_d$. Then two situations are possible. First $i(T_d) = g$, then according to property (4) there exists a switching time instant T_l, such, that $\mathbf{x}(T_l) \in X_l$ and signal i is switched to position l. Then basing on property B from Assumption 1 one can conclude, that system possesses output attractiveness property. Second $i(T_d) = l$. Then either $\mathbf{x}(T_d) \in X_\rho$ and trajectory leaves set X_ρ, in such case there exists a switching time instant T_l, such, that $\mathbf{x}(T_l) \in X_l$ and signal i is switched to position l; or trajectory stays in set X_ρ for all $t \geq T_d$ and according to property B of Assumption 1 local attractiveness of output $\mathbf{y}$ is obtained. All statements of the theorem are proved. ■

Proposed in Theorem 1 solution does not handle third property listed in control goals in Section 1. It is hard to specify some neighborhood of the origin, where uniting controller equals to local one. It is a consequence of dwell time technique, indeed, starting outside of set X_ρ a trajectory of system (3) with global controller can in general reaches the origin for time less than τ_D.

It is worth to note, that forward completeness property stated in Assumption 1 and dwell time were not really needed in proof of Theorem 1. Forward completeness property was imposed to guarantee existence of system (2) solutions for $\tau \le \tau_D$. In fact, forward completeness requirement can be weakened to merely existence of system (2) solution for all initial conditions from set X_ρ on finite time interval with length is bigger than τ_D. It is obvious, that proof of Theorem 1 can be used in such way without any modifications. Compact set $\mathcal{N}$ plays a role of buffer, where trajectories of system (2) spend time while crossing this set, such time depends on L_∞ norm of input $\mathbf{d}$ and can be considered as a dwell time analogue for the system. Such technique was used in (Prieur, 2001; Prieur, and Praly, 1999) for construction of uniting controllers for case of external disturbances $\mathbf{d}$ absence and it is called hysteresis switching in logic-based control theory (Morse, 1995):

$$i(t)=\begin{cases} l, & \text{if } \mathbf{x}(t)\in X_l;\\ g, & \text{if } \mathbf{x}(t)\notin X_\rho;\\ i(t_j), & \text{otherwise,}\end{cases} \quad (7)$$

where t_j, $j=1,2,3,...$ are moments of switching, j is number of the last switching. As in algorithm (5), the system is equivalent to (2) on set X_l and to system (3) on set R^n / X_ρ. Term “otherwise” in such context means that signal $i(t)$ is not changing on set $\mathcal{N}$. Set $\mathcal{N}$ prevents fast switching arising and due to its compactness helps to avoid problem of unboundedness of system (2) solutions for case of disturbance $\mathbf{d}$ presence. Hence, it is possible to modify Assumption 1 as follows.

A s s u m p t i o n 3. *There exist a constant* $K>0$ *and a continuous local control* $\mathbf{u}_l : R^n \to R^p$, *such, that system* (2) *with* $\mathbf{d}(t)\equiv 0$ *for all* $t\ge 0$ *admits the following properties*:

A. *Solution of system (2) is locally stable:*

$$\mathbf{x}(t_0)\in X_l \Rightarrow \mathbf{x}(t)\in X_\rho .$$

B. *Solution of system (2) is locally output asymptotically stable, i.e. for all* $t_0\ge 0$ *and* $\mathbf{x}(t_0)\in X_\rho$, *there exists some function* $\beta\in\mathcal{KL}$, *such, that*

$$|\mathbf{y}(t,\mathbf{x}(t_0),0)| \le \beta(|\mathbf{x}(t_0)|, t-t_0) \text{ for all } t\ge t_0 . \ ■$$

In the last assumption all requirements to system (2) deal with its unperturbed by disturbance version. Such modification allows to simplify problem of local controller construction for system (1).

T h e o r e m 2. *Let Assumptions 2 and 3 be true and* $k<K$, $\rho(s)>s$. *Then system* (6) *with supervision algorithm* (7) *possesses the following properties for all* $\mathbf{x}(t_0)\in R^n$ *and* $t_0\ge 0$:

1. *Global stability with respect to set* X_g *for any* $\mathbf{d}\in\mathcal{M}_q$:

$$|\mathbf{x}(t)|_{X_g} \le \sigma(R(\mathbf{x}(t_0)))+\sigma(\|\mathbf{d}\|),$$
$$R(\mathbf{x}(t_0))=\max\{\rho(K),\ \mathbf{x}(t_0)\}.$$

2. *Global output attractiveness for* $\mathbf{d}(t)\equiv 0$, $t\ge t_0$: $\forall\, \varepsilon>0\ \exists\, T_y=T_y(\mathbf{x}(t_0))$: $|\mathbf{y}(t)|\le\varepsilon$ *for all* $t\ge T_y$.

3. *For all* $t\ge t_0$:

$$\mathbf{x}(t)\in X_l \Rightarrow i(t)=l .$$

P r o o f. The main steps of proof of the present theorem coincide with the same from proof of Theorem 1. Differences are originated by implicit dwell time technique using. Let us say that $\mathbf{d}\in\mathcal{M}_q^r$, if $\mathbf{d}\in\mathcal{M}_q$ and additionally $\|\mathbf{d}\|\le r$, $r\in[0,+\infty)$. Then it is possible to introduce an upper bound for right hand sides of systems (2) and (3) on set $\mathcal{N}$ as follows:

$$F(r)=\max\left\{\begin{array}{c}\sup\limits_{\mathbf{x}\in\mathcal{N},\mathbf{d}\in\mathcal{M}_q^r}\{\mathbf{f}_l(\mathbf{x},\mathbf{d})\},\\ \sup\limits_{\mathbf{x}\in\mathcal{N},\mathbf{d}\in\mathcal{M}_q^r}\{\mathbf{f}_g(\mathbf{x},\mathbf{d})\}\end{array}\right\}.$$

Number $F(r)$ always exists and finite due to continuity property of functions $\mathbf{f}_l$, $\mathbf{f}_g$ and compactness of sets $\mathcal{N}$ and $\mathcal{M}_q^r$, it can infinitely increase with $r\to+\infty$, but it always stays finite for finite r. Hence, the smallest time $\tau_r>0$ that systems (2) or (3) need to cross set $\mathcal{N}$ can be estimated in the following manner:

$$\tau_r=\frac{\rho(K)-K}{F(r)} .$$

Further one can repeat all steps of Theorem 1 proof with substitution $\tau_D=\tau_r$. On any time interval $[T_s,T_e)$ with $T_s\ge t_0\ge 0$, $T_e>T_s$, a finite number of switches $N_{[T_s,T_e)}$ is possible and the following upper estimate holds:

$$N_{[T_s,T_e)}\le 1+\frac{T_e-T_s}{\tau_r} .$$

Between switches uniting controller is continuous and equals to local $\mathbf{u}_l$ or global $\mathbf{u}_g$ controllers, which are continuous for all $\mathbf{x}\in R^n$. Thus resulting control is piecewise continuous function of time and solution of system (6), (7) is continuous and defined at the least locally $[t_0,T)$.

C l a i m 3. *Let* $t\in\Omega_1$. *Then* $\mathbf{x}(t)\in X_\rho$.

P r o o f. The statement simply follows from definition of algorithm (7).

C l a i m 4. *Let* $t\in\Omega_2$. *Then all trajectories are bounded*

$$|\mathbf{x}(t)|_{X_g} \le \sigma(R(\mathbf{x}(t_0)))+\sigma(\|\mathbf{d}\|).$$

P r o o f. According to property A of Assumption 2 solution in this case is bounded:

$$|\mathbf{x}(t)| \le \sigma\left(\left|\mathbf{x}\left(t_j\right)\right|\right) + \sigma\left(\|\mathbf{d}\|\right) + k,$$

where t_j, $j = 0,1,2,...$ are instants of time, then global controller was plugged to the plant (1). For $j > 0$, $\mathbf{x}\left(t_j\right)$ by continuity property admits estimate from Claim 3 and

$$|\mathbf{x}(t)| \le \sigma\left(R\left(\mathbf{x}\left(t_0\right)\right)\right) + \sigma\left(\|\mathbf{d}\|\right) + k.$$

It is clear, that $\left[t_0, T\right) = \Omega_1 \cup \Omega_2$ and according to Claims 3 and 4 solution of system (6), (7) is bounded, thus, $T = +\infty$ and the following inequality is satisfied for all $t \ge 0$:

$$|\mathbf{x}(t)|_{X_g} \le \sigma\left(R\left(\mathbf{x}\left(t_0\right)\right)\right) + \sigma\left(\|\mathbf{d}\|\right).$$

Output attractiveness for system (6), (7) for case of disturbance $\mathbf{d}$ absence can be obtained by the same line of consideration as in proof of Theorem 1 taking into account Assumption 3. The third point of the theorem is a corollary of supervisory algorithm (7) properties. ■

Due to results of Theorem 2 system (6), (7) possesses all requirements posed in control goals. In practice applications $\mathbf{d} \in \mathcal{M}_q^r$ with $r \le R < +\infty$ and with proper choice of function ρ (which defines the size of set $\mathcal{N}$) it is possible to obtain desired minimal value of time τ_r for any $\mathbf{d} \in \mathcal{M}_q^R$. Such minimal value of time τ_r is needed in practice to realize supervisory algorithm (7) due to it includes a part working in discrete time (supervision algorithm (7)) and its implementation can be performed only with some minimum time step.

R e m a r k 2 . Let global controller provide practical stability and attractiveness properties, stated in Assumption 2, only on some finite time interval, i.e. there is a time instant $t_1 > t_0$, such, that inequalities

$$|\mathbf{x}(t, \mathbf{x}(t_0), \mathbf{d})| \le \sigma\left(\left|\mathbf{x}\left(t_0\right)\right|\right) + \sigma\left(\|\mathbf{d}\|\right) + k;$$

$$|\mathbf{x}(t_1, \mathbf{x}(t_0), \mathbf{d})| \le k$$

hold for $t_0 \ge 0$, initial conditions $\mathbf{x}\left(t_0\right) \in R^n$, $\mathbf{d} \in \mathcal{M}_q$, $t \in \left[t_0, t_1\right]$, $\sigma \in \mathcal{K}$ and $k \ge 0$. In other words, global controller ensures, that from any initial conditions solution stays bounded on some finite time interval $\left[t_0, t_1\right]$ and in the end of this time interval trajectory enters into set X_l; in common case for $t > t_1$ system can produce unstable solution. Then it is enough to prove a version of Theorem 2 with such requirements imposed on global controller, due to into set X_l system always should be switched to local variant. ■

4. CONCLUSION

The paper develops a solution of uniting controller synthesis problem for case of essentially bounded disturbances presence.

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