ADVANCE THEORETICAL MECHANICS OF GALILEAN

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Abstract

The Monograph contains description for fundamentals of Galilean mechanics, under conditions rejecting discreteness of «analytic» mechanics rejecting presence rotating and deformation particles in misconnected locally continuous media. The main properties of Galilean mechanics Universe are formulated as axioms of solidity for 6-demensional vector and scalar measures. This leads principally new architecture of Galilean mechanics, as geometry of generalize Galilean group.

For mathematicians and mechanicals.

Preface

Conventional mechanics of continuum media, a rigid body and systems of rigid bodies is based on Newtonian axiomatics (formulated for finite sets of points without volume, but with concentrated masses - not existing in the nature), on analytical mechanics and so-called integral theorems of changing momentum of movement of allocated volumes of continuum medium, their moments and energy. One may prove these theorems for the points mentioned above, but it is not possible mathematically strictly for continuum media.

The problem of constructing the continuum mechanics without using these theorems is put in the given book. Naturally, 'game rules' with finite and uncountable infinite sets are essentially different, the architecture of the theoretical mechanics constructed on these bases essentially differs from the traditional one.

The mechanics worked out here is defined as the geometry (invariant) of Galilean generalized group, *i.e.*, the group of 6–dimensional shifts with constant vectors or the vectors being products of the constant velocity of points and time, and the constant rotations in the screw vector space. That is why this mechanics is called that of Galileo.

Galilean mechanics is worked out on axioms. As they are elements of Galilean mechanics, *i.e.*, they are invariant w.r.t. the generalized Galilean group, then all results mathematically obtained on its basis are its elements also.

As result the mechanics of Galileo is constructed without ideologically its division into traditional parts: that of continuum medium, absolutely rigid body and systems of rigid bodies. The two last parts are constructed on the same bases, but in view of specific requirements. Their working tools are focused on direct use of computers on the basis of modern software in order to solve applied problems more economically (in the computing relation) than earlier (at 15-20 times).

In every theoretical investigation of a phenomenon (and in mechanics, too) we are always forced to idealize its properties. For a certain phenomenon, theoretically, there are many models. To study the admissibility of any model of a certain phenomenon we must set that its properties are not in contradictory with the experience or the theory constructed for a more common model (Andronov *et al.* 1937). That is why we may use Newtonian and Galilean mechanics or mechanics of Minkovsky depending on problems under consideration.

Basically, the monograph is aimed to acquaint mathematicians with the theoretical mechanics developed by its authors as Galilean mechanics. At the same time it is not forgotten about interests of professional mechanicians. For this reason Chapter 1 and detailed Comments are included in the monograph (they can be omitted by the readers who are not possessing sufficient experience or preparation in the field of mechanics).

In the last decades many monographs have been published, where the authors vest the standard ideology of mechanics construction (Buhgoltz 1969) with the modern mathematical language (*e.g.*, Arnold 1989, Truesdell 1972) in order to do mechanics more modern or contemporary. Surely, it is not natural to present the scientific literature of the 21st century (including the mechanical one) in terms of the applied mathematics of the 19th century. However, it is necessary to mark two essential circumstances. First, this is the inertia of the mechanical education (note that this is not a problem of the pupils but of their teachers). Second, the language should not be the aim of its own, but the main point is the ideas pursued.

It would be better if the reader, no matter what he thinks about these questions, in the psychological attitude he has prepared himself to understand that the given monograph has nothing at all in common with the monographs mentioned above and with similar ones. This monograph is devoted to a discussion of the proposed axiomatics and of the new architecture of Galilean mechanics that follows from it. The language corresponds to the problems but it could be different without doing any harm to the content, *i.e.*, the apparatus of Lee groups and algebras (Nikulin *et al.* 1983, Suprunenko 1972), of differential geometry (Vershik *et al.* 1975, Milnor 1965, Wallace 1968, Hirsch 1976), and so on.

In the ideological attitude the monograph does not include anything, that it would be possible to find in the well-known courses concerning the discussed questions (Buhgoltz 1969, Kochin *et al.* 1964, Lur'e 1961 and 1970, Sedov 1972). But it includes many things that make possible for one to understand the problems of the classical mechanics and the solution of applied problems in the beginning of 21st century. Naturally, in the monograph there is no fantasy, regarding 'wonderful' phenomena, paradoxes and related with them attempts to invent 'new' mechanics. Surely, the mathematically intelligent reader has to verify all these points. This is not very difficult: all the results are formulated correctly and are proved. But if no principal contradiction would found, the same reader will agree that many results known to him/her from (Buhgoltz 1969, Kochin *et al.* 1964, Lur'e 1961 and 1970, Sedov 1972) should be reconsidered, deeply and thoroughly.

As for the works of some specialists in the field of continuous media, one could find useful the mutual symbiosis between the results of this monograph and some results of the group of W. Noll, B.D. Coleman, in particular C. Trusdell (Truesdell 1972).

We usually use:

- lower-case letters for numbers, variables, vectors;
- capital letters for matrices, operators;
- bold letters for sets, spaces;
- calligraphic letters for groups and aggregated objects;
- $x^{\cdot}$ means the time derivative of a function $x = x(t)$.

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Chapter 1

Introduction

The introduction to a monograph with such a title could be a separate work including a detailed analysis of the state of the discussed questions here. The real attempts to create such a text have led to several variants of a vast material whose main defect is the fact that this material could be understood by specialists with the traditional education in the field of the mechanics not 'before' but after reading the monograph. Thus, there is no sense in making it. That is why the decision is accepted to abridge this part to a minimum and to put more detailed discussions of particular questions in the comments relevant to the main statements of the theory.

In the monograph, it is continued with improving the worked out axiomatics of Galilean mechanics (Konoplev 1996a) that principally differs from the known ones by the following.

1. A mathematical model of a medium is postulated to be a measured by Lebesgue pointwise absolutely continuous w.r.t. Lebesgue measure (further on – only continuous) continuum with a directing oriented Euclidean vector space, and the relation between them is determined (as it has to be) by three separate axioms (17 axioms of G. Weil as a whole) (Dieudonne 1972, Yaglom 1980, Weyl 1950).

It is proved that seemingly equivalent notions characterizing the medium, such as 'continuity', connectedness ('uniformity') or 'continualness' (Buhgoltz 1969, Kochin *et al.* 1964, Lur'e 1961, Lur'e 1970, Sedov 1972), describe different properties of the medium. By definition, here a medium is continual if the pointwise set mentioned above has the power of continuum. For example, the medium of the analytical mechanics is not continual (Lur'e 1961).

At the same time a medium is not obliged to be continuous or connected (for example, the medium of a perfect Cantor set kind (Kolmogorov *et al.* 1957)).

The concept of 'continuity' of a medium is not related to the medium itself (the pointwise continuum). This property is completely determined by the distribution of the scalar and vector measures of the mechanics on the continuum. The notion of 'entirety, uniformity' of a medium is equivalent to the requirement for conservation (in time) of the scalar measures of inertia and gravity on σ–algebra of the mechanical systems. This notion is not directly related to the property of 'continuity'. This confusion is clarified to a certain degree by the postulate of the disconnection property of the Galilean mechanics Universe postulated in the monograph.

2. A vicinity of any point of the continuous continuum is constructed where the transformation of medium differs a little from the linear transformation of the corresponding directrix vector space. There exists one–to–one correspondence between all linear approximations of the medium transformations in the vicinity mentioned above and the elements of some subgroup of the complete linear group. This four–parameter group of matrices is the group of the kinematical deformators (k–deformators) that are kinematical mathematical models of the linear approximations of the medium motion in the vicinity of each point which performs a shift w.r.t. the basic (inertial, if needed) frame.

The matrices pointed out, *i.e.*, solutions of some matrix differential equations on the k–deformator group, are the unique bearer of information about the medium motion in the each point vicinity. The continuous medium model is an unconnected locally continuous continuum of points (Bourbaki 1966, Kolmogorov *et al.* 1957, Natanson 1955), where in correspondence with the reality there are neither isolated points (Buhgoltz 1969, Lur'e 1961) nor subsets which could be treated in any physical sense as rotating 'particles' and after that deformable 'particles', nor 'material points' considered as mechanical systems whose 'dimensions' could be ignored.

The mechanical systems, traditionally called absolutely rigid, are considered only as a continuous continual medium with some additional properties of thransforming (Konoplev 1996a).

In the continuum mechanics as a whole and in the locally continuous one in particular, there exist its own 'rules' that have no analogy with the mechanics of the finite set of points (*i.e.*, the analytical mechanics) or with the deformable 'particles' mechanics. For example, the question related to the transfer of the theorems about 'momentum change and moment of momentum change' is not discussed (proved for finite sets of points using second Newton's law (Buhgoltz 1969, Lur'e 1961) on the (non–existing in the real continual medium) deformable 'parts' whose boundaries are nowhere differentiable or do not exist at all). If some definite properties of the Galilean mechanics Universe are imposed the necessity of such theorems for the continuous medium does not arise at all. It means that the theorems are not 'laws of the nature' but elements of an unsuitable mechanics axiomatics determining 'properties of the nature'. For example, the statement about the symmetry of the tension 'tensor', that follows from the 'integral form of the theorem of moment of momentum conservation of a continuous medium' only, has neither mathematical foundations, nor physical ones.

3. The following statement plays the main role in the monograph: any 'nature laws' for a locally continuous continual medium, used as primary properties of the Galilean mechanics Universe, could be formulated only as axioms, only as statements about a balance of the densities of the corresponding vector and scalar mechanics measures defined on σ–algebra of the medium subsets, and only in the inertial frame.

Moreover, we do not use such 'habitual' but really meager concepts as: 'force applied at a point' (here it always equals zero), 'point with a concentrated mass' (its mass is always equal to zero), 'angular velocity of a point' (this velocity principally does not exist since the rotations group is only defined on vector spaces whose dimensions are not less then 2), 'momentum (and moment of momentum) of a point', and therefore all Newton axiomatics that is based on the concepts pointed out, as well as all that follows in the train of it (Lagrange equations of first order, the dynamics general equation, Lagrange central equation, the general central equation (Lur'e 1961) and so on).

The traditionally used condition (that has not a physical foundation) for the continuous medium equilibrium (in the form of the rigid body equilibrium condition) is replaced by the condition of equilibrium of a dynamically continuous continuum having the mathematical content (the dynamical measure density w.r.t. Lebesgue measure is equal to zero).

4. The concepts of inertia and gravity masses as scalar measures on σ–algebra of mechanical systems are introduced in the monograph. Such a separation has been made earlier in some cases but then their equality has been postulated (Devis 1982). Here they are affirmed to be proportional with a coefficient equal to the square root of the gravity constant. This enables one to consider again a lot of habitual results of the 'classical' mechanics that concern the concepts of inertia and gravity. The gravity mass (essentially less than the inertial one) becomes a physically tangible concept. Thus, for example, writing the dynamic screw of gravity in the Earth vicinity as a dynamic screw of inertia with acceleration equal to gravity, one does not give a rise of the internal resistance.

5. One of the main positions in the developed theory is occupied by a principally new object of the mechanics, namely, the dynamic screw being 6–dimensional skew–symmetric Radon bi–measure on σ–algebra of mechanical systems (Konoplev 1996a). Making sense to this concept, one may realize the simple fact: there exist no objects in the nature whose mathematical models are 'moments' of any origin (kinematical ones, dynamical ones, *etc.*).

The appearance of the moment as an individual concept is due to the artificial division of 6–dimensional screws into two 3–dimensional parts. Here the matter is not in the outward form of writing: the physical objects (whose mathematical models are dynamical, kinematical and kinetic screws) are 6–dimensional ones in principle. Indeed, the 'moments' are parts of these objects which carry in themselves the information for producing more wide factorization of the free vector space (in particular, defining the 'action line' of the line vector if speaking of geometrical vectors as classes of equivalence of line vectors having the same straight line along which this vector is directed and the same sense and length). Note again that any mathematical objects that appear in one kind of axiomatics and disappear in others being equivalent to the former ones, cannot be mathematical models of a process of any nature.

As a result, Galilean mechanics is defined here as the geometry (invariant) of Galilean generalized group, *i.e.*, the group of 6–dimensional shifts with constant vectors or the vectors being products of the constant velocity of the points and time, and the constant rotations in the screw vector space. Moreover, it is clear that only two mechanics are defined objectively, namely: the above mentioned and the geometry of Lorentz group (Minkovski geometry) (Yaglom 1980).

6. It is shown that in Galilean mechanics there exist no dynamical screws that differ from the inertial, gravitational and deformation ones. Here the mentioned screws are identified with 6–dimensional vector measures of impacts upon mechanical systems (the actions of mechanical system supplements in their σ–algebra).

As to all other known dynamic screws (*i.e.*, fiction, supplement, vibration, surface, reaction ones, *etc.*) there are two possibilities: they do not exist or they are the veiled form of those mentioned above.

7. As for the field of continuous media kinematics, the following results are obtained:

7.1 It is shown that the representation of the velocity of motion of an arbitrary point of a medium 'particle' as a sum of translation, rotation and deformation terms (attributed to Cauchy and Helmholtz) (Kochin *et al.* 1964, Lur'e 1970, Sedov 1972) has neither physical, nor mathematical sense since:

- If the mathematical model of a medium is a locally continuous continuum there are no points with concentrated masses and no 'particles' that could be 'frozen', be rotated, be deformed, *etc.* In this sense, the 'continuous' medium mechanics is not at all a generalization of the rigid body motion concept – the rigid body considered as a 'frozen particle' with added dilation deformation to its shift and rotation. Indeed, quite the contrary is shown in the monograph: the absolute rigid body (Konoplev 1996a) is a particular case of motion of a continuous continual medium with additional properties (in the case of systems being bodies, the group of k–deformators is a group of rotations).
- 'Caughy–Helmholtz formulas' (Kochin *et al.* 1964, Lur'e 1970, Sedov 1972) are not equations in fact, and hence they cannot be mathematical models of a certain physical process. They are represented by the identities where the right-hand part is another notation of the left-hand one. That is why they can neither give any information about the character of medium motion, in general, nor about medium rotation and dilation, in particular.
- There exists an infinite number of decompositions (expansions) of the velocity of an arbitrary point in the vicinity of any point of the medium, and therefore no element of these decompositions is a mathematical model of any of medium transformation (by analogy with the decomposition of a point motion path as series whose elements are not mathematical models of this motion).
- 'Cauchy–Helmholtz formulas' give the relation not between medium transformations but between transformation velocities (w.r.t. coordinates and time). Their form gives no information about medium transformations (as any relation connecting velocities). Such transformations cannot be obtained from the formulas, but only after integrating either velocities or some differential equations with coefficients depending on these velocities.
- With describing the deforming of a 'particle' as a symmetrical matrix (Kochin *et al.* 1964, Lur'e 1970, Sedov 1972), we exclude from consideration any 'shifts' of a medium, since there are no shifts in this matrix in principle.

7.2. There is worked out an algorithm for constructing the group of k–deformators that includes only operators principally being linear approximations of continual medium deformation (as symmetry, mirror image, *etc.*). The group of k–deformators is a four–parameter group (its elements depend on the time and on 3 space coordinates) of matrices that are represented as a sum of the unit matrix and a matrix infinitesimally small at a given point. In some conditions, the last matrix is the medium deformation one at this point.

7.3. Fundamentals of the locally continuous medium kinematics are worked out, they having nothing at all in common with the similar ones that one can find in the traditional works concerning the mechanics of 'connected (continuous) medium' (Kochin *et al.* 1964, Lur'e 1970, Sedov 1972). The studies reduce to a research of the structure of the k–deformator group at an arbitrary point of the medium (Gantmacher 1964, Dieudonne 1972 and 1972, Suprunenko 1972, Skornyakov 1980, Yaglom 1980).

The following points are made and clarified:

- The kinematics equation is constructed on the k–deformator group, *i.e.*, the matrix differential equation that connects k–deformators with their velocities and the derivatives of the medium point velocity w.r.t. the space coordinates. Having the last ones as a result of solving the dynamic part of the problem (or by experiment), the medium k–deformator is found. This k–deformator is a linear mathematical model of the medium motion in the considered vicinity. By the action of this operator on the vectors belonging to the vector space 'local set', the trajectories of medium points are determined, the stage of the preliminary definition of the velocities field in the chosen point vicinity being omitted.
- As in the case of 'Cauchy–Helmholtz formulas' there does not exist a way to determine (understand, see) the character of linear approximation of the medium deforming with the help of the outward form of k–deformators.
- All generatrices of the k–deformator group are obtained and studied. It is proved that there exists an infinite number of expansions of k–deformators in the form of the simplest factors: transvections and dilations (Gantmacher 1964, Dieudonne 1955 and 1969, Konoplev *et al.* 2001, Nikulin 1983, Suprunenko 1972).
- There exists an infinite number of 'enlarged' expansions of k–deformators where the simplest factors mentioned above, as bricks, are arranged in 'blocks'. Such blocks, for example, are Cavalieri groups of first and second type, the elements of Bruhat expansions, *etc.* (Ambartzumjan *et al.* 1989, Dieudonne 1955 and 1969).
- A special attention is paid to k–deformator expansions that include rotations. They are called polar and are products of a rotation matrix and a dilator (a symmetrical matrix with a positive spectrum) (Gantmacher 1964, Dieudonne 1972, Suprunenko 1972, Skornyakov 1980). Such expansions are two in number: left–hand and right–hand ones. These expansions (naturally identical to each other) refer to the class of 'enlarged' ones. If substituting these expansions in the kinematics equations, two expansions of an arbitrary point velocity in a vicinity of an earlier chosen point are obtained on the k–deformator group. None of them coincides with 'Cauchy–Helmholtz formulas', as by the outward form of writing so by the sense: as it should be expected, the velocity of an arbitrary point belonging to a continuous continual medium could not be principally decomposed on 'pure rotation' and 'pure dilation', since each of the two terms in the velocity decomposition includes both rotation and dilation. In other words, no pure rotations and dilations of the continual medium exist: a continuous 'cocktail' consisting of mathematical models of these motions exists, moreover such 'cocktails' are two and an infinite number of other 'cocktails' of other components exist and they are equivalent to the mentioned ones when a superposition is made.

- In both polar expansions, the three–dimensional rotations are (again) decomposed into the simplest rotations. Several such expansions are well known: 'airplane', 'ship', the Euler expansion in the three–dimensional space or in a space of another type (Lur'e 1961), *etc.* Naturally, all they are equivalent, too.
- It is shown that none of the expansions is a mathematical model of medium deformation. They are also specific 'cocktails' of different operators that have no physical sense if separately considered but leading to one and the same linear model of medium transformation.
- Some questions concerning the kinematics of locally continuous medium in particular cases of deforming (as noncircularity, incompressibility and so on) are considered. For example, it is proved that a medium is incompressible if the group of its k–deformators is a subgroup of a special linear group (of matrices with a determinant equal to 1).
- The synthesis of matrices of medium deformations (non–symmetric, in general) and of its velocity is studied. What is obtained is a relation between these matrices and k–deformators, their velocities, the velocity of point velocity change w.r.t. the space coordinates, the displacement vectors, and so on.

In particular, it is shown that the continuous medium deformations are not sums of the partial derivatives of the displacement vector w.r.t. the space coordinates, as it is usually assumed (Kochin *et al.* 1964, Lur'e 1970, Sedov 1972, Truesdell 1972), but they are derivatives themselves. The concepts outlined coincide only in the case of the noncircular medium motion (as a flow, a deformation).

This is the first time that the displacement vector is determined analytically. This permits one to formalize a lot of statements of the theory, and in particular to obtain the analytical condition for the medium continuity in the case of deformation: the condition has nothing at all with Saint–Venant one (Lur'e 1970) that is usually applied in this case. The former condition affirms that the divergence of the displacement vector is a constant, in particular equal to 0.

7.4. A principally new mathematical formalism about the kinematics of an absolute rigid body is worked out (Konoplev 1996a). The complex motion of the rigid body is considered as a motion of the final element of the kinematical chain in terms of kinematical screws and of 6–dimensional quasi–velocities regarding the possible constructive shifts and rotations in the kinematical couples of this chain (Konoplev 1984, Konoplev 1996a). (In the last case) simple matrix formulas, easily realized on computers, are obtained; they have a simple form for any number of intermediate motions (Konoplev 1996a).

7.5. A simple free motion and a motion with the simplest holonomic constraints are considered out as partial cases of the complex motion. Matrix equations of the kinematics over the k–deformators group are obtained (Konoplev *et al.* 2001).

7.6. The foundations of the theory for representing the groups of rigid body rotations in Rodrigues–Hamilton group, Cayley–Klein group and a body of quaternions are worked out (Konoplev 1996a).

8. The following results are obtained in the continuous medium dynamics.

8.1. The divergences of the rows of the matrices of tensions or tensors velocities but not the matrices themselves enter the continuous medium motion equations

(Kochin *et al.* 1964, Lur'e 1970, Sedov 1972, Truesdell 1972). Thus, there is no basis to represent the relations between the pointed tensions and the elements of the deformations matrix (in the case of Hooke–elastic media) and the elements of this matrix derivative (in the case of fluids) in the form of matrix functions of matrix arguments. Besides, when such functions are used (and called constitutive relations), the essential questions about resolving these relations w.r.t. the elements of matrices–arguments (non–symmetric in the most common case) come out of site.

A new concept is introduced in the monograph: 'mechanical state equations' of a medium. In the initial variant they are implicit functions of the elements of the matrices (elements) mentioned above and of the rheological coefficients (of elasticity and of stiffness) that satisfy the following requirements:

- the writing form of the functions has to be invariant w.r.t. the inertial frame choice;
- rheological coefficients values have to be invariant w.r.t. the inertial frame choice;
- implicit functions must be resolved in an unique way w.r.t. any set of variables (tensions, deformations or their velocities).

The first and second requirements guarantee that the equations belong to Galilean mechanics (as an invariant of Galilean group), the third one guarantees the equations correctness (Tihonov *et al.* 1979). If the last requirement is not fulfilled (*i.e.*, the solution does not exist or solutions are more than 1), the mechanical state equations are called incorrect; if the first and second requirements are not satisfied, the medium cannot be studied by means of Galilean mechanics methods.

In other words, the incorrect equations of the medium dynamics appear when one solves jointly the motion equations and the incorrect equations of the mechanical state. The question is to estimate what results can be produced by the integration of the incorrect equations of the continuous medium dynamics. This question requires a separate study (Tihonov *et al.* 1979, Ladyzhenskaya 1987) especially in the cases when (under some propositions conditions) the dynamics equations of correct and incorrect media coincide.

8.2. The partial derivatives of tensions w.r.t. the space coordinates (that come in the divergence of the tensions matrices rows) are calculated with the help of the rules of implicit functions differentiating where the correctness of mechanical state equations plays an important role.

8.3. Every kind of mechanical state equations generates a class of equivalent continuous media. Each class of continuous media has dynamics equations (when one jointly resolves the mechanical state equations and the motion equations) and thermodynamical equations of its own. A large part of correct media classes have not constitutive relations. In other words, the continuous media set is characterized by a two–dimensional array of elements that depend on a media class and on rheological coefficients (of viscosity, of stiffness) values.

The outlined approach to the synthesis of mechanical state equations and consequently to the synthesis of new classes of correct continuous media practically has unlimited possibilities.

8.4. In details are studied the equations which include (among the other terms) quasi–linear combinations of the elements of deformations matrices or their velocities with rheological coefficients depending on the invariants of these matrices, on the space coordinates and on temperature. In this case the requirement of equations correctness is reduced to the requirement that the vector and matrix of rheological coefficients (of viscosity, of stiffness) should belong to some group.

It is found that in this case, similarly to the fact that every geometry is generated by some group ('Erlangenian program' of F. Klein (Dieudonne 1972, Yaglom 1980)), each class of quasi–linear continuous media is also generated by some group of matrices whose elements are rheological coefficients (of viscosity, of stiffness). Having the structure of a full linear group of matrices, we have therefore the structure of possible classes of quasi–linear continuous media.

8.5. At present, it is cleared up that:

- The class of ideal fluids occupies a separate place.
- Six classes of quasi–linear continuous viscous fluids and six classes of quasi–linear elastic materials exist.
- There exist classes of quasi–linear two–dimensional viscous fluids and elastic materials, not having 3–dimensional analogues. This fact is essential for one to understand when 'plane' problems are to be studied (Konoplev 1996a). The existence of such media is explained by the fact that the requirement that medium rheological coefficients should be invariant (w.r.t. 2–dimensional rotations group) is 'weaker' than those referred to 3–dimensional rotations group.
- There exists a class of quasi–linear incorrect media of Navier–Stokes–Lame (Kochin *et al.* 1964, Lur'e 1970, Sedov 1972, Truesdell 1972) (matrices of rheological coefficients are either singular, or rectangular). These media are not generated by any group. Hence, it could follow that they either occupy an essential place among the continuous media, or they are not at all models of real media, being incorrect approximations of correct media.
- The subclasses of the main classes of media are studied under conditions that motion is incompressible and non–circular (potential), and so on.
- The most subtle results are obtained for the subclasses of continuous media whose rheological coefficients of the first type are proportional to divergence of the displacement vector or its velocity. In this case, using the property of correctness of the mechanical state equations and the group structure of the set of matrices of rheological coefficients, it has been possible to obtain in explicit form (as elements of inverse matrices) the fluid viscosity moduli for all correct media, and to obtain for all quasi–linear elastic media the elasticity moduli and then Young moduli, the shift moduli and Poisson coefficients. It is proved that Navier–Stokes–Lame media do not have the above mentioned moduli (defined for the medium non–circular deforming, only) in the general case.
- It is proved that the medium rheological coefficients (of viscosity and of stiffness) do not depend on the medium dimension and hence they are the medium correct characteristics. The corresponding moduli (the elements of the inverse matrices of the rheological coefficients) do not have such properties

and in this sense they are incorrect characteristics of a medium. The last fact is essential when one solves ‘plane’ and ‘one–dimensional’ problems of strength where Young moduli, shift moduli and Poisson coefficients (obtained for three–dimensional media) are traditionally used without any basis.

– A new concept about the equivalence of media classes is introduced: two classes of media are dynamically equivalent if their dynamical equations coincide; two classes of media are energetically equivalent if their thermodynamics equations coincide.
– In general, all classes of media are not dynamically and energetically equivalent.
– In some particular cases (concerning the media properties and their motion character) some classes of media are dynamically equivalent.
– In the case of non–circular motion of an incompressible fluid, under condition that the mixed second partial derivatives of the velocity vector of a point w.r.t. the space coordinates should be continuous, all viscous fluids are dynamically (but no energetically) equivalent to the ideal fluid. First, in particular, it follows that the viscous fluids could not have potential flows, and, second, this could explain the satisfactory coincidence between theoretical and experiment results when potential flows of fluids ‘not too really viscous’ with almost constant temperature (for example, water) have been studied (with using dynamics equations of the ideal fluid).
– It follows from the above points that media, having different (in principle) mechanical state equations, could have (under given conditions) the same dynamics equations, and *vice versa*. This fact can explain the disagreement between the theory and the experiment for some real media and the agreement between them for others when one and the same dynamics equations are used (Ladyzhenskaya 1987). This is the reason that one could be unable to study the continuous medium deformation using only mechanical state equations or constitutive relations.

It is possible that the same medium may belong to different classes (in the case of different conditions) and thus may have principally different behavior. A special attention should be paid to those cases when the correct media and the incorrect ones are dynamically equivalent. They also should be specially studied.

8.6. The matrix equations of rigid body motions in the space of quasi–velocities and generalized velocities are obtained and studied if the constructive parameters and simplest holonomic constraints exist or do not exist. These equations are in the form suitable for direct use in computer programs (Konoplev 1985, Konoplev 1987, Konoplev 1996a). This means that when solving practical problems, the stage of ‘working out equations of the body motion’ is omitted (one knows that this stage is often an origin of errors and illusions about the ‘scientific approach’ to the problem).

8.7. This is the first time when the matrix algorithms for obtaining Lagrange equations of second order referred to continual rigid bodies are worked out (without using the apparatus of analytical mechanics and Newton formalism) if holonomic constraints are present or absent. Six–dimensional continual analogs of the basic properties of Universe mechanics are formulated and proved in the part related to the absolute rigid bodies (Konoplev 1996a).

8.8. The main results obtained in the multibody system mechanics are given as reference material for solving the applied problems (those results are published in whole by Konoplev (1984–2001)).

Chapter 2

The geometrical Universe of mechanics and its primary properties

The primary properties of the geometrical Universe of mechanics as well as the concept itself are defined by axioms, definitions and propositions that are consequent from them.

2.1. Axioms of kinematics

Points, free vectors and relations between them are initial notions, not given by definitions. The first 5 axioms below define their properties (Edwards 1969, Nikulin *et al.* 1987, Truesdell 1972, Yaglom 1969 and 1986).

Axiom K 1 *A point wise continuum* $\mathbf{D}_3$ *exists.*

Comments

1. Any point can be considered for example as a geometric one (when illustrations are needed) or as a triple of real numbers (when calculations are needed) (Dieudonne 1969).
2. The continuum $\mathbf{D}_3$, if desired, can be considered as a mathematical model of the 'physical space'. Everything that happens with the above continuum below can be represented by its transformations.
3. It is not recommended to confuse the property of continuity of $\mathbf{D}_3$ determined by its capacity only, with the 'continuity' in the traditional mechanics of continuum media.

Axiom K 2 *There is defined Lebesgue measure* μ_3 *on* $\mathbf{D}_3$.

Henceforth we use the notation $\mathbf{D}_3^\mu \equiv \{\mathbf{D}_3,\ \sigma_3^\mu,\ \mu_3\}$ where σ_3^μ is σ–algebra.

Axiom K 3 *The space* $\mathbf{V}_3$ *of* 3*–dimensional free vectors (*8 *axioms defining* $\mathbf{V}_3$ *and* 2 *axioms of the dimension) is determined over the field of real numbers. This space is a directrix of* $\mathbf{D}_3$ *(*3 *axioms defining relations with* $\mathbf{V}_3$*).*

Comments

1. $\mathbf{V}_3$ is a universal tool for fixing position of points of $\mathbf{D}_3$ in some frame (special one in required cases).
2. If necessary (for example in illustrations) an element of $\mathbf{V}_3$ can be treated as a class of equivalence that consists of all oriented segments having equal lengths and one and the same direction.
3. In the number representation of $\mathbf{V}_3$ (considered as triples of real numbers) when the canonical basis $[\mathbf{e}^0] = (e_1^0, e_2^0, e_3^0)$, $e_1^0 = \{1,0,0\}$, $e_1^0 = \mathrm{col}\{1,0,0\}$, $e_2^0 = \mathrm{col}\{0,1,0\}$, $e_3^0 = \mathrm{col}\{0,0,1\}$ is given, the representative of a free vector x coincides with its coordinate column $x = [\mathbf{e}^0]\ x^0 = x^0$. Thus, it is possible to not distinguish coordinate and invariant formulations of the theory fundamentals. In what follows, one uses the number space $\mathbf{R}_3$ (as representation of $\mathbf{V}_3$) and its Cartesian product.

Axiom K 4 *The space of free vectors is Euclidean.*

Comment By Axiom K 4 the affine vector space $\mathbf{A}_3^\mu = \mathbf{V}_3 \cup \mathbf{D}_3^\mu$ is transformed in a metric one.

Axiom K 5 *The space* $\mathbf{V}_3$ *of free vectors is an oriented one.*

Comments

1. Axioms K 3 – K 5 postulate the existence of the oriented Euclidean space, and consequently, of the normed and metric affine vector space $\mathbf{A}_3^\mu = \mathbf{V}_3 \cup \mathbf{D}_3^\mu$ (17 axioms of H. Weyl) (Dieudonne 1969, Yaglom 1969 and 1986, Weyl 1950) where the index μ reminds that the continuum $\mathbf{D}_3^\mu$ is a Lebesgue measured one.
2. Everything that is modeled in the nature with the help of the above mentioned continuum under condition that definite agreements are satisfied (see Axiom K 8) can be mathematically represented by transformations of the above vector space (in geometric or number variants of the vector space).
3. The elements of σ–algebra σ_3^μ are 'particles' of $\mathbf{D}_3^\mu$ neither in mathematical sense, nor in physical one. But they can be used for 'measuring' any bounded subset of $\mathbf{D}_3^\mu$ (in Euclidean metrics) in the sense of G. Vitali (Kolmogorov *et al.* 1957, Natanson 1955).
4. The orientation of the number space $\mathbf{R}_3$ is defined by (Dieudonne 1969)

$$(x, y, z) = x\cdot < y > z = x^T < y > z \tag{2.1}$$

where the skew–symmetric matrix

$$< y >= \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix} \tag{2.2}$$

is generated by the vector $y = \mathrm{col}\{y_1, y_2, y_3\}$ (the triple of vectors x, y, z is a right one if $(x, y, z) > 0$; if $(x, y, z) < 0$ then the triple is left one, if $(x, y, z) = 0$ then the triple x, y, z is linearly dependent).

Axiom K 6 *There exists a subset (a ray* $\mathbf{T}_+ = \mathbf{V}_{t+} \cup \mathbf{D}_t^\mu$ *) of the one–dimensional affine vector space* $\mathbf{T}_t = \mathbf{V}_t \cup\ \mathbf{D}_t^\mu$ *(the time axis) where* $\mathbf{D}_t^\mu = \{\mathbf{D}_t,\ \sigma_t,\ \mu_t\}$ *is the continual set of time instants that is a measured one in Lebesgue sense.*

Definition 2.1 *Let: 1. the time* $\mathbf{T}_+$ *do not depend on points* $y \in \mathbf{D}_3^\mu$, $t(y) = t$;

2. *the following relation of equivalence be defined on the time axis: two instants* t *and* t' *are equivalent if*

$$t = kt' + t_0 \tag{2.3}$$

where $k > 0$.

Then 1. the time $\mathbf{T}_+$ *is called absolute, uniform and isotropic;*

2. *the constants* t_0 *and* k *are called a reference origin and a time scale (Devis 1982, Weyl 1950, Yaglom 1969).*

Axiom K 7 *The time* $\mathbf{T}_+ = \mathbf{V}_{t+} \cup\ \mathbf{D}_t^\mu$ *is absolute, uniform and isotropic.*

Notation Henceforth

1. $\mathcal{P}_q$ is n–parametric body of quadratic matrices, where $q \in \mathbf{Q}_n$, $\mathbf{Q}_n$ is n–dimensional manifold;
2. $\mathcal{P}_{q+}$ and $\mathcal{P}_{q\times} = \mathcal{P}_{q+} \backslash \{0\}$ are additive and multiplicative groups of the body $\mathcal{P}_q$;
3. 0_+, $1_\times$ are unit elements of the algebraic operations of additions and multiplication in the above groups.

Definition 2.2 *1. The matrix function* $A(q) \in \mathcal{P}_q$ *is called infinitesimal at the point* $q_0 \in \mathbf{Q}_n$ *if*

$$\lim_{q \to q_0} A(q) = 0_+ \tag{2.4}$$

The set of such points q_0 *is called an infinitesimal one for* $A(q)$.

2. *Matrices* $A(q)$ *and* $B(q)$ *are called equivalent at the point* q_0 *if*

$$\lim_{q \to q_0} A(q) B^{-1}(q) = \lim_{q \to q_0} B^{-1}(q) A(q) = 1_\times \tag{2.5}$$

The set of such points q_0 *is called the set of equivalence of* $A(q)$ *and* $B(q)$.

Notation Henceforth

1. $\mathbf{E}_0 = (o_0, [\mathbf{e}^0])$ is a fixed orthonormal frame in $\mathbf{A}_3^\mu$, $o_0 \in \mathbf{D}_3^\mu$, $[\mathbf{e}^0] = (e_1^0, e_2^0, e_3^0) \in \mathbf{V}_3$ where e_i^0 are vectors of unit norm;
2. for points x and $y \in \mathbf{D}_3^\mu$, x^{00} and y^{00} are the coordinate columns of their radius–vectors x^0 and $y^0 \in \mathbf{V}_3$ in $\mathbf{E}_0$ (inside upper index) with the basis $[\mathbf{e}^0]$ (upper outside index);

3. $\mathbf{V}_{3y}$ is the vector space

$$\mathbf{V}_{3y} = \{h_y : h_y = x^0 - y^0,\ x \in \mathbf{D}_3^\mu\} \tag{2.6}$$

consisting of the representatives of vectors belonging to $\mathbf{V}_3$ that have the common 'origin' at the point y; h_y^0 is the coordinate column of the vector $h_y = x^0 - y^0$ in the basis $[\mathbf{e}^0]$

$$h_y^0 = x^{00} - y^{00} \tag{2.7}$$

4. $\mathcal{GL}_{yt}(\mathcal{R}, 3)$ is the complete four–parametric linear group of automorphisms (matrices) of $\mathbf{V}_{3y} \cong \mathbf{R}_3$ in the instant $t \in \mathbf{T}_+$ at the point $y \in \mathbf{D}_3^\mu$ (Dieudonne 1969 and 1974);
5. E is the identity matrix.

Consider the set of matrices $D_d^{00} = D_d^{00}(q(t, y)) \in \mathcal{GL}_{yt}(\mathcal{R}, 3)$ which describe transformations of the basis $[\mathbf{e}^0]$ (upper inside index) to any non–orthonormal basis $[\mathbf{e}^d]$ (lower index), being calculated in the basis $[\mathbf{e}^0]$ (upper outside index)

$$[\mathbf{e}^d] = [\mathbf{e}^0] d_d^{00} \tag{2.8}$$

Proposition 2.1 *Let there exist some point $q_0(t, y) \in \mathbf{Q}_n$ so that for any D_d^{00} the matrix*

$$\Delta D_d^{00} = D_d^{00} - E \tag{2.9}$$

is infinitesimal. Then the set (see (2.4))

$$\begin{aligned}\mathcal{GD}_{yt}^q(\mathcal{R}, 3) = \{D_d^{00} : D_d^{00} = D_d^{00}(q(t, y)) \in \mathcal{GL}_{yt}(\mathcal{R}, 3),\\ D_d^{00} = E + \Delta D_d^{00},\ \Delta D_d^{00} \to 0,\ q \to q_0(t, y)\}\end{aligned} \tag{2.10}$$

is a group.

Proof 1. Let $A(q),\ B(q) \in \mathcal{GD}_{yt}^q(\mathcal{R}, 3)$ then $A(q) = E + \Delta A(q),\ B(q) = E + \Delta B(q),\ \Delta A(q) \to 0,\ \Delta B(q) \to 0,\ q \to q_0(t, y) \Longrightarrow A(q)B(q) = E + \Delta A(q)\, E + E\, \Delta B(q) + \Delta A(q)\ \Delta B(q) \cong E + \Delta(A(q)\ B(q)) \Longrightarrow A(q)B(q) \in \mathcal{GD}_{yt}^q(\mathcal{R}, 3)$;

2. $A^{-1}(q) = (E + \Delta A)^{-1} = E + \sum\limits_{k=1}^{\infty} (-\Delta A)^k,\ \|\Delta A\| < 1,\ \sum\limits_{k=1}^{\infty} (-\Delta A)^k \to 0$ when $q \to q_0(t, y) \Longrightarrow A^{-1}(q) \in \mathcal{GD}_{yt}^q(\mathcal{R}, 3)$.

Comments

1. According to the definition, the group $\mathcal{GD}_{yt}^q(\mathcal{R}, 3)$ does not include improper automorphisms of the type of mirror reflections, axial symmetries, *etc.*, that, in principle, cannot be adequate models of transformations of the medium.

2. The infinitesimal matrix (2.9) plays an important role in the constructions that follow: it will be proved (see (3.26)) that under definite conditions (see (3.21)) it approximately coincides with the matrix of medium deformation in the vicinity of the point $y \in \mathbf{D}_3^\mu$ (see (3.22)); and the relation

$$D_d^{00} = E + \Delta D_d^{00} \tag{2.11}$$

defines the connection between the deformation matrix ΔD_d^{00} and the matrix D_d^{00}.

Definition 2.3 *Let* $\mathbf{U}$ *be a set of arbitrary (non–linear) transformations* u *of the affine vector space* $\mathbf{A}_3^\mu$ *such that for each point* $x \in \mathbf{A}_3^\mu$ *there exists such point* $y \in \mathbf{A}_3^\mu$ *that* $y = ux$. *Then the continuum* $\mathbf{D}_3^\mu$ *is called locally changeable.*

Notation Henceforth $\mathcal{GTD}_{yt}^q(\mathcal{R},3) = \mathcal{T}_{yt}(\mathcal{R},3)\mathcal{GD}_{yt}^q(\mathcal{R},3)$ is a 4–parameter subgroup of the complete affine group $\mathcal{GA}_{yt}^q(\mathcal{R},3)$ on $\mathbf{A}_3^\mu$ where $\mathcal{T}_{yt}(\mathcal{R},3)$ is the group of shifts in $\mathbf{A}_3^\mu$.

Definition 2.4 *Let: 1. the continuum* $\mathbf{D}_3^\mu$ *be locally changeable;*

2. *for every* $u_1 \in \mathbf{U}$, $\varepsilon \in \mathbf{R}_1$, $\varepsilon > 0$ *and each point* $y \in \mathbf{D}_3^\mu$ *there exist a vicinity* $\varepsilon_y \in \sigma_3^\mu$ *of this point, an affine transformation* $\nu \in \mathcal{GA}_{yt}^q(\mathcal{R},3)$, *and a transformation* $u_2 \in \mathbf{U}$ *such that*

$$u_1 x^{00} = \nu x^{00} + u_2 x^{00} \quad , \ \|u_2 x^{00}\| \, \|\nu x^{00}\|^{-1} < \varepsilon \tag{2.12}$$

Then: 1. the continuum $\mathbf{D}_3^\mu \equiv \{\mathbf{D}_3,\ \sigma_3^\mu,\ \mu_3\}$ *is called a locally linearly changeable medium (further – medium simply);*

2. *the four–parameter group* $\mathcal{GD}_{yt}^q(\mathcal{R},3)$ *is called a group of linear kinematical deformators (further – that of k–deformators) of the medium, the matrices* $D_d^{00} \in \mathcal{GD}_{yt}^q(\mathcal{R},3)$ *are called k–deformators of the medium in the vicinity* $\varepsilon_y \in \sigma_3^\mu$ *of the point* $y \in \mathbf{D}_3^\mu$ *(further – medium* ε_y*);*
3. *the couple (consisting of* $\mathbf{A}_3^\mu$ *and of the group* $\mathcal{GTD}_{yt}^q(\mathcal{R},3)$ *on* $\mathbf{A}_3^\mu$*), i.e.,*

$$\mathfrak{B} = \{\mathbf{A}_3^\mu,\ \mathcal{GTD}_{yt}^q(\mathcal{R},3)\} \tag{2.13}$$

is called a geometrical Universe of mechanics;

4. *the sets of* σ*–algebra* σ_3^μ *are called geometric systems;*
5. *the geometrical Universe of mechanics* $\mathfrak{B}$ *is called a locally linearly changeable one;*
6. *a transformation* $v \in \mathcal{GTD}_{yt}^q(\mathcal{R},3)$ *is called* ε*–affine one for the system* $\varepsilon_y \in \sigma_3^\mu$*;*
7. *the basis* $[\mathbf{e}^d] = [\mathbf{e}^0]d_d^{00}$ *is called a basis accompanying the linear deformation of* $\varepsilon_y \in \sigma_3^\mu$ *(produced by k–deformator* D_d^{00}*).*

Comments

1. At the stage of constructing the kinematics, the elements of σ–algebra σ_3^μ have no 'dynamic' characteristics, and the used frames have not special properties of the type of inertialness. The sets of σ_3^μ are geometric objects consisting of points that could be transformed by means of elements of $\mathbf{U}$ as well as geometric objects that are affine linear transformations from the group $\mathcal{GTD}_{yt}^q(\mathcal{R},3)$. Otherwise, there is nothing in the geometrical Universe of mechanics except μ_3–measured point–wise continuum and its transformations (linear and non–linear).
2. The concept of 'basis $[\mathbf{e}^d]$ accompanying deformation' is one of the main ones in the considered theory: the coordinate column of the radius–vector of any point belonging to a linearly changeable medium in the frame $\mathbf{E}_d = (o_0, [\mathbf{e}^d])$ remains constant and equal to the coordinate column in the initial time, and, hence, to the coordinate column of the same radius–vector in the initial frame $\mathbf{E}_0 = (o_0, [\mathbf{e}^0])$ if in the instant $t = 0$

$$a^d(t) = a^d(0) = a^0(0) \tag{2.14}$$

   This permits us to substitute the research of variation of these vectors with studying groups of matrices $u_d^0(t)$ that lead to their variation

$$a^0(t) = u_d^0(t)\ a^d(t) = u_d^0(t)\ a^d(0) = u_d^0(t)\ a^0(0) \tag{2.15}$$

   It makes possible to formulate all problems of 'linear' medium kinematics in terms of the linear algebra. In this way the medium transformations can be treated by means of the contemporary algebra (Edwards 1969, Gantmacher 1964, Skornyakov 1980, Suprunenko 1972). The corresponding problems could be formulated in the algebraic form being suitable for computer realization.
3. Relation (2.11) shows that k–deformator D_d^{00} should not be confused to the matrix of medium deformation.

Axiom K 8 *The geometrical Universe of mechanics is locally linearly changeable.*

Comments

1. Axiom K 8 affirms the following: (arbitrary) transformations of the affine vector space $\mathbf{A}_3^\mu$ are such that for each $y \in \mathbf{A}_3^\mu$ and any $\varepsilon > 0$ there exists μ_3–measured vicinity $\varepsilon_y \in \sigma_3^\mu$ of the point y where in the fixed instant of time $t \in \mathbf{T}_+$ the result of this transformation differs from the result of action of the affine transformation (called ε–affine) and the norm of this difference (in the uniform metrics $\mathbf{V}_3$) is less than ε.
2. From above it does not follow that a transform $u \in \mathbf{U}$ differs small from an affine one $v \in \mathcal{GTD}_{yt}^q(\mathcal{R},3)$ in the corresponding vicinity $\varepsilon_y \in \sigma_3^\mu$ of the point $y \in \mathbf{D}_3^\mu$ because here the notion 'small' is not defined (the non–linear transform u and the linear one y cannot be compared, here only their actions are able to be compared).

3. From Axiom K 8 it does not follow that in the geometric Universe of mechanics one cannot study non–linear models of physical phenomena. Here it is clear that for every point $y \in \mathbf{D}_3^\mu$ there is a vicinity $\varepsilon_y \in \sigma_3^\mu$ where the existence of a linear ε–approximation is guaranteed for these models (in sense of relation (2.12)).
4. The point–wise set $\varepsilon_y \in \sigma_3^\mu$ is a topological (open) vicinity of the point $y \in \mathbf{D}_3^\mu$ (but it is not an open ε–sphere) whose elements are chosen in every fixed instant of time, $t \in \mathbf{T}_+$, according to (2.12). This set cannot be represent graphically in general (only in some particular cases).
5. Vicinities $\varepsilon_y \in \sigma_3^\mu$ of points $y \in \mathbf{D}_3^\mu$ (as well as other elements of σ–algebra σ_3^μ) are not 'medium particles'; ε–affine transformation $v \in \mathcal{GTD}_{yt}^q(\mathcal{R},3)$ acts on the medium ε_y such that it shifts the point $y \in \varepsilon_y$ (not the all set ε_y). It simultaneously transforms number representations of vector spaces $\mathbf{V}_{3y}$ (of vector–radiuses from changing sets ε_y – due to (2.12) – with common 'origins' in changing points y) with the help of k–deformator $D_d^{00} \in \mathcal{GD}_{yt}^q(\mathcal{R},3)$ when y and t are varying.
6. The notion 'k–deformator' does not coincide with well–known one of 'deformation tensor' (Kochin *et al.* 1964, Lur'e 1970, Sedov 1972), but they connect one to other under some condition by the (2.11)–kind relation.
7. The notion 'linear medium deformation' given here is broader than the analogical traditional one (linear medium transformation with changing distance between points): it is any linear transformation $D_d^{00} \in \mathcal{GD}_{yt}^q(\mathcal{R},3)$ of the medium ε_y (it may be rotation if ε_y is a rigid body – see section 5.1). This permits us to construct the medium deformation theory starting from the general mathematical apparatus and the common point of view.
8. No properties of the conventional 'continuity' kind are supposed at this stage of the theory constructing.

2.2. Axioms of dynamics

Notation Henceforth σ_3^μ is Borell σ–algebra generated by μ_3–measured open subsets of the set $\mathbf{D}_3^\mu$.

Definition 2.5 *Let $\phi(\cdot)$ be a function on σ_3^μ whose values are n–ples of Radon measure (Edwards 1969) (sign–variable scalar measures, charges (Kolmogorov et al. 1957)).*

Then: 1. the function $\phi(\cdot)$ is called a vector measure of Radon on σ_3^μ;

2. a function $\pi(\cdot,\cdot)$, defined on $\sigma_3^\mu \times \sigma_3^\mu$ and being a Radon vector measure by each of arguments, is called a Radon vector bi–measure;

3. the Radon vector bi–measure $\pi(\cdot,\cdot)$ is called skew–symmetric if

$$\pi(A,B) = -\pi(B,A) \tag{2.16}$$

for each A and $B \in \sigma_3^\mu$.

Comment The function $\phi(\cdot)$ is not an element of some vector space. In practice it is always an element of n–dimensional manifold $\mathbf{Q}_n$.

Henceforth we shall use notions of the screw theory given in Chapter 7 (see Index, too).

Definition 2.6 *1. A screw $F_0^0(A,B)$ (see (7.3)) in $\mathbf{E}_0$ with properties of 6–dimensional vector skew–symmetric bi–measure of Radon on $\sigma_3^\mu \times \sigma_3^\mu$ is called a dynamic bi–measure (force) on $\sigma_3^\mu \times \sigma_3^\mu$ of the geometric Universe of mechanics*

$$F_0^0(A,B) = -F_0^0(B,A) \tag{2.17}$$

2. the geometric Universe of mechanics on which the dynamic bi–measure is defined, is called a dynamic one.

Axiom D 1 *the geometric Universe of mechanics is a dynamic one.*

Comment The dynamic bi–measure $F_0^0(A,B)$ is, in principal, 6–dimensional vector (a screw, a dynamic screw, wrench, twist). There is the well known statement that there exists a 'moment of vector', and in particular a 'main moment of force' (the second triple of coordinates in $F_0^0(A,B)$), as a mathematical model of some physical object, but this has neither physical nor mathematical foundations. The second triple $\langle r_a^0\rangle^0 x^0$ in the sliding vector definition (see (7.1)) gives information that only together with the vector x^0 defines the sliding vector. One has to understand that the sum of second triples is not a triple of the same kind: $\langle r_a^0\rangle^0 x^0 + \langle r_b^0\rangle^0 y^0 = \langle ?\rangle ?$ if $x^0 \neq \lambda y^0$ (where λ is arbitrary real number). The number multiplication is not defined here correctly: $\lambda\langle r_a^0\rangle^0 x^0 = \langle \lambda r_a^0\rangle^0 x^0 = \langle r_a^0\rangle^0 \lambda x^0$.

The following three axioms define fundamental properties of the geometric Universe of mechanics connected with dynamical bi–measures.

Definition 2.7 *Let the dynamic bi–measure $F_0^0(A,B)$ on $\sigma_3^\mu \times \sigma_3^\mu$ in $\mathbf{E}_0$ be absolutely continuous w.r.t. the Lebesgue measure μ_3 by both arguments, e.g., for any points x and y from arbitrary A and $B \in \sigma_3^\mu$ there exist μ_3–integrable w.r.t. both arguments sliding vectors (see (7.1)) $\rho_0^{f(x,dy),0}$ and $\rho_0^{f(dx,y),0}$ in $\mathbf{E}_0$ such that*

$$F_0^0(dx,dy) = \rho_0^{f(x,dy),0}\ \mu_3(dx) = \rho_0^{f(dx,y),0}\ \mu_3(dy) \tag{2.18}$$

Then 1. the medium $\mathbf{D}_3^\mu$ is called a dynamically continuous medium;

2. $\rho_0^{f(x,dy),0}$ and $\rho_0^{f(dx,y),0}$ are called densities of the dynamic bi–measure $F_0^0(dx, dy)$ on $\sigma_3^\mu \times \sigma_3^\mu$ at points x and y w.r.t. the Lebesgue measure $\mu_3(dx)$ and $\mu_3(dy)$, respectively.

Axiom D 2 *the geometric Universe of mechanics is dynamically continuous .*

Comments

1. Axiom D 2 affirms that in the geometric Universe of mechanics there are not geometric systems consisting of one point with non–zero dynamic measures (by any argument). In these systems only the densities of measures relative to the Lebesgue measure can differ from zero.

2. The concept of Newton force applied to a one–point geometric system, without a volume, is out of sense here as it is always equal to 0.
3. The definition of a dynamically continuous medium differs principally from the traditional and intuitive definition of continuous medium. Here the property pointed out is defined by the corresponding property of the dynamic measure: the medium is dynamically continuous if the dynamic measure (the existence of which on $\sigma_3^\mu \times \sigma_3^\mu$ is guaranteed by Axiom D 1) is absolutely continuous relative to the Lebesgue measure $\mu_3(d(\cdot))$ on σ_3^μ w.r.t. both arguments (Axiom D 2).

Definition 2.8 *Let: 1. for σ_3^μ there exist such a decomposition that*

$$\sigma_3^\mu = \sigma_+^\mu \cup \sigma_-^\mu, \ \sigma_+^\mu \cap \sigma_-^\mu = \emptyset \tag{2.19}$$

2. σ_+^μ be a dynamically continuous medium and be the set of concentration of the dynamical bi–measure $F_0^0(dx, dy)$, i.e.,

$$\rho_0^{f(x,dy),0} \neq 0, \ \rho_0^{f(dx,y),0} \neq 0 \tag{2.20}$$

for arbitrary x, $y \in \sigma_+^\mu$;

3. $\sigma_-^\mu = \sigma_3^\mu \backslash \sigma_+^\mu$ be a dynamically continuous medium and

$$\mu_3(B) \neq 0, \ \rho_0^{f(x,dy),0} = 0, \ \rho_0^{f(dx,y),0} = 0 \tag{2.21}$$

for an arbitrary geometric system $B \in \sigma_-^\mu$ and for arbitrary points $x \in \mathbf{D}_3^\mu$, $y \in B$;

4. the infinite number of geometric systems $B \in \sigma_+^\mu$ exist such that for each of them there is a geometric system $A \in \sigma_3^\mu$ with the properties

$$B \subset A, \ A \backslash B \in \sigma_+^\mu$$

5. there exist geometric open systems $B \in \sigma_3^\mu$ and a system $A \in \sigma_-^\mu$ such that $A \subset B$, $B \backslash A \in \sigma_+^\mu$.

Then the geometric Universe of mechanics is called dynamically disconnected.

Axiom D 3 *the geometric Universe of mechanics is dynamically disconnected.*

Comments

1. Axioms D 2 and D 3 affirm the following: Borell σ–algebra of geometric systems belonging to the geometric Universe of mechanics consists only of systems that are dynamically continuous media whose part from σ_-^μ does not interact, does not act on systems of σ_+^μ with non–zero densities of dynamic measures, is not exposed by any action from them and is disconnected.
2. The set σ_+^μ of concentration of the dynamic bi–measure $F_0^0(dx, dy)$ is disconnected: it contains the infinite number of isolated geometric systems that are 'separated' each other by systems from σ_-^μ (a part of these systems contains in itself systems of σ_-^μ). At the level of present knowledge (Devis 1982) for the physical Universe the Lebesgue measure of σ_+^μ is neglected little w.r.t. the Lebesgue measure of σ_-^μ, namely $\mu_3(\sigma_+^\mu)/\mu_3(\sigma_-^\mu) \approx 10^{-30}$.

Definition 2.9 *Let $A \in \sigma_+^\mu$ be a geometric system, χ_A be the characteristic function of A. Then the Lebesgue integral of the dynamic bi–measure $F_0^0(dx, dy)$ by the first argument for the system A of the kind*

$$F_0^0(A, dy) = \int \chi_A \; \rho_0^{f(x,dy),0} \mu_3(dx) = \rho_0^{f(A,y),0} \; \mu_3(dy) \tag{2.22}$$

is called a dynamic measure (a dynamic screw, a force) of the action of A on the systems belonging to σ_+^μ with the density $\rho_0^{f(A,\ y),0}$ at the point $y \in \mathbf{D}_3^\mu$.

Comment The dynamic measure $F_0^0(A, dy)$ is a screw about the first argument and a sliding vector about the second one (see Chapter 7).

Notation $\rho_0^{f(\urcorner A,y),0}$ is the density of the dynamic screw of action of the complement $\urcorner A \in \sigma_+^\mu$ (of any system $A \in \sigma_+^\mu$) on A

$$\begin{aligned} F_0^0(\urcorner A,\ A) \quad &= \quad \int \chi_A \; \rho_0^{f(\urcorner A,\ y),0} \; \mu_3(dy) \\ \rho_0^{f(\urcorner A,y),0} \quad &= \quad \int \chi_{\urcorner A} \; \rho_0^{f(x,\ y),0} \; \mu_3(dx) \end{aligned} \tag{2.23}$$

Definition 2.10 *Let $\rho_0^{f(\urcorner y,y),0}$ be equal to zero in each point $y \in \mathbf{D}_3^\mu$, i.e.,*

$$\rho_0^{f(\urcorner y,y),0} = 0 \tag{2.24}$$

Then the geometric Universe of mechanics (2.13) is called dynamically balanced.

Axiom D 4 *the geometric Universe of mechanics is dynamically balanced.*

Comment Definition (2.24) of balance differs from the relation $F_0^0(\urcorner A,\ A) = 0$, in principle. The last one is possible even when $\rho_0^{f(\urcorner y,y),0} \neq 0$ (Truesdell 1972).

The axioms that follow postulate the existence (on σ–algebra of the geometric Universe of mechanics) of three different (in principle) dynamic measures (forces) that express the action of complements in σ_3^μ on these systems (they have different densities $\rho_0^{f(\urcorner y,y),0}$, in principle).

Definition 2.11 *Let: 1. on σ_3^μ there be defined Borell measure $m_i(dy)$ such that*

1.1. it is absolutely continuous w.r.t. the Lebesgue measure $\mu_3(dy)$ on σ_3^μ with a density ρ_y^i, i.e.,

$$m_i(dy) = \rho_y^i \; \mu_3(dy) \tag{2.25}$$

1.2. for each system $B \in \sigma_-^\mu$

$$m_i(B) = 0 \tag{2.26}$$

2. v_y^{00} be the coordinate column of the vector v_y^0 representing the velocity of a point $y \in \mathbf{D}_3^\mu$ w.r.t. $\mathbf{E}_0$ in $[\mathbf{e}^0]$, $v_y^{00\cdot}$ is its derivative in $\mathbf{E}_0$ (i.e., the acceleration of $y \in \mathbf{D}_3^\mu$ w.r.t. $\mathbf{E}_0$ in $[\mathbf{e}^0]$), $\rho_0^{v^\cdot(y),0}$ is the sliding vector generated by $v_y^{00\cdot}$ (see (7.1)) in $\mathbf{E}_0$ (Konoplev 1987a, Konoplev 1996a)

$$\rho_0^{v^\cdot(y),0} = G_0^0 \; \mathrm{col}\,\{v_y^{00\cdot},\ v_y^{00\cdot}\} \tag{2.27}$$

3. *such a dynamic bi–measure* $F_0^0(dx, dy) \equiv I_0^0(dx, dy)$ *be defined on* $\sigma_3^\mu \times \sigma_3^\mu$ *that the dynamic screw*

$$F_0^0(\urcorner dy, dy) \equiv I_0^0(\urcorner dy, dy) \tag{2.28}$$

of action on systems of σ_3^μ *by their complements with the density* $\rho_0^{i(\urcorner y, y),0}$ *relative to the measure* $m_i(dy)$ *(see (2.25)) is of the form (due to (2.26))*

$$I_0^0(\urcorner dy, dy) = \rho_0^{i(\urcorner y, y),0}\ m_i(dy) \tag{2.29}$$

$$\rho_0^{i(\urcorner y, y),0} = -\rho_0^{v^{\cdot}(y),0} \tag{2.30}$$

Then 1. the frame $\mathbf{E}_0$ *(where relation (2.29) is true), is called an inertial frame;*

2. *the Borell measure* $m_i(dy)$ *on* σ_3^μ *for which relation (2.29) is fulfilled, is called a scalar measure of inertia on* σ_3^μ *(for simplicity – inertial mass);*
3. *the dynamic screw* $I_0^0(\urcorner y, dy)$ *defined by relation (2.29) is called a vector dynamic measure of inertia (a dynamic screw of inertia, a vector field of inertia) on* σ_3^μ*;*
4. *the sliding vector*

$$\rho_0^{i(\urcorner y, y),0} = -\rho_0^{v^{\cdot}(y),0} \tag{2.31}$$

generated by the vector $v_y^{00\cdot}$ *(with sign '–') of the point relative to inertial frame* $\mathbf{E}_0$*, is called a density of the dynamic measure (of dynamic screw, of vector field) of inertia on* σ_3^μ *relative to the measure* $m_i(dy)$*;*

5. *the geometric Universe of mechanics , where the inertial frame* $\mathbf{E}_0 = (o_0, [\mathbf{e}^0])$ *as well as the scalar measure* $m_i(dy)$ *of inertia and the vector dynamic measure* $I_0^0(\urcorner y, dy)$ *of inertia are defined, is called an inertial one.*

Axiom D 5 *the geometric Universe of mechanics is inertial.*

Comments

1. Axiom D 5 postulates the existence of an inertial frame and two measures of inertia (a scalar one – the inertial mass and a vector one – the dynamic screw of inertia). If anyone of these concepts is eliminated from the definition, then the rest statements are wrong.
2. Relation (2.25) affirms that 'inertial' one–point geometric systems (without volume) do not exist in the geometric Universe of mechanics . For the mentioned systems, only the density of the inertial mass referred to the Lebesgue measure can differ from zero.
3. Statement (2.26) confirms the non–inertialness of the systems belonging to σ_-^μ.
4. The proposed mechanism for introduction of the concepts of an inertial frame and of inertia is logically equivalent to the first and the second laws of Newton, but it advantageously differs from them because:

   4.1. this mechanism does not use the concepts of (non–existing in the nature) points without volume with non–zero inertial mass;

4.2. it does not use the concept of forces that act at (non–existing in the nature) ‘inertial’ points without volume and which ones are always equal to zero;

4.3. it correctly introduces the mentioned above concepts of a continuous medium without mathematically and physically inconsistent ‘forming’ a continuous continuum from a finite set of (really non-existing) points with ‘concentrated inertial masses’ and ‘particles’;

4.4. this mechanism guarantees the possibility to construct a correct theory of the multibody system mechanics based on study of a continuous medium with additional properties without replacing bodies by finite sets of (physically non–existing) points with concentrated masses;

4.5. this mechanism does not use the concept of momentum and moment of momentum that are traditionally introduced in the case of a locally changeable continuous medium.

5. The dynamic measure of inertia is introduced by using the dainty hypothesis of E. Mach. According to it, in the geometric Universe of mechanics the inertia appearance is due not to ‘inborn force of the matter, included in itself’ but to acceleration of mechanical systems belonging to σ_+^μ (which have inertial mass, *i.e.*, inertial systems) w.r.t. other inertial systems (Devis 1982).

Definition 2.12 *Let:*

1. a scalar measure $m_g(dy)$ be defined in σ_3^μ such that:

1.1. the measure $m_g(dy)$ is absolutely continuous w.r.t. the Lebesgue measure $m_g(dy)$ on σ_3^μ with the density ρ_y^g

$$m_g(dy) = \rho_y^g\ \mu_3(dy) \tag{2.32}$$

1.2. for each geometric system $A \in \sigma_-^\mu$

$$m_g(A) = 0 \tag{2.33}$$

2. there be defined the dynamic bi–measure on $\sigma_3^\mu \times \sigma_3^\mu$ in $\mathbf{E}_0$

$$F_0^0(dx, dy) = G_0^0(dx, dy) \equiv l_0^{g(dx,dy),0} \tag{2.34}$$

which is generated by the following three–dimensional vector–function

$$g^0(dx, dy) = (y^{00} - x^{00})\frac{m_g(dx)\ m_g(dy)}{\|y^{00} - x^{00}\|^3} \tag{2.35}$$

where y^{00} and x^{00} are the coordinate columns in the basis $[\mathbf{e}^0]$ of the radius–vectors y^0 and x^0 of points y and x in $\mathbf{E}_0$.

Then: 1. the dynamic bi–measure $F_0^0(dx, dy)$ is called a dynamic bi–measure (force) of gravity on $\sigma_3^\mu \times \sigma_3^\mu$;

2. the measure $m_g(dy)$ is called a scalar measure of gravity on σ_3^μ (for simplicity a gravitational mass only);

3. *the vector dynamic measure on σ_3^μ (the dynamic screw (2.22)) of action on the system $B \in \sigma_3^\mu$ by its complement $\urcorner B$ in σ_3^μ*

$$G_0^0(\urcorner dy, dy) = \rho_0^{g(\urcorner y,y),0} \; \rho_y^g \; \mu_3(dy) \qquad (2.36)$$
$$g^0(\urcorner y, y) = \int \chi_{\urcorner y} \; (y^{00} - x^{00}) \frac{\rho_x^g \; \mu_3(dx)}{\|y^{00} - x^{00}\|^3} \qquad (2.37)$$

is called a vector dynamic measure (a dynamic screw, a force, a vector field) of gravity on σ_3^μ;

4. *the geometric Universe of mechanics is called gravitational if the gravitational dynamic bi–measure (force) and the scalar measure of gravity are defined on $\sigma_3^\mu \times \sigma_3^\mu$ and σ_3^μ, respectively.*

Axiom D 6 *the geometric Universe of mechanics is gravitational.*

Comments

1. At this stage of the theory construction, the scalar measure of gravity has no relation with the scalar measure of inertia (an inertial mass).
2. Relations (2.34) and (2.35) do not represent Newton's law of gravitation (here, the concept of a gravity measure has a different sense; the Universe gravity constant is equal to 1).
3. It follows from (2.33) that the mass of gravity (as the mass of inertia) is concentrated on σ_+^μ.
4. Here the inertia and gravity masses on σ_3^μ are independent concepts. For any relations between them, it is necessary a special agreement.

Axiom D 7 *The inertia and gravity masses are proportional.*

Definition 2.13 *Let the inertia and gravity masses be proportional in the geometric Universe of mechanics , i.e.,*

$$m_g(dy) = k \; m_i(dy) \qquad (2.38)$$

Then 1. the positive constant $\gamma = k^2$ is called a gravitational constant of the geometric Universe of mechanics ;

2. *for simplicity, the inertia mass is called a mass only, and it is written as follows*

$$m_i(dy) \equiv m(dy) \qquad (2.39)$$

Proposition 2.2 *The vector dynamic measure σ_3^μ of gravity on σ–algebra of the systems belonging to the geometric Universe of mechanics (if (2.36) and (2.38) are taken into account) is calculated by the law (of Newton)*

$$G_0^0(\urcorner y, dy) = \rho_0^{g(\urcorner y,y),0} \quad \rho_y \; \mu_3(dy) \qquad (2.40)$$
$$g^0(\urcorner y, y) = \int \chi_{\urcorner y} \; (y^{00} - x^{00}) \frac{\rho_x \; \mu_3(dx)}{\|y^{00} - x^{00}\|^3} \qquad (2.41)$$

where the index g in the density ρ is removed due to Axiom D 6.

Proof is based on substituting (2.38) in relations (2.34) and (2.35).

Comment The obtained results are not trivial. The constructions permit us making the following conclusions:

1. If relation (2.38) is accepted, then in traditional forms of Newton's law of gravity the inertia mass (the scalar measure of inertia) takes place.
2. If statement (2.38) is accepted, other scalar measures on σ_3^μ related to the concepts of gravity and of inertia do not exist with exception of proportional to masses $m_g(dy)$ and $m_i(dy)$. It means that both notions (Axioms D 4 and D 5) could be formulated in terms of one measure $m_g(dy)$ or $m_i(dy)$. The question consists only in the habit of the form of writing and of notation.
3. If statement (2.38) is accepted, then one can substitute the measure of inertia by $m_i(dy) \equiv m(dy)$ and call it a measure of 'quantity of matter'. But then it is necessary to answer the question: "What is this 'matter'?". In the scope of the formulated axioms this question has not an answer. It is represented that it is not present and in other axiomatics. Everything that is in connection with the mass, considered as a measure of quantity of matter, in fact is connected with the concept of the mass of inertia considered as a measure. Both masses mentioned above, not connected with any substances ('matter', 'medium quantity', *etc.*), represent properties of the the geometric Universe of mechanics .

Notation $\mathbf{D}_2^\mu$ is a section between the point–wise set $\mathbf{D}_3^\mu$ and an arbitrary affine vector plane $\mathbf{P}_2$.

Definition 2.14 *Let:*

1. *$D_0^0(dx, dy)$ be a dynamic bi–measure defined on $\sigma_3^\mu \times \sigma_3^\mu$ and be such that*

$$\begin{aligned} D_0^0(\rceil dy, dy) &= \rho_0^{\Delta(\rceil y,y),0}\ \mu_3(dy) = \rho_0^{\Delta(\rceil y,y),0}\ \rho_y^{-1}\ \rho_y\ \mu_3(dy) \\ &= \rho_0^{\Delta(\rceil y,y),0}\ \rho_y^{-1}\ m_i(dy) \end{aligned} \tag{2.42}$$

represents the action on a system $B \in \sigma_3^\mu$ by its complement in σ_3^μ with a density $\rho_0^{\Delta(\rceil y,y),0}$ w.r.t. the measure $\mu_3(dy)$ at a point $y \in \mathbf{D}_3^\mu$;

2. *on the set $\mathbf{D}_2^\mu$ there be defined the density $\rho_0^{\delta(\rceil y,y),0}$ (in $\mathbf{E}_0$) of the dynamic measure $D_0^0(\rceil dy, dy)$ on σ_3^μ in $\mathbf{E}_0$ w.r.t. Lebesgue two–dimensional measure $\mu_2(dy)$ on Borell σ–algebra σ_2^μ of subsets of $\mathbf{D}_2^\mu$*

$$D_0^0(\rceil dy, dy) = \rho_0^{\delta(\rceil y,y),0}\ \mu_2(dy) \tag{2.43}$$

3. *there be defined 3×3–matrix function $T_y^0 : \mathbf{R}_3 \times \mathbf{T}_+ \to \mathbf{R}_9$ which can be differentiable by y^{00} the necessary number of times and such that the free vector $\delta^0(\rceil y, y)$ which generates the sliding vector $\rho_0^{\delta(\rceil y,y),0}$ of density is a result from the linear transformation of the normal n_y^0 to the plane $\mathbf{P}_2$ at the point y w.r.t. the basis $[\mathbf{e}^0]$:*

$$\delta^0(\rceil y, y) = T_y^0 n_y^0 \tag{2.44}$$

4. 3–dimensional free vector $\Delta^0(\rceil y, y)$ generating the sliding vector of the 3–dimensional density from (2.42) be connected with the rows $T^0_{yj} = (T^0_{1j},\ T^0_{2j}, T^0_{3j})$ (where $j = 1, 2, 3$) of the matrix T^0_y from (2.44) by the following relation

$$\Delta^0(\rceil y, y) = \mathrm{col}\,\{\mathrm{div}_0\, T^0_{y1}, \mathrm{div}_0\, T^0_{y2}, \mathrm{div}_0\, T^0_{y3}\} \equiv \mathrm{Div}_0\, T^0_y \tag{2.45}$$

Then 1. the dynamic measure $D^0_0(\rceil dy, dy)$ is called a dynamic measure (a screw, a vector field) of deformation of the continuous medium;

2. the matrix T^0_y from (2.44) is called a dynamic deformator (d–deformator) of the medium at the point $y \in \mathbf{D}^\mu_3$;

3. the elements T^0_{ij} of T^0_y are called tensions of medium deformation at the point $y \in \mathbf{D}^\mu_3$;

4. if a dynamic measure of medium deformation is determined on σ^μ_3, then the the geometric Universe of mechanics is called a deformable one.

Axiom D 8 *The geometric Universe of mechanics is deformable.*

Comments

1. In the definitions mentioned above, no surfaces of geometric systems exist (these surfaces do not exist in the nature) and, therefore, i–columns of the matrix $T^0_{iy} = (T^0_{i1}, T^0_{i2}, T^0_{i3})^T$ as vectors of tensions on the coordinate planes of frames accompanying medium deformation have no physical reason. The notions as 'tangent' and 'normal' tensions to really non–existing (in the case of open systems) as well as imagined nowhere differentiated surfaces of closed geometric systems are not under consideration, too.
2. Respectively, there are no surface forces that have been introduced in continuum mechanics by analogue with surface forces of friction in classical mechanics.
3. Converting to an other inertial basis $[\mathbf{e}^f]$ we get

$$\delta^0(\rceil y, y) = T^0_y n^0_y \to c^0_f\, \delta^f(\rceil y, y) = T^0_y\ c^0_f\ n^f_y,\ T^f_y = c^{0,\ T}_f T^0_y c^0_f \tag{2.46}$$

4. The term of 'dynamic deformator' (the causal characteristic of medium deformation) is used in connection with the introduced concept of 'kinematic deformator' (see (2.10)) (*i.e.*, the geometric characteristic of medium deformation which is a consequence of the dynamic deformator action). Such a 'dynamic deformator' is the operator $D^{00}_d \in \mathcal{GD}^q_{yt}(\mathcal{R}, 3)$ related to the dynamic deformator by the mechanical state equations (section 4.1).

Definition 2.15 *Let both the scalar measure of inertia (the inertial mass $m_i(dy) = m(dy)$ – see (2.39)) and hence (in accordance with Axiom D 7) the scalar measure of gravity (the gravitational mass $m_g(dy)$ – see (2.38)) be invariant w.r.t. the time change at any point $y \in \mathbf{D}^\mu_3$*

$$\frac{d}{dt} m_i(dy) = 0 \Longleftrightarrow \frac{d}{dt} m_g(dy) = 0 \tag{2.47}$$

Then the geometric Universe of mechanics is called inertially (gravitationally) balanced.

Proposition 2.3 *Let: 1.* $\rho_0^{i(\urcorner y,y),0} \rho_y^i$ *be the density of the dynamic screw of inertia on* σ_3^μ *w.r.t. Lebesgue measure* $\mu_3(dy)$ *(see (2.29));*

2. $\rho_0^{g(\urcorner y,y),0} \rho_y^i$ *be the density of the dynamic screw of gravity on* σ_3^μ *w.r.t. Lebesgue measure* $\mu_3(dy)$ *(see (2.36));*
3. $\rho_0^{\Delta(\urcorner y,y),0}$ *be the density of the dynamic screw of deformation w.r.t. Lebesgue measure* $\mu_3(dy)$ *(see (2.42)).*

Then the sum of these densities is equal to zero at any point $y \in \mathbf{D}_3^\mu$*, i.e.,*

$$\rho_0^{i(\urcorner y,y),0} \rho_y^i + \rho_0^{g(\urcorner y,y),0} \rho_y^i + \rho_0^{\Delta(\urcorner y,y),0} = 0 \tag{2.48}$$

Proof follows from the additivity of the dynamic measures on σ_3^μ and from Axiom D 4 (see (2.44)).

Definition 2.16 *Let: 1.* $\Phi_y : \mathbf{R}_3 \times \mathbf{R}_3 \to \mathbf{R}_1$ *be a scalar function of the form*

$$\Phi_y = (T_y^0) \cdot V_y^0 = \operatorname{trace}\left(dv_y^{00}/dy^{00}\right)(T_y^0)^T = \sum_j T_{yj}^0 \operatorname{grad}_0 v_j^{00} \tag{2.49}$$

2. *on* σ_3^μ *there be defined a scalar measure* $U(dy)$ *invariant w.r.t. Galilean generalized group which is absolutely continuous relative to the scalar measure* $m(dy)$ *of inertia and to Lebesgue measure* $\mu_3(dy)$ *(with the densities* u_y *and* $u_y\rho_y$*), and its velocity under the condition* $\frac{d}{dt}m(dy) = 0$*, respectively, by the relations*

$$\begin{aligned} U(dy) &= u_y\ m(dy) = u_y\ \rho_y\ \mu_3(dy) && (2.50)\\ U^{\boldsymbol{\cdot}}(dy) &= u_y^{\boldsymbol{\cdot}}\ m(dy) = u_y^{\boldsymbol{\cdot}}\ \rho_y\ \mu_3(dy) && (2.51) \end{aligned}$$

3. *3. at each point* $y \in \mathbf{D}_3^\mu$ *there be defined a vector* q_y *with the coordinate column* q_y^0 *in the basis* $[\mathbf{e}^0]$ *and a scalar function* φ_y *such that*

$$-\rho_y\ u_y^{\boldsymbol{\cdot}} + \Phi_y + \operatorname{div}_0 q_y^0 + \rho_y\ \varphi_y = 0 \tag{2.52}$$

Then: 1. the geometric Universe of mechanics is called thermodynamically balanced;

2. *the measure* $U(dy)$ *is called inner (heat) energy of the medium;*
3. *the vector* q_y *is called a vector of heat flow at the point* $y \in \mathbf{D}_3^\mu$ *passing through a unity of area that is normal to the flow plane per a time unity;*
4. *the scalar function* φ_y *is called a heat quantity that is separated (absorbed) at the point* $y \in \mathbf{D}_3^\mu$ *of the medium per a unity of medium mass and per a time unity;*
5. *the scalar function* Φ_y *is called a density of power of tensions of d–deformator* T_y^0 *w.r.t. the measure* $m(dy)$*;*
6. *the matrix* dv_y^{00}/dy^{00} *(column* V_y^0*) is called a matrix (column) of velocities of medium deformation at the point* $y \in \mathbf{D}_3^\mu$*.*

Comments

1. The function Φ_y represents the work that the tensions of d–deformator T_y^0 perform on medium deformations (corresponding to the tensions) per a time unity.
2. d–deformator of the continuous medium depends on Pascal pressure and on the tensions of viscous or elastic origin (see D 4.1).
3. The matrix of velocities of deformations of a linearly changeable continuous medium is the whole matrix dv_y^{00}/dy^{00} but not its symmetric part $[dv_y^{00}/dy^{00}]$ which can be seen in Cauchy–Helmholtz identities as it is assumed in Kochin *et al.* 1964, Ladyzhenskaya 1987, Sedov 1972.
4. Any ways to define the medium deformation velocities,without using the tensions power concept (2.49) have no mathematical nor physical bases (for example, as elements of the symmetric matrix $[dv_y^{00}/dy^{00}]$ in Cauchy–Helmholtz identities (3.5), (3.78) in Kochin *et al.* 1964.
5. When heat transfer and heat detachment or absorption are absent ($\varphi_y = 0,\ q_y^0 = 0$) then

   $$\rho_y\ u_y^{\cdot} = \Phi_y$$

   i.e., the velocity $u_y^{\cdot}$ of change of the medium inner energy density w.r.t. the measure $m(dy)$ at a point is determined only by the power of tensions of d–deformator and by the density ρ_y of mass of this point relative to Lebesgue measure $\mu_3(dy)$.

Definition 2.17 *Let the geometric Universe of mechanics be 1. dynamically balanced (see (2.24)), 2. inertially (gravitationally) (see (2.47)) and 3. thermodynamically balanced (see (2.52)). Then the geometric Universe of mechanics is called balanced.*

Axiom D 9 *The geometric Universe of mechanics is balanced.*

Comment The introduced concept of balance differs, in principle, from the similar concept presented in Truesdell 1972, where the first requirement in the above definition is determined for a 3–dimensional 'force', but not for density of 6–dimensional measure, and the last two requirements are not present at all.

Definition 2.18 *Let:*

1. *on $\sigma_3^\mu \times \sigma_3^\mu$ there be defined a dynamic bi–measure $F_0^0(dx, dy)$ with properties (2.17) and (2.18);*
2. *on σ–algebra σ_3^μ there be defined a measure $U(dy)$ of the medium inner energy;*
3. *on σ–algebra σ_3^μ there be defined an inertial (gravitational) mass;*
4. *the geometric Universe of mechanics be balanced (Axiom D 9).*

Then 1. the geometric Universe of mechanics is called the mechanics Universe and it is noted as

$$\mathfrak{B} = \{\mathbf{A}_3^\mu,\ \mathcal{GTD}_{yt}(\mathcal{R},3),\ F_0^0(dx,dy),\ U(dy),\ m(dy)\} \tag{2.53}$$

2. elements of σ–algebra σ_3^μ of the mechanics Universe are called mechanical systems.

Comments

1. According to the ideology developed in the monograph, in the mechanics Universe there do not exist any points without a volume with concentrated scalar and vector measures (mass, dynamic screw, inner energy, *etc.*). In each point of Universe only densities of these (and others) measures differ from 0. That is why in terms of these measures the fundamental properties of Universe can be formulated only for densities of these measures either w.r.t. Lebesgue measure, or w.r.t. other measures, as for example the masses (inertial or gravitational).
2. Introduced in standard courses of mechanics, the concepts of non–zero Newton forces 'applied' at non–existing mass points without volume, of a momentum and of moment of momentum of really a finite set of (non–existing) points mentioned above – they do not arise here at all. Out of sense there is to prove theorems about the change of the mentioned above quantities. It poins out the fact that these quantities and theorems refer not to the fundamental properties of the 'nature', but to the common assumed variant (Newton variant) of the axiomatics of mechanics.
3. The 'integral form of laws for mass and energy conservation' does not arise in the case of continuous medium since:

   3.1. In the nature there do not exist forms of continuous medium which are limited by real surfaces except a finite number of natural continuous media (as gas in a container with plunger, a fluid with a free boundary, *etc.*).

   3.2. If for such forms there be used elements of the defined above Borell σ–algebra, then, in the first place, there exists a continual subset of open mechanical systems which have not boundaries at all, and second, there exists a continual subset of systems which have non–derivative boundaries. For such systems and for other ones 'integral laws of mass and energy conservation' could be formally formulated but the theorem of Ostrogradsky-Gauss for transfer to local formulation of the laws above (of type (2.48) and (2.52)) cannot be used. The statements of the theory that are formulated for a part of the mechanical systems of Universe only, and that are 'convenient' for an use of any mathematical tools, could not be considered as 'nature laws'.
4. The 'integral form of the conservation of the momentum and of moment of momentum' does not arise here since:

   4.1. Only a part of the mechanical systems has piecewise–differentiable boundaries.

   4.2. The real mechanical systems (except a finite set of naturally bounded) have no physical boundaries, and, therefore, in the real locally changeable medium, 'surface forces' do not exist.

   4.3. The pointed 'integral theorems' cannot be proved as independent propositions, they cannot be obtained by any mathematical transformation of these theorems which indeed are proved for a finite set of (physically non–existing) mass points without volume.

4.4. In general, it is not true that the moment of momentum in an arbitrary mechanical system of a continuous medium is conserved because this conservation demands symmetry of d–deformator T_y^0 (this symmetry is valid in the particular case of medium motion) (see Chapter 3). The symmetry of d–deformator is a consequence of Ostrogradsky–Gauss theorem when transforming the 'integral form of equations for moment of momentum conservation' to the local one, and, therefore, it cannot be valid for continual sets of mechanical systems without boundaries and with boundaries anywhere non–differentiable. Besides, no properties of a real medium can be obtained on the basis of analysis of the recording form of mathematical models of medium motion.

5. The definition of the vector measure of inertia (with analogy to vector measures of gravity and of deformation) excludes from discussion the questions about the physical sense of such concepts as 'd'Alembert principle', 'reality', or 'non–reality' of forces of inertia, 'matter', 'quantity of matter', *etc.*
6. There do not exist any dynamic screws different from the considered in Axioms D 5 – D 9 (as 'surface', 'complemental', 'vibrational', of reactions, *etc.*).
7. Any mechanical system is a geometric system for which such concepts as inertia, gravity, deformation and inner energy are defined.
8. The formulated above system of 17 axioms (8 of kinematics and 9 of dynamics) is complete, consistent and it completely determines the primary properties of geometric Universe of mechanics represented only continuous continual locally and linearly locally changeable μ–measured medium on σ–algebra of mechanical systems where dynamic bi–measure, measure of inner energy and scalar measure of inertia or gravity (mass) are determined.

2.3. Generalized Galilean group on mechanics Universe

To end the formulation of primary properties of the geometric Universe of mechanics let us prove the invariance of the above axioms w.r.t. the generalized Galilean group.

Definition 2.19 *Let: 1.* $\mathbf{E}_l$ *and* $\mathbf{E}_k$ *be Cartesian frames in* $\mathbf{A}_3^\mu$*;*

2. v_k^{ll} *be the coordinate column of the velocity vector* v_k^l *of shift of the origin* o_k *w.r.t. the frame* $\mathbf{E}_k$ *in the basis* $[\mathbf{e}^l]$*,* $v_k^{ll} = o_k^{ll\bullet}$*;*

3. the velocity vector v_k^l *be constant;*

4. p_k^{ll} *be the coordinate column of the constant vector* p_k^l *of shift of the origin* o_k *of the frame* $\mathbf{E}_l$ *(with the basis* $[\mathbf{e}^l]$*);*

5. $\mathcal{T}_t(\mathcal{R}, 3)$ *be the one–parameter group of shifts in* $\mathbf{A}_3^\mu$ *by the vectors*

$$o_k^{ll} = p_k^{ll} + v_k^{ll} t \tag{2.54}$$

where $t \in \mathbf{T}_+$*;*

6. $\mathcal{SO}(\mathcal{R}, 3)$ *be the group of constant rotations of the directrix vector space* $\mathbf{V}_3$ *of the affine–vector space* $\mathbf{A}_3^\mu$*;*

7. *$\mathcal{ML}(\mathcal{R},6) \subset \mathcal{L}(\mathcal{R},6)$ be a subgroup of the motions of the vector space of screws (see P 7.4) that is introduced by the groups $\mathcal{T}_t(\mathcal{R},3)$ and $\mathcal{SO}(\mathcal{R},3)$*

$$\mathcal{ML}(\mathcal{R},6) = \mathcal{T}_t(\mathcal{R},6)\mathcal{SO}(\mathcal{R},6) \tag{2.55}$$

where $\mathcal{T}_t(\mathcal{R},6)$ is the multiplicative group (remind that $\mathcal{T}_t(\mathcal{R},3)$ is not a multiplicative group) generated by shifts with vectors (2.54) ($\mathcal{M}$ reminds the word 'motion'); $\mathcal{SO}(\mathcal{R},6)$ is the group of constant rotations of the (7.8)–kind.

Then the group $\mathcal{ML}(\mathcal{R},6)$ is called a Galilean generalized group on the vector space consisting of kinematic and dynamic screws of any nature (inertial, gravitational and deformable ones).

Definition 2.20 *The set of all objects, being invariant w.r.t. the Galilean generalized group (i.e., the geometry of the Galilean generalized group), is called Galilean mechanics.*

Comments

1. According to the definition, Galilean mechanics consists of objects of different nature (basic undefined concepts, axioms, definitions, propositions, and so on) whose mathematical formalization are not changed when doing transformations on the generalized Galilean group.
2. Galilean mechanics is worked out on axioms. If they are elements of Galilean mechanics, *i.e.*, they are invariant w.r.t. the generalized Galilean group, then all results mathematically obtained on their basis are Galilean mechanics elements also.

Proposition 2.4 *Axioms of dynamics (D 1 – D 9) are invariant w.r.t. the generalized Galilean group.*

Proof Let $L_k^0 : \mathbf{E}_0 \to \mathbf{E}_k$, $L_k^0 \in \mathcal{ML}(\mathcal{R},6)$. It is necessary to prove that mathematical formulations of all dynamics axioms in the frames $\mathbf{E}_0$ and $\mathbf{E}_k$ are written in the same way.

1. As the mass density $\rho(y)$ w.r.t. Lebesgue measure $\mu_3(dy)$ on σ_3^μ is scalar and it does not depend on the frame choice then the proof that the density $\rho_k^{i(\urcorner y,y),k}$ of the inertial dynamic measure is invariant in $\mathbf{E}_0$ w.r.t. generalized Galilean transformation $L_k^0 \in \mathcal{ML}(\mathcal{R},6)$ is reduced to the proof that the dynamic measure density is invariant about L_k^0 w.r.t. the measure $m(dy)$ – the sliding vector $l_0^{v^\cdot(y),0}$. Otherwise, according to definition (2.30), it is necessary to show that the relation

$$\rho_k^{i(\urcorner y,y),k} = -l_k^{v^\cdot(y),k},\ v^\cdot(y) \equiv v_y^{kk\cdot} \tag{2.56}$$

is fulfilled in $\mathbf{E}_k$.

For the left-hand side part of (2.30), in accordance with (7.7), we get

$$\rho_0^{i(\urcorner y,y),0} = L_k^0\ \rho_k^{i(\urcorner y,y),k}$$

Let us prove that $l_0^{v^\cdot(y),0} = L_k^0\ l_k^{v^\cdot(y),k}$. Indeed, $y^{00} = o_k^{00} + y^{k0} \to y^{00} = o_k^{00} + c_k^0 y^{kk} \to v_y^{00} = v_k^{00} + c_k^0\ v_y^{kk}$, $c_k^0 = \text{const}$, $v_k^{00} = o_k^{00\cdot} \Leftrightarrow (o_k^{00} = p_k^{00} + v_k^{00}t)^\cdot$,

$p_k^{00} = \text{const}$, $v_y^{00\cdot} = v_k^{00\cdot} + c_k^0\ v_y^{kk\cdot} \to v_y^{00\cdot} = c_k^0 v_y^{kk\cdot}$, $v_k^{00} = \text{const}$, $l_0^{v^\cdot(y),0} = G_{y0}^0$ $\text{col}\{v_y^{00\cdot},\ v_y^{00\cdot}\} = G_{y0}^0\,\text{col}\{c_k^0 v_y^{kk\cdot},\ c_k^0 v_y^{kk\cdot}\} = G_{y0}^0\ [c_k^0]\,\text{col}\{v_y^{kk\cdot}, v_y^{kk\cdot}\} = T_k^0\ G_{yk}^0[c_k^0]$ $\text{col}\{v_y^{kk\cdot}, v_y^{kk\cdot}\} = T_k^0\ [c_k^0][c_k^0]^T\ G_{yk}^0[c_k^0]\,\text{col}\{v_y^{kk\cdot}, v_y^{kk\cdot}\} = T_k^0\ [c_k^0]\ G_{yk}^k\,\text{col}\{v_y^{kk\cdot}, v_y^{kk\cdot}\}$ $= L_k^0\ G_{yk}^k\,\text{col}\{v_y^{kk\cdot}, v_y^{kk\cdot}\} = L_k^0\ l_k^{v^\cdot(y),k}$ – see (7.7).

Substituting the obtained results in (2.30), and then dividing by L_k^0, $\det L_k^0 \neq 0$, we obtain (2.56).

2. For the density of the gravitational dynamic bi–measure (2.34) we get $l_0^{g(dx,dy),0} = G_{y0}^0\,\text{col}\{g^0,\ g^0\} = G_{y0}^0\,\text{col}\{c_k^0 g^k, c_k^0 g^k\} = G_{y0}^0[c_k^0]\,\text{col}\{g^k, g^k\} =$ $T_k^0\ G_{yk}^0\ [c_k^0]\,\text{col}\{g^k, g^k\} = T_k^0\ [c_k^0][c_k^0]^T\ G_{yk}^0[c_k^0]\,\text{col}\{g^k, g^k\} = T_k^0\ [c_k^0]\ G_{yk}^k\,\text{col}\{g^k,$ $g^k\} =$
$L_k^0\ G_{yk}^k\,\text{col}\{g^k,\ g^k\} = L_k^0\ l_k^{g(dx,dy),k}$.

3. For the density of the dynamic measure of medium deformation at a point y taking into account (7.7) we get

3.1. $\rho_k^{i(\urcorner y,y),k} = L_k^0\ \rho_0^{\Delta(\urcorner y,\ y),0} = L_k^0\ G_0^0\,\text{col}\{T_y^0 n_y^0, T_y^0 n_y^0\} = T_0^{kk}\ [c_0^k]\ G_0^0\ [c_k^0][c_0^k]$ $\text{col}\{T_y^0\ c_k^0\ c_0^k\ n_y^0, T_y^0\ c_k^0\ c_0^k\ n_y^0\} = T_0^{kk}\ G_0^k\,\text{col}\{T_y^k n_y^k, T_y^k n_y^k\}$ and at last $G_k^k\,\text{col}\{T_y^k\ n_y^k,$

$T_y^k\ n_y^k\}$, where G_0^0 and G_k^k are the matrices of sliding vector representations in the frames $\mathbf{E}_0$ and $\mathbf{E}_k$.

3.2. For the vector $\Delta^k(\urcorner y, y)$ in accordance with (2.45), we get

$$\Delta^k(\urcorner y, y) = \Delta^k(\urcorner y^k, y^k) = c_0^k\ \Delta^0(\urcorner y^0, y^0) \to c_0^k\,\text{Div}_0\,T_y^0 = \text{Div}_k\,T_y^k \tag{2.57}$$

with $\text{Div}_0\,T_y^0 = \text{col}\{\text{div}_0\,T_{y1}^0,\ \text{div}_0\,T_{y2}^0, \text{div}_0\,T_{y3}^0\}$, $\text{Div}_k\,T_y^k = \text{col}\{\text{div}_k\,T_{y1}^k, \text{div}_k\,T_{y2}^k,$ $\text{div}_k\,T_{y3}^k\}$.

Really, let: 3.2.1. $\mathbf{E}_0$ and $\mathbf{E}_k$ be two inertial frames, $L_k^0 :\ \mathbf{E}_0 \to\ \mathbf{E}_k, L_k^0 \in \mathcal{ML}(\mathcal{R}, 6) = \mathcal{T}_t(\mathcal{R}, 6)\,\mathcal{SO}(\mathcal{R}, 6)$;

3.2.2. T_y^0 and T_y^k be the dynamic deformators of the medium (see (2.44)) in the bases $[\mathbf{e}^0]$, $[\mathbf{e}^k]$ of the frames $\mathbf{E}_0$ and $\mathbf{E}_k$, (see (2.46))

$$T_y^0 = c_k^0\ T_y^k\ c_k^{0,T},\quad T_y^k = c_k^{0,T} T_y^0\ c_k^0 \tag{2.58}$$

3.2.3. $\text{Div}_0\,T_y^0$ and $\text{Div}_k\,T_y^k$ be 2×1–columns of divergences of the rows of T_y^0 and T_y^k in $\mathbf{E}_0$ and $\mathbf{E}_k$, respectively:

$$\begin{aligned}
\text{Div}_0\,T_y^0 &= \text{col}\{\text{div}_0\,T_{y1}^0,\ \text{div}_0\,T_{y2}^0\},\ \text{Div}_k\,T_y^k = \text{col}\{\text{div}_k T_{y1}^k,\ \text{div}_k T_{y2}^k\}\\
\text{div}_0\,T_{y1}^0 &= \partial T_{y11}^0/\partial y_1^0 + \partial T_{y21}^0/\partial y_2^0,\ \text{div}_0\,T_{y2}^0 = \partial T_{y12}^0/\partial y_1^0 + \partial T_{y22}^0/\partial y_2^0\\
\text{div}_k\,T_{y1}^k &= \partial T_{y11}^k/\partial y_1^k + \partial T_{y21}^k/\partial y_2^k,\ \text{div}_k\,T_{y2}^k = \partial T_{y12}^k/\partial y_1^k + \partial T_{y22}^k/\partial y_2^k\\
\partial T_{ij}^k/\partial y_p^0 &\equiv \partial T_{ij}^k/\partial y_1^k\cdot\partial y_1^k/\partial y_p^0 + \partial T_{ij}^k/\partial y_2^k\cdot\partial y_2^k/\partial y_p^0
\end{aligned} \tag{2.59}$$

where

$$y^k = c_k^{0,T} y^0 \tag{2.60}$$

Then the following relation

$$\mathrm{Div}_0\, T^0_y = c^0_k\ \mathrm{Div}_k\, T^k_y \tag{2.61}$$

is true.

The proof of (2.61) is achieved by simple calculations due to (2.45). For example, in the two–dimensional case, the first coordinate $\mathrm{div}_0\, T^0_{y1}$ of the column $\mathrm{Div}_0\, T^0_y$ is of the form $\mathrm{div}_0\, T^0_{y1} = \partial T^0_{y11}/\partial y^0_1 + \partial T^0_{y21}/\partial y^0_2 = \partial/\partial y^0_1(T^k_{11}\cos^2\theta - T^k_{12}\cos\theta\sin\theta - T^k_{21}\cos\theta\sin\theta + T^k_{22}\sin^2\theta) + \partial/\partial y^0_2(T^k_{11}\cos\theta\sin\theta - T^k_{12}\sin^2\theta - T^k_{21}\cos^2\theta + T^k_{22}\cos\theta\sin\theta)$.

Using (2.46) in accordance with (4.51), we get $\mathrm{div}_0\, T^0_{y1} = \ \mathrm{div}_k\, T^k_{y1}\cos\theta - \mathrm{div}_k T^k_{y2}\sin\theta$.

Similarly, the relation $\mathrm{div}_0\, T^0_{y2} = \ \mathrm{div}_k\, T^k_{y1}\sin\theta - \mathrm{div}_k\, T^k_{y2}\cos\theta$ is proved.

The matrix representation of the obtained results is what we need, *i.e.*, (2.61).

In the three–dimensional case, the proof is similar but more awkward.

Let us prove the invariance of relation (2.49) w.r.t. the generalized Galilean group using the relation $v^{00}_y = v^{00}_k + c^0_k v^{kk}_y = v^{00}_k + v^{k0}_y,\ v^{00}_k = \mathrm{const}$, here

$$\sum_{ij} T^0_{ij}\partial v^{00}_{yj}/\partial y^{00}_i = \sum_{ij} T^0_{ij}\partial(v^{00}_{kj} + v^{k0}_{yj})/\partial y^{00}_i = \sum_{ij} T^0_{ij}\partial v^{k0}_{yj}/\partial y^{00}_i$$

It remains in the scalar equalities (2.49) to pass to the basis $[\mathbf{e}^k]$: $T^0_{yj}\ \mathrm{grad}_0\, v^{k0}_{yj} = T^k_{yj}\ \mathrm{grad}_k\, v^{kk}_{yj}$.

The following proposition is a consequence of the proved above.

Proposition 2.5 *Let: 1.* $\mathbf{E}_0$ *be an inertial frame;*
2. $L^0_k:\ \mathbf{E}_0 \to\ \mathbf{E}_k, L^0_k \in \mathcal{ML}(\mathcal{R}, 6)$ *– see (7.7) and (2.55).*
Then the frame $\mathbf{E}_k$ *is inertial one.*

Comments

1. Comparing the proved results with the definition of an inertial frame (D 2.11-1) and with proposition P 2.4 the following proposition can be formulated: the set of inertial frames is a class of equivalence w.r.t. the generalized Galilean group, *i.e.*, $\mathbf{E}_l \approx \mathbf{E}_k$ if $L^l_k : \mathbf{E}_k \to \mathbf{E}_1,\ L^l_k \in \mathcal{ML}(\mathcal{R}, 6) = \mathcal{T}_t(\mathcal{R}, 6)\mathcal{SO}(\mathcal{R}, 6)$. The factor–structure that is defined upon the set of all frames in $\mathbf{A}^\mu_3$ in according to the pointed equivalence, consists of two elements: the set of inertial frames and the set of non–inertial ones.
2. The formalization of all objects of Galilean mechanics does not depend on the inertial frame choice.

2.4. Secondary properties of geometric Universe of mechanics

Secondary properties of geometric Universe of mechanics are consequences of the primary ones that were formulated as axioms.

2.4.1. *Dynamics homogeneity and isotropy of geometric Universe of mechanics*

Definition 2.21 *Let: 1.* $\mathcal{L}(\mathcal{R},6) = \mathcal{T}(\mathcal{R},6)\ \mathcal{SO}(\mathcal{R},6)$ *be the group of motions on the screw space (Konoplev 1987a, Konoplev et al. 2001);*

2. dynamics axioms D 1 – D 9 be invariant w.r.t. the group $\mathcal{L}(\mathcal{R},6)$.

Then the mechanics Universe is called dynamically uniform and dynamically isotropic, respectively.

Proposition 2.6 *The mechanics Universe is dynamically uniform and isotropic (6–dimensional generalization of Neutter theorem (Devis 1982, Yaglom 1969)).*

Proof coincides with the proof of the previous proposition since it is its particular case when $v_k^{00} \equiv 0$.

2.4.2. *Equations of motion of locally changeable continuous medium*

Proposition 2.7 *Let: 1.* $\mathbf{D}_3^\mu$ *be a locally changeable continuous medium;*

2. $\rho_0^{i(\urcorner y,y),0} \rho_y^i \equiv -\rho_0^{v^{\cdot}(y),0} \rho_y$ *be the sliding vector of the density of the dynamic measure of inertia w.r.t.* $\mu_3(dy)$ *at a point* $y \in \mathbf{D}_3^\mu$ *(see (2.29));*

3. $\rho_0^{g(\urcorner y,y),0} \rho_y$ *be the sliding vector of the density of gravity w.r.t.* $\mu_3(dy)$ *at a point* $y \in D_3^\mu$ *(see (2.36));*

4. $\rho_0^{\Delta(\urcorner y,y),0}$ *be the sliding vector of the density of the dynamic measure of continuous medium deformation w.r.t.* $\mu_3(dy)$ *at a point* $y \in D_3^\mu$ *(see (2.42)).*

Then at each point $y \in D_3^\mu$ *of the changeable continuous medium the following relation is fulfilled*

$$-\rho_0^{v^{\cdot}(y),0}\ \rho_y + \rho_0^{g(\urcorner y,y),0}\ \rho_y + \rho_0^{\Delta(\urcorner y,y),0} = 0 \tag{2.62}$$

Proof is based on Axiom D 4 due to (2.48).

Proposition 2.8 *Let: 1.* $\rho_y v_y^{00\cdot}$ *be the main vector of the sliding vector of the density of the dynamic measure (screw) of inertia (see (2.31), (7.4))*

$$\rho_y\ v_y^{00\cdot} = mv\ (-\rho_0^{v^{\cdot}(y),0} \rho_y) \tag{2.63}$$

2. $\rho_y g_t^0(\urcorner y, y)$ *be the main vector of the sliding vector of the density of the dynamic measure (screw) of gravity (see (2.36), (7.4))*

$$\rho_y\ g_t^0(\urcorner y, y) = mv\ (\rho_0^{g(\urcorner y,y),0} \rho_y) \tag{2.64}$$

3. $\mathrm{Div}_0\, T_y^0 \equiv \mathrm{col}\{\mathrm{div}_0\, T_{y1}^0,\ \mathrm{div}_0\, T_{y2}^0,\ \mathrm{div}_0\, T_{y3}^0 \ldots\}$ *be the main vector of the sliding vector of the density of the dynamic measure (screw) of deformation (see (2.45)).*

Then the equation of the locally changeable continuous medium motion at the point is of the form

$$-\rho_y\ v_y^{00\cdot} + \rho_y\ g_t^0(\urcorner y, y) + \mathrm{Div}_0\, T_y^0 = 0 \tag{2.65}$$

Proof The sum of sliding vectors whose directrixes cross at one point is a sliding vector generated by a sum of free vectors generating terms with directrixes passing through the same point; for arbitrary free vectors $x_1, x_2, \ldots, x_n \in \mathbf{V}_3$ (see (7.2)) (Konoplev 1987a, Konoplev *et al.* 2001)

$$\sum_{p=1}^{n} l_{0p}^{x0} = l_0^{(x_1+x_2+\ldots+x_n),0} \tag{2.66}$$

In this case the relations $\sum_{p=1}^{n} l_{0p}^{x0} = 0$ and $x_1 + x_2 + \ldots + x_n = 0$ are equivalent (see (7.2)) that proves the proposition.

Comments

1. From P 2.8 follows that the motion of a locally changeable continuous medium is not determined by dynamic measures distributed on σ_3^μ (as it is usually assumed since the properties of the finite set of 'mass' points are carried over a continuous medium).

   This motion is determined by 3–dimensional free vectors $f^0(\cdot,\cdot)$, generating these measures. Besides, although $f^0(\cdot,\cdot)$ are free vectors, their μ_3–densities depend on the point $y \in \mathbf{A}_3^\mu$, and therefore they belong to $\mathbf{V}_{3y}$ (see (2.6)) but not to $\mathbf{V}_3$.
2. The passage from equation (2.48) to equation (2.62) is possible for a locally changeable medium only, whose properties and characteristics of properties variation are local ones.
3. The linear changeability of this medium does not follow from medium local changeability. That is why the medium motion equation has an universal form. If the medium is linearly locally changeable (Axiom K 8), the medium motion could be treated under some conditions by using linear mathematical models.
4. Neither the motion equation (2.65) nor their proof allow to conclude anything about properties of d–deformator T_y^0 of the continuous medium, for example about its symmetry.

2.4.3. *Equation of balance of inertial mass for locally changeable continuous medium*

Proposition 2.9 *1. The following equation written in two equivalent forms is true*

$$\rho_y^{\boldsymbol{\cdot}} + \rho_y \operatorname{div}_0 v_y^{00} = 0 \tag{2.67}$$

$$\partial \rho_y / \partial t + \operatorname{div}_0 \rho_y v_y^{00} = 0 \tag{2.68}$$

2. Equations (2.67) and (2.68) are called differential equations of the balance of inertial (gravitational) mass of a locally changeable continuous medium at a point $y \in D_3^\mu$.

Proof According to Axiom D 9 (see (2.47))

$$d/dt\; m(dy) = 0 \tag{2.69}$$

we arrive at equation (2.67) from $dm(dy)/dt = d(\rho_y\,\mu_3(dy))/dt = \rho^{\boldsymbol{\cdot}}_y\ \mu_3(dy)+$ $\rho_y\mu^{\boldsymbol{\cdot}}_3(dy) = 0 \to \rho^{\boldsymbol{\cdot}}_y + \rho_y\ \mu^{\boldsymbol{\cdot}}_3(dy)/\mu_3(dy) = 0$ taking into account $\mu^{\boldsymbol{\cdot}}_3(dy)/\mu_3(dy)$ $= \mathrm{div}_0\, v^{00}_y$. Equation (2.68) is obtained from equation (2.67) with the help of $\rho^{\boldsymbol{\cdot}}_y = \partial\rho_y/\partial t + v^{00}_y\ \mathrm{grad}_0\,\rho_y$.

Comments

1. Equalities (2.67) and (2.68) do not depend on the kind of mass (inertial or gravitational) since the gravitational constant does not depend on the time.
2. The name of the equations corresponds to physical essentialness of the question: the velocity of the mass density at a point coincides with the characteristic of the velocity of mass input/output at a point (from a point).

2.4.4. *Equations of energy balance for locally changeable continuous medium*

Notation Henceforth

1. $v^{00}_y \in \mathbf{R}_3$ is the coordinate column (in $[\mathbf{e}^0]$) of the motion velocity $v^0_y \in \mathbf{V}_3$ of a point $y \in D^\mu_3$w.r.t. the inertial frame $\mathbf{E}_0 \in \mathbf{A}^\mu_3$;
2. $v^{00}_y \cdot v^{00}_y = v^{00,T}_y\ v^{00}_y = v^{00,2}_y$is the inner product;
3. $m(dy)$ is the scalar measure of inertia (mass) on σ^μ_3– see (2.25).

Definition 2.22 *1. The scalar measure $K(dy)$ on σ^μ_3, absolutely continuous w.r.t. $m(dy)$,with non–negative density $\rho_v = \frac{1}{2}\ v^{00,2}_y \geqslant 0$,*

$$K(dy) = \frac{1}{2}\ v^{00,2}_y\ m(dy) = \frac{1}{2}\ v^{00,2}_y\ \rho_y\ \mu_3(dy) \tag{2.70}$$

is called a kinetic energy of a locally changeable continuous medium.

2. The density $\rho_k$of $K(dy)$w.r.t. Lebesgue measure $\mu_3(dy)$at a point $y \in \mathbf{D}^\mu_3$

$$\rho_k = \frac{1}{2}\ \rho_y\ v^{00,2}_y \tag{2.71}$$

is called a velocity pressure of the medium at a point $y \in \mathbf{D}^\mu_3$.

Definition 2.23 *Let:*

1. *$\rho_g = \rho_y\ g^0_y(\urcorner y, y) \in \mathbf{R}_3$ be the density of the main vector of the dynamic measure of gravity w.r.t. Lebesgue measure $\mu_3(dy)$ at a point $y \in \mathbf{D}^\mu_3$ (see (2.34), (2.36));*
2. *$\Delta^0(\urcorner y, y) = \mathrm{Div}_0\, T^0_y \in \mathbf{R}_3$ be the density of the main vector $v^0_y \in \mathbf{R}_3$ of the dynamic measure of medium deformation w.r.t. Lebesgue measure $\mu_3(dy)$ at a point $y \in \mathbf{D}^\mu_3$ (see (2.42), (2.45)).*

Then the inner products

$$\rho_{gv} = \rho_g \cdot v^{00}_y = \rho_y g^0_y(\urcorner y, y) \cdot v^{00}_y \tag{2.72}$$

$$\rho_{\Delta v} = \Delta^0(\urcorner y, y) \cdot v^{00}_y \tag{2.73}$$

are called powers $\rho_g = g^0_y(\urcorner y, y)$ and $\Delta^0(\urcorner y, y)$ of the densities of the main vectors (2.35) and (2.45) of screws (2.40), (2.42) w.r.t. Lebesgue measure $\mu_3(dy)$ at a point $y \in \mathbf{D}^\mu_3$.

Proposition 2.10 *Let:*

1. $\rho_{gv} = \rho_y g_y^0(\urcorner y, y)\cdot v_y^{00}$, $\rho_{\Delta v} = \Delta^0(\urcorner y, y)\cdot v_y^{00}$ *be powers of the vectors* $\rho_y g_y^0(\urcorner y, y)$ *and* $\Delta^0(\urcorner y, y)$ *from (2.72), (2.73);*
2. $\frac{1}{2}\ \rho_y v_y^{00,2\bullet}$ *be the density of the medium kinetic energy change velocity w.r.t. Lebesgue measure* $\mu_3(dy)$ *at a point* $y \in \mathbf{D}_3^\mu$ *(velocity pressure of the medium).*

Then: 1. the following relation is true

$$\frac{1}{2}\ \rho_y\ v_y^{00,2\bullet} = \rho_y g_y^0(\urcorner y, y)\cdot v_y^{00} + \Delta^0(\urcorner y, y)\cdot v_y^{00} \tag{2.74}$$

2. *relation (2.74) is called a differential equation of the balance of medium mechanical system energy at a point* $y \in \mathbf{D}_3^\mu$*.*

Proof is achieved by doing scalar multiplication of equation (2.65) and the vector v_y^{00} and with the help of $\rho_y v_y^{00\bullet}\cdot v_y^{00} = \frac{1}{2}\ \rho_y\ v_y^{00,2\bullet}$.

Comments

1. From (2.74) follows that the density of the velocity of change of the medium kinetic energy w.r.t. Lebesgue measure $\mu_3(dy)$ at a point $y \in \mathbf{D}_3^\mu$ is determined by the sum of powers of the main vector of the sliding vectors of the densities $\rho_y g_y^0(\urcorner y, y)$ and $\Delta^0(\urcorner y, y)$ of the medium gravity and deformation dynamic measures w.r.t. the same measure at the same point.
2. The equation of balance of the medium mechanical energy at a point $y \in \mathbf{D}_3^\mu$ is equivalent to the medium motion equation (2.65) at the same point.

Proposition 2.11 *Let:*

1. $\delta^0(\urcorner y, y) = T_y^0 n_y^0$ *be* 3*–dimensional free vector from relation (2.44)*

$$\delta^0(\urcorner y, y) = mv\ \rho_0^{\delta(\urcorner y, y),0} = T_y^0\ n_y^0 \tag{2.75}$$

2. $\rho_{\delta v}$ *be the power of the vector* $\delta^0(\urcorner y, y)$ *at a point* $y \in \mathbf{D}_3^\mu$

$$\rho_{\delta v} = \delta^0(\urcorner y, y)\cdot v_y^{00} = T_y^0 n_y^0\cdot v_y^{00} \tag{2.76}$$

Then the power $\rho_{\delta v}$ *is represented as follows*

$$\rho_{\delta v} = T_y^{0,T} v_y^{00}\cdot n_y^0 \tag{2.77}$$

where $T_y^{0,T} v_y^{00}$ *is the vector of the powers* $T_{iy}^{0,T} v_y^{00}$ *of i–columns* $T_{iy}^{0,T} = (T_{i1}^0,\ T_{i2}^0,\ T_{i3}^0)^T$ *of d–deformator* $T_y^0, i = 1, 2, 3.$

Proposition 2.12 *Let: 1. the velocity of change of the medium kinetic energy (2.70) at a point* $y \in \mathbf{D}_3^\mu$ *be of the form*

$$K^\bullet(dy) = \frac{1}{2}\ v_y^{00,2\bullet}\ m(dy) = \frac{1}{2}\ v_y^{00,2\bullet}\ \rho_y\ \mu_3(dy) \tag{2.78}$$

2. the velocity of change of the medium inner energy (2.51) at a point $y \in \mathbf{D}_3^\mu$ be

$$U^{\cdot}(dy) = u^{\cdot}_y \; m(dy) = u^{\cdot}_y \; \rho_y \; \mu_3(dy) \tag{2.79}$$

Then 1. the following equation is fulfilled

$$\rho_y(\frac{1}{2}\; v_y^{00,2} + u_y)^{\cdot} = \rho_{gv} + \rho_{\Delta v} + \Phi_y + \mathrm{div}_0\, q_y^0 + \rho_y\; \varphi_y \tag{2.80}$$

or, due to (2.72) and (2.73), (2.49)

$$\begin{aligned}\rho_y(\frac{1}{2}\; v_y^{00,2} + u_y)^{\cdot} \;\; &= \;\; \rho_y g_y^0(\urcorner y, y)\cdot v_y^{00} + \Delta^0(\urcorner y, y)\cdot v_y^{00} + \\ &\sum_{ij} T_{ij}^0\; \partial v_j^0/\partial y_i^{00} + \mathrm{div}_0\, q_y^0 + \rho_y \varphi_y \end{aligned}\tag{2.81}$$

2. relation (2.81) is called a differential equation of the medium energy balance at a point $y \in \mathbf{D}_3^\mu$.

Proof The result is obtained by adding the mechanical energy equation (2.74) with the energy thermodynamic equation (2.52).

Comments

1. From relation (2.81) follows that the sum of the densities of the velocity of change of the mechanical energy and the heat one of the system w.r.t. the measure $\mu_3(dy)$ at a point coincides with the sum of powers of the main vectors of the dynamic measures densities, the power of d–deformator tensions, divergence of heat flow vector and heat apportionment or heat absorption per unit of time at the same point.
2. Equation (2.81) can be written in another form.

Proposition 2.13 *For divergence of powers $T_y^{0,T} v_y^{00}$ of d–deformator, the following presentation is true*

$$\mathrm{div}_0 \left(T_y^{0,T} v_y^{00}\right) = \rho_{\Delta v} + \Phi_y \tag{2.82}$$

or in the developed form

$$\mathrm{div}_0\, T_y^{0,T} v_y^{00} = \Delta^0(\urcorner y,\; y)\cdot v_y^{00} + \sum_{ij} T_{ij}^0\; \partial v_j^0/\partial y_i^{00} \tag{2.83}$$

Proof Indeed, we have $\mathrm{div}_0\, T_y^{0,T}\; v_y^{00} = \mathrm{div}_0(T_{1y}^{0,T} v_y^{00}, T_{2y}^{0,T} v_y^{00}, T_{3y}^{0,T} v_y^{00})^T =$ $(\partial T_{1y}^0/\partial y_1^{00} + \partial T_{2y}^0/\partial y_1^{00} + \partial T_{3y}^0/\partial y_1^{00}) v_y^{00} + \sum_{ij} T_{ij}^0 \partial v_j^0/\partial y_i^{00} = [(\partial T_{11}^0/\partial y_1^{00},$ $\partial T_{12}^0/\partial y_1^{00},\; \partial T_{13}^0/\partial y_1^{00})^T + (\partial T_{21}^0/\partial y_2^{00},\; \partial T_{22}^0/\partial y_2^{00},\; \partial T_{23}^0/\partial y_2^{00})^T + (\partial T_{31}^0/\partial y_3^{00},$ $\partial T_{32}^0/\partial y_3^{00},\; \partial T_{33}^0/\partial y_3^{00})^T] v_y^{00} + \sum_{ij} T_{ij}^0 \partial v_j^0/\partial y_i^{00} = [(\partial T_{11}^0/\partial y_1^{00} + \partial T_{21}^0/\partial y_2^{00} +$ $\partial T_{31}^0/\partial y_3^{00})^T, (\partial T_{12}^0/\partial y_1^{00} + \partial T_{22}^0/\partial y_2^{00} + \partial T_{32}^0/\partial y_3^{00})^T, (\partial T_{13}^0/\partial y_1^{00} + \partial T_{23}^0/\partial y_2^{00} +$ $\partial T_{33}^0/\partial y_3^{00})^T] v_y^{00} + \sum_{ij} T_{ij}^0 \partial v_j^0/\partial y_i^{00} = (\mathrm{div}_0\, T_{y1}^0,\; \mathrm{div}_0\, T_{y2}^0,\; \mathrm{div}_0\, T_{y3}^0)^T v_y^{00} +$ $\sum_{ij} T_{ij}^0 \partial v_j^0/\partial y_i^{00} = \rho_{\Delta v} + \Phi_y$.

Proposition 2.14 *The equation of medium energy balance at a point* $y \in \mathbf{D}_3^\mu$ *(2.79) can be written as follows*

$$\rho_y(\frac{1}{2}\ v_y^{00,2} + u_y)^{\cdot} = \rho_y\ g_y^0(\urcorner y, y) \cdot v_y^{00} + \mathrm{div}_0\, T_y^{0,\ T} v_y^{00} + \mathrm{div}_0\, q_y^0 + \rho_y \varphi_y \qquad (2.84)$$

Comment In Galilean mechanics the energy balance equation is equivalent to the conditions (2.24) and (2.52), respectively, of dynamic and thermodynamic balance of Universe (see (2.84)).

Chapter 3

Kinematics of locally linearly changeable medium

3.1. Kinematics equation on k–deformator group

Remind that

1. $\mathbf{E}_0 = (o_0, [\mathbf{e}^0])$ is a fixed orthonormal frame in A_3^μ, $o_0 \in \mathbf{D}_3^\mu$, $[\mathbf{e}^0] \in \mathbf{V}_3$;
2. $v_y^{00} = y^{00\bullet}$ is the coordinate column of the velocity vector $v_y^0 \in \mathbf{V}_3$ of a point $y \in \varepsilon_y$ (the lower index) w.r.t. $\mathbf{E}_0$ (the upper internal index) in the basis $[\mathbf{e}^0]$ (the upper external index);
3. dv_y^{00}/dy^{00} is derivative of $v_y^{00} \in \mathbf{R}_3$ w.r.t. the radius–vector $y^{00} \in \mathbf{R}_3$ of the point $y \in \varepsilon_y$ defined at the point y in the basis $[\mathbf{e}^0]$, *i.e.*, the matrix of deformation velocities (see D 2.16-6); V_y^0 be 9×1–column consisting of columns of the matrix $d\nu_y^{00}/dy^{00}$.

Proposition 3.1 *k–deformator $D_d^{00} \in \mathcal{GD}_{yt}^q(\mathcal{R}, 3)$ of the medium ε_y (see D 2.4-3) satisfies the following matrix differential equation*

$$D_y^{00\bullet}(t) = dv_y^{00}/dy^{00}\, D_y^{00}(t) \tag{3.1}$$

Proof For any $x \in \varepsilon_y$ let us decompose the vector $h_y = x^0 - y^0 \in \mathbf{V}_{3y}$ w.r.t. $[\mathbf{e}^0]$ and to the basis $[\mathbf{e}^d]$ accompanying deformation (see D 2.4-7)

$$h_y = [\mathbf{e}^0]\, h_y^0 = [\mathbf{e}^d]\, h_y^d \tag{3.2}$$

Substituting relation (2.8) in the above equality and taking into account the uniqueness of the vector decomposition at any basis, we arrived at

$$h_y^0(t) = D_d^{00}(t)\, h_y^d \tag{3.3}$$

When differentiating equation (3.3) w.r.t. the time and taking into account that the vector h_y is ‘deformed’ together with the medium (*i.e.*, with the basis $[\mathbf{e}^d] = [\mathbf{e}^0]\, D_d^{00}(t)$ accompanying the medium deformation), and therefore $h_y^d = \text{const}$, and using (3.3) in the form $h_y^d = (D_d^{00}(t))^{-1} h_y^0(t)$, we come to the following expression for the distribution of velocities of ε_y w.r.t. the frame $\mathbf{E}_0$ in its basis $[\mathbf{e}^0]$

$$v_x^{00}(t) = v_y^{00} + D_d^{00\bullet}(t)\, D_d^{00,-1}(t)\, h_y^0(t) \tag{3.4}$$

Comparing the linear part of the velocity $v_x^{00}(t)$ decomposition in ε_y

$$v_x^{00}(t) = v_y^{00} + dv_y^{00}/dy^{00}\, h_y^0(t)$$

with equality (3.4), we obtain the wanted result.

Definition 3.1 *Equation (3.1) is called kinematics equation on the group of k–deformators.*

Comments

1. From equations (3.1) and (3.4) we cannot make a conclusion about the possibility for representing the velocity vector $v_x^{00}(t)$ and the transformation of $h_y^0(t)$ with the help of symmetric operators as it is usually assumed to be (Kochin *et al.* 1964, Lur'e 1970, Sedov 1972).
2. Unlike Cauchy–Helmholtz identity (Kochin *et al.* 1964, Lur'e 1970, Sedov 1972)

$$\begin{aligned} v_x^{00}(t) &= dv_y^{00}/dy^{00}\, h_y^0 \equiv v_y^{00} + \frac{1}{2}(dv_y^{00}/dy^{00} - dv_y^{00}/dy^{00})\, h_y^0(t) + \\ & \frac{1}{2}(dv_y^{00}/dy^{00} + dv_y^{00}/dy^{00})\, h_y^0(t) \end{aligned} \tag{3.5}$$

that determines nothing and whose left–hand side is another notation of its right–hand side, equality (3.1) is the kinematic differential equation for defining k–deformator $D_y^{00}(t)$. In fact, this matrix is the sole 'adequate' linear mathematical model of medium deformation.
3. k–deformator $D_y^{00}(t)$ cannot be a priori presented (before integrating equation (3.1)) by a matrix of any definite kind (for example a shift, torsion, bend, *etc.*).
4. If the deformation velocities dv_y^{00}/dy^{00} are known and Cauchy conditions $D_y^{00}(t_0)$, $h_y^0(t_0)$ are given, the set of equalities

$$D_y^{00\cdot}(t) = dv_y^{00}/dy^{00}\, D_y^{00}(t) \tag{3.6}$$

$$x^{00}(t) = y^{00} + D_y^{00}(t)\, h_y^d, \quad h_y^d = h_y^0(t_0) \tag{3.7}$$

leads to the solution of the kinematical problem of the linear medium deformation at the point $y \in \mathbf{D}_3^\mu$.

3.2. Medium deformation matrix and its relation with medium displacement and with k–deformator

Definition 3.2 *Let: 1. (δ^0) be 9×1–column of the functions of the point $y \in \mathbf{D}_3^\mu$ and of the time $t \in \mathbf{T}_+$, $\delta_{ij} = \delta_{ij}^0(t,\, y)$, $\Delta_y^0 = \{\delta_{ij}^0\}$, $i,\, j, = 1,2,3,$ be the corresponding 3×3–dimensional matrices in the basis $[\mathbf{e}^0]$;*

2. the derivative of the column (δ^0) (of the matrix $\Delta_y^0 = \{\delta_{ij}^0\}$) w.r.t. the time at the point $y \in \varepsilon_y$ in the instant $t \in \mathbf{T}_+$ coincide with the column V_y^0 (with the matrix dv_y^{00}/dy^{00})

$$(\delta^0)^\cdot = V_y^0, \quad \Delta_y^{0\cdot} = dv_y^{00}/dy^{00} \tag{3.8}$$

Then: 1. the elements of the 9×1–column (δ^0) (of the matrix $\Delta_y^0 = \{\delta_{ij}^0\}$) are called deformations, and the column (δ^0) (the matrix $\Delta_y^0 = \{\delta_{ij}^0\}$) is called a column (matrix) of the medium deformation at the point $y \in \varepsilon_y$ in the instant $t \in \mathbf{T}_+$;

2. *the column* (V_y^0) *and the matrix* $(\Delta_y^0)^{\boldsymbol{\cdot}} = dv_y^{00}/dy^{00}$ *are called a column and a matrix of the medium deformation velocities at the point* $y \in \varepsilon_y$ *in the instant* $t \in \mathbf{T}_+$ *(see D 2.16-6);*
3. *the inner product*

$$\Phi_{yt} = dW_y/dt = (T_y^0) \cdot V_y^0 = \operatorname{trace} T_y^0 \, (dv_y^{00}/dy^{00})^T \tag{3.9}$$

is called a power of the medium d–deformators tensions T_y^0 *at the point* $y \in \varepsilon_y$ *in the instant* $t \in \mathbf{T}_+$ *(see (2.44));*

4. *the inner product*

$$dW_y = \Phi_{yt}\, \mu_t(dt) = (T_y^0) \cdot d(\delta^0) = \operatorname{trace} T_y^0 \, (d\Delta_y^0)^T \tag{3.10}$$

is called elementary work of the tensions T_{ij}^0 *on corresponding elementary deformations* $d\delta_{ij}^0$ *of the medium at the point* $y \in \varepsilon_y$ *in the instant* $t \in \mathbf{T}_+$ *(here* $\mu_t(dt)$ *is from Axiom K 6).*

Proposition 3.2 *Let the thermodynamic equation of locally linearly changeable continuous medium be of form (2.52)*

$$(T_y^0) \cdot V_y^0 = \operatorname{trace} T_y^0 \, (dv_y^{00}/dy^{00})^T = \rho_y u_{yt}^{\boldsymbol{\cdot}} - \operatorname{div}_0 q_{yt}^0 - \rho_y \varphi_{yt} \tag{3.11}$$

and heat transfer and heat separation (heat absorption) be absent, i.e., $\operatorname{div}_0 q_{yt}^0 = 0$ *and* $\varphi_{yt} = 0$*. Then the elementary work of the tensions* T_{ij}^0 *on medium elementary deformations* $d\delta_{ij}^0$ *at the point* $y \in \varepsilon_y$ *in the instant* $t \in \mathbf{T}_+$ *is equal to the product of the medium mass density and the differential of the medium internal energy density w.r.t. the mass* $m(dy)$ *at the same point and in the same instant:*

$$dW_y = (T_y^0) \cdot d(\delta^0) = \operatorname{trace} T_y^{\,0} \, (d\Delta_y^0)^T = \rho_y \, du_{yt} \tag{3.12}$$

Proof is achieved by substituting (3.10) in the equation and by multiplying with $\mu_t(dt)$.

Comments

1. From relations (3.11) and (3.8) follows that the k–deformator at the point $y \in \varepsilon_y$ in the instant $t \in \mathbf{T}_+$ satisfies equation (3.6) whose matrix coefficient is the matrix of medium deformation velocities at the same point and in the same instant.
2. Any way of the medium deformation definition without using the tensions work has under itself neither mathematical, nor physical reasons, for example by using such concept as distance between the medium points, or as elements of symmetric part of the matrix dz_y^{00}/dy^{00} (Lur'e 1970, Sedov 1972).
3. The final object of studying the medium deformation process is the path of the point $y \in \varepsilon_y$, *i.e.*, the vector–function $y^{00}(t) \in \mathbf{R}_3$ when its value $y^{00}(0)$ in the instant $t = 0$ is known. The mentioned vector–function is known if the vector $y^{00}(t)$ is given, then the vector $z^{00}(t)$, such that $y^{00}(t) = y^{00}(0) + z^{00}(t)$, is known, too. Thus, if the vector $y^{00}(t)$ is known, the final object of the medium deformation process study is the vector function $z^{00}(t)$ under condition that $z^{00}(0) = 0$.

4. It is not necessary to insert any 'practical' sense in the definition of 'vector of deformations' and of 'matrix of deformations'. In particular, it is not necessary to look for (really non–existing) 'tangent' and 'normal' deformations to (really non–existing) surfaces of the mechanical systems. Due to the results in section 3.2 it is possible to consider δ^0_{ji} in some 'geometrical sense'. The deformations δ^0_{ji} of a locally linearly changeable continuous medium are 9 functions that satisfy equality (3.8) due to (3.6), and nothing more.

Definition 3.3 *Let $z^{00}_y(t) \in \mathbf{R}_3$ be the coordinate column (in the basis $[\mathbf{e}^0]$) of the vector $z^0_y(t)$ that*

$$z^{00\cdot}_y(t) = v^{00}_y(t), \quad z^{00}_y(t) = 0 \tag{3.13}$$

Then: 1. the vector $z^0_y(t)$ is called a displacement vector of the point $y(t) \in \varepsilon_y$ w.r.t. the frame $\mathbf{E}_0$ in the instant $t \in \mathbf{T}_+$;

2. the column $z^{00\cdot}_y(t) \in \mathbf{R}_3$ is called a velocity of the coordinate column of the displacement vector of the point $y(t) \in \varepsilon_y$ w.r.t. the frame $\mathbf{E}_0$ in the instant $t \in \mathbf{T}_+$.

Comment The velocities of change of the medium vectors $y^0(t)$ and $z^0_y(t)$ at the point $y(t) \in \varepsilon_y$ in the instant $t \in \mathbf{T}_+$ coincide (see (3.13)) but the vectors differ each other in principle: the vector $z^0_y(t) = 0$, the vector $z^0_y(t + \Delta t)$ represents the increment of $y^0(t)$ at the point $y \in \varepsilon_y$ for a time $\Delta\ t \in \mathbf{T}_+$, *i.e.*,

$$z^{00}_y(t + \Delta t) = \int \chi_{[t+\Delta t]} v^{00}_y(t)\mu_t(dt) = y^{00}(t + \Delta t) - y^{00}(t) \tag{3.14}$$

Dividing (3.14) by $\Delta(t) \to 0$, we obtain (3.13).

Proposition 3.3 *Let: 1. the medium be deformed ($\Delta^0_y(t) \neq 0$) and deformable ($\Delta^{0\cdot}_y(t) \neq 0$) in the instant $t \neq 0$;*

2. the coordinates of the column v^{00}_y and the entries v^i_j, $i,\ j = 1, 2, 3$ of the matrix $dv^{00}_y/d\ y^{00}$ be continuous w.r.t. $(y^{00},\ t)$ on $\mathbf{D}^\mu_3 \times [t, t + \Delta t]$ which is equivalent to (see Kolmogorov et al. 1957)

$$v^{00}_y \in \mathbf{C}^1_{D\times[t,\,t+\Delta t]} \tag{3.15}$$

Then: 1. the velocity of the derivative dz^{00}_y/dy^{00} of the shift vector z^{00}_y of $y(t) \in \varepsilon_y$ in the instant $t \in \mathbf{T}_+$ (in terms of coordinate columns) coincides with the matrix of medium deformation velocities and with the matrix dv^{00}_y/dy^{00} in the same point and in the same instant $t \in \mathbf{T}_+$:

$$(dz^{00}_y/dy^{00})^{\cdot} = \Delta^{0\cdot}_y = dv^{00}_y/dy^{00} \tag{3.16}$$

2. the matrices $dz^{00}_y/dy^{00}(t)$ and $\Delta^0_y(t)$ differ each other on the constant matrix $dz^{00}_y/dy^{00}(0)-\ \Delta^0_y(0)$

$$dz^{00}_y/dy^{00}(t) = \Delta^0_y(t) + dz^{00}_y/dy^{00}(0) - \Delta^0_y(0) \tag{3.17}$$

Proof Integrating the medium shift vector defined by (3.13) on the interval $\Delta t \equiv [t+\Delta t]$, the we obtain the following

$$z_y^{00}(t+\Delta t) - z_y^{00}(t) = \int \chi_{\Delta t} v_y^{00} \mu_t(dt) \tag{3.18}$$

When differentiate the above equality w.r.t. the vector y^{00} and when rearrange derivative (3.16) and integral (3.18), then

$$\begin{aligned} dz_y^{00}(t+\Delta t)/dy^{00} - dz_y^{00}(t)/dy^{00} &= \int \chi_{\Delta t} dv_y^{00}/dy^{00} \mu_t(dt) = \\ \int \chi_{\Delta t} \Delta_y^{0\cdot} \mu_t(dt) &= \int \chi_{\Delta t} d\Delta_y^0 = \Delta_y^0(t+\Delta t) - \Delta_y^0(t) \end{aligned} \tag{3.19}$$

Dividing (3.19) on $\Delta t \to 0$, we have (3.16).

Comments

1. Requirement (3.15) is basic for the theory presented here. If it is not satisfied, then equalities (3.16) and (3.19) have no sense and everything is wrong. In particular, condition (3.15) is not fulfilled if there are cracks, jumps of condensation in supersonic currents of gas, heterogeneous inclusions, *etc.*
2. When substituting (3.16) in the differential equation (3.6) we obtain the relation between velocities of the k–deformator and of the deformation matrix

$$D_y^{00\cdot}(t) = \Delta_y^{0\cdot} D_y^{00}(t) \tag{3.20}$$

Proposition 3.4 *Let the medium ε_y be initially non–deformed, but deformable in each instant, i.e.,*

$$D_y^{00}(0) = E,\ \Delta_y^{00}(0) = 0,\ D_y^{00\cdot}(0) \neq 0,\ D_y^{00\cdot}(t) \neq 0 \tag{3.21}$$

Then: 1. in the vicinity of the point $t = 0$, the deformation matrix $\Delta_y^0(t)$ and the derivative $dz_y^{00}/dy^{00}(t)$ coincide

$$dz_y^{00}/dy^{00}(t) = \Delta_y^0(t) \tag{3.22}$$

2. the medium k–deformator $D_y^{00}(t)$ and the matrices dz_y^{00}/dy^{00} and $\Delta_y^0(t)$ (that represent different notations of the deformation matrix) are related by the relation

$$\begin{aligned} D_y^{00} &= E + dz_y^{00}/dy^{00} + o(||dz_y^{00}/dy^{00} \Delta D_y^{00}(t)||) = \\ &\quad E + \Delta_y^0 + o(||\Delta(\Delta_y^0) \Delta D_y^{00}(t)||) \end{aligned} \tag{3.23}$$

where $\Delta D_y^{00}(t)$ is the infinitesimal matrix in the instant $t = 0$ (see (2.11)).

Proof 1. As the medium is not deformed in the instant $t = 0$, we obtain $\Delta_y^0(0) = 0$ and $z_y^{00} \equiv 0 \to dz_y^{00}/dy^{00}(0) = 0 \to \Delta_y^0(t) = dz_y^{00}(t)/dy^{00}$ (see (3.17)).
2. Using the kinematics equation (3.6) in a small vicinity of $t = 0$, we have

$$\begin{aligned} D_y^{00\cdot}(t) &= dv_y^{00}/dy^{00}(t) D_y^{00}(t) = dv_y^{00}/dy^{00}(t)(E + \Delta D_y^{00}(t)) = \\ &\quad dv_y^{00}/dy^{00}(t) + dv_y^{00}/dy^{00}(t) \Delta D_y^{00}(t) \end{aligned} \tag{3.24}$$

When integrating (3.24) w.r.t. $\Delta t = t - 0$ and taking into account (3.17), we obtain

$$\begin{aligned}
D_y^{00} - E &= \int \chi_{\Delta t} dv_y^{00}/dy^{00} \mu_t(dt) + \int \chi_{\Delta t} dv_y^{00}/dy^{00} \Delta D_y^{00}(t) \mu_t(dt) \\
&= \int \chi_{\Delta t} \Delta_y^{0\cdot}(t) \mu_t(dt) + \int \chi_{\Delta t} \Delta_y^{0\cdot}(t) \Delta_y^{0\cdot}(t) \Delta D_y^{00}(t) \mu_t(dt) \\
&= \int \chi_{\Delta t} d\Delta_y^{0}(t) + \int \chi_{\Delta t} d\Delta_y^{0}(t) \Delta D_y^{00}(t) \\
&= \Delta_y^{0}(t) + \int \chi_{\Delta t} d\Delta_y^{0}(t) \Delta D_y^{00}(t) = \Delta_y^{0}(t) + \Delta \Delta_y^{0}(t) \overline{\Delta D}_y^{00}(t) \\
&= \Delta_y^{0}(t) + o(||\Delta(\Delta_y^{0}) \Delta D_y^{00}(t)||)
\end{aligned}$$

where $\overline{\Delta D}_y^{00} = \frac{1}{2}[\Delta D_y^{00}(t) + 0] = \frac{1}{2}\Delta D_y^{00}(t)$ is the mean value of ΔD_y^{00} on a small interval $\Delta t = t - 0$, $\Delta(\Delta_y^0) = \Delta_y^0 - 0 = \Delta_y^0$.

Comments

1. The conditions $dv_y^{00}/dy^{00}(0) = dz_y^{00\cdot}/dy^{00}(0) = \Delta_y^{0\cdot}(0) \neq 0$, $dz_y^{00}/dy^{00}(0) = \Delta_y^0(0) = 0$ are essential in all proofs.
2. From (3.23) in a small vicinity of $t = 0$, for initially non–deformed deformable at the point medium (see 3.21)), the following is presented

$$D_y^{00} \approx E + dz_y^{00}/dy^{00} = E + \Delta_y^0 \tag{3.25}$$

   where the error has order of $||\Delta(\Delta_y^0)\, \Delta D_y^{00}(t)||$.
3. Comparing (2.11) and (3.25) we may conclude that in a small vicinity of the point $t = 0$ for initially non–deformed medium being deformable at the point y (see (3.21)), the infinitesimal matrix ΔD_y^{00} from k–deformator definition (see (2.11)) is approximately equal to the medium deformations matrix at the same point, *i.e.*,

$$\Delta D_y^{00}(t) \approx dz_y^{00}/dy^{00} = \Delta_y^0 \tag{3.26}$$

   where the error has order of $||\Delta D_y^{00}(t)||^2$.

3.3. Generatrices of k–deformator group

Definition 3.4 *The set of group $\mathcal{GD}_{yt}^q(\mathcal{R}, 3)$ elements such that each element $D_y^{00} \in \mathcal{GD}_{yt}^q(\mathcal{R}, 3)$ represented as a superposition (a product, in particular) of these elements, is called a set of generatrices.*

Comment The generatrices of $\mathcal{GD}_{yt}^q(\mathcal{R}, 3)$ are not mathematical models of transformations of media. Here the situation is similar to that we have with expansions in functions series of whose graphics, *e.g.*, are mathematical models of the movement paths of material system points. In presence of definite properties, there are Taylor series ('w.r.t. parabolas'), Fourier series (w.r.t. sines, cosines), *etc.*, w.r.t.

any complete system of functions (Kolmogorov *et al.* 1957, Natanson 1955). But none of these functions is a mathematical model of motion. These expansions are a lot, and hence every researcher can see the desired 'physics' of what has happened. In fact, in all cases the mechanical system motion information contains in k–deformator and no information contains in the mentioned generatrices or in the mentioned series elements. Each of such decompositions is in the definite sense (if a diversity in $\mathbf{L}_2$ is considered) or in the direct sense (in uniform diversity) an identity whose right–hand part is another notation of the left–hand part; k–deformator is superposition of the mentioned generatrices. Besides, the number of such superpositions is infinite and all of them are equivalent.

That is why all this is of theoretical interest relative to k–deformator properties, but not to the properties of motions of locally linearly changeable media.

Definition 3.5 *Let: 1.* $\mathbf{A}^{\mu}_{2y} \equiv \mathbf{V}_{2y} \cup \mathbf{D}^{\mu}_{2}$ *be the affine vector plane that includes the point* $y \in \mathbf{D}^{\mu}_{2}$*;*

2. $H^0_{yt}(q^0_y)$ *be a homothety with a coefficient* $q^0_y > 0$ *(a positive scalar) which does not depend on the vector* $h^0_y = x^{00} - y^{00}$ *for each point* $x \in \varepsilon_y \in \sigma^{\mu}_3$*:*

$$H^0_{yt}(q^0_y) = q^0_y E \tag{3.27}$$

3. w *be an arbitrary free vector of* $\mathbf{V}_2$*,* w^0 *be its coordinate column in the basis* $[\mathbf{e}^0]$*;*

4. the matrix $u^{00}_y \in \mathcal{GD}^q_{yt}(\mathcal{R}, 3)$ *in the basis* $[\mathbf{e}^0]$ *be determined by the following conditions*

$$\begin{aligned} u^{00}_y h^0_y &= H^0_{yt}(q^0_y) h^0_y + w^0, \; h^0_y \notin \mathbf{V}_{2y} && (3.28) \\ u^{00}_y h^0_y &= h^0_y, \qquad\qquad\quad h^0_y \in \mathbf{V}_{2y} && (3.29) \end{aligned}$$

where $q^0_y \in \mathbf{Q}_1$ *(see (2.7));* h^0_y *is the coordinate column of the vector* $h^0_y \in \mathbf{V}_3$ *(see (2.6)) in the basis* $[\mathbf{e}^0]$*.*

Then: 1. the matrix u^{00}_y *is called an extension (when* $q^0_y > 1$*) or compression (when* $q^0_y < 1$*) of* $\mathbf{A}^{\mu}_{3y}$ *from the hyperplane* $\mathbf{A}^{\mu}_{2y}$ *(to* $\mathbf{A}^{\mu}_{2y}$ *in the case of compression) in direction of its own subspace with an eigenvalue* q^0_y *(i.e., the straight line which is complement to* $\mathbf{A}^{\mu}_{2y}$ *in* $\mathbf{A}^{\mu}_{3y}$*) (Dieudonne 1969 and 1974, Skornyakov 1980, Suprunenko 1972);*

2. the homothety

$$H^0_{yt}(q^0_y) = q^0_y E \equiv d^{00}_y(q^0_y) \tag{3.30}$$

defined by (3.27) is called a central dilator.

Comments

1. The central dilatator $d^{00}_y(q^0_y)$ (that is another 'mechanical' notation of the homothety) is a mathematical model of a medium motion if other components of the k–deformator are absent. This dilation extends ($q^0_y > 1$) or compresses ($q^0_y < 1$) the medium $\varepsilon_y \in$ from/to the point $y \in \varepsilon_y$.

2. When $h_y^0 \notin \mathbf{V}_{2y}$, the matrix u_y^{00} displaces the result of dilation action (3.28) parallel to the hyperplane $\mathbf{A}_{2y}^{\mu}$ by a vector $w \in \mathbf{V}_2$; when $h_y^0 \in \mathbf{V}_{2y}$, the result of the dilation action leaves it at the same place.
3. The matrix u_y^{00} is k–deformator only if it differs from the identity matrix.
4. k–deformator u_y^{00} is determined on the vicinity ε_y of the point y only, it is not determined on all medium $\mathbf{D}_3^{\mu}$.

Proposition 3.5 *Let: 1.* $u_y^{00} \in \mathcal{GD}_{yt}^q(\mathcal{R}, 3)$ *be an extension (a compression)* $q_y^0 > 1$ *(*$q_y^0 < 1$*) of the medium* ε_y *from/to the hyperplane* $\mathbf{A}_{2y}^{\mu}$ *at the point* $y \in \varepsilon_y$*;*

2. $w = [\mathbf{e}^0]w^0,\ w \in \mathbf{V}_{2y},\ w^0 = \mathrm{col}\{w_1^0,\, w_2^0,\, w_3^0\}$.

Then u_y^{00} *has the following matrix form*

$$u_y^{00} = \begin{bmatrix} q_y^0 + w_1^0 & w_2^0 & w_3^0 \\ w_1^0 & q_y^0 + w_2^0 & w_3^0 \\ w_1^0 & w_2^0 & q_y^0 + w_3^0 \end{bmatrix} \tag{3.31}$$

Proof $u_y^{00}e_1^0 = q_y^0e_1^0 + w_1^0e_1^0 + w_2^0e_2^0 + w_3^0e_3^0$, $u_y^{00}e_2^0 = q_y^0e_2^0 + w_1^0e_1^0 + w_2^0e_2^0 +$ $w_3^0e_3^0$, $u_y^{00}e_3^0 = q_y^0e_3^0 + w_1^0e_1^0 + w_2^0e_2^0 + w_3^0e_3^0$.

Proposition 3.6 *Let: 1.* $w = w_1^f f_1^0 + w_2^f f_2^0$ *be the expansion of* $w \in \mathbf{V}_2$ *w.r.t. the basis* $(f_1^0,\, f_2^0) \subset \mathbf{V}_2$*,* $w_1^f,\ w_2^f \in \mathbf{R}_1$*;*

2. $[\mathbf{f}^0] = (h_y^0,\, f_1^0,\, f_2^0)$ *be a non–orthogonal non–inertial basis of the number representative of* $\mathbf{V}_3 \supset \mathbf{V}_2$*.*

Then the k–deformator u_y^{00} *(in the basis* $[\mathbf{f}^0]$*) is of the form*

$$u_y^{00} = E + \Delta u_y^{0f}, \quad \Delta u_y^{0f} = \begin{bmatrix} q_y - 1 & 0 & 0 \\ w_1^f & 0 & 0 \\ w_2^f & 0 & 0 \end{bmatrix} \tag{3.32}$$

where Δu_y^{0f} *is infinitesimal matrix at the point* $(q_y^f,\, w^f) = \mathrm{col}(1, 0, 0) \in \mathbf{Q}_3$*.*

Proof $u_y^{00}h_y^0 = q_y^f h_y^0 + w_1^f f_1^0 + w_1^f f_2^0$, $u_y^{00}f_1^0 = 0h_y^0 + 1f_1^0 + 0f_2^0$, $u_y^{00}f_2^0 = 0h_y^0 +$ $0f_1^0 + 1f_2^0$.

Comments

1. There exist two more non-orthogonal bases which keep the orientation of $\mathbf{V}_2$ (see (2.1)), $[\mathbf{f}^0] = (f_1^0,\, h_y^0,\, f_2^0)$, $[\mathbf{f}^0] = (f_1^0,\, f_2^0,\, h_y^0)$ where k–deformator u_y^{0f} has the following simple forms

$$u_y^{0f} = \begin{bmatrix} 1 & w_1^f & 0 \\ 0 & q_y^f & 0 \\ 0 & w_2^f & 1 \end{bmatrix} \text{ and } u_y^{0f} = \begin{bmatrix} 1 & 0 & w_1^f \\ 0 & 1 & w_2^f \\ 0 & 0 & q_y^f \end{bmatrix}, \text{ respectively.}$$

2. $\det u_y^{00} = q_y^0$.

Definition 3.6 *Let: 1.* $\mathbf{A}^{\mu}_{2y} \equiv \mathbf{V}_{2y} \cup D^{\mu}_{2}$ *be the affine vector plane containing the point* $y \in \mathbf{D}^{\mu}_{2}$ *(see D 2.14);*

2. w *be an arbitrary free vector of* $\mathbf{V}_2$*;*

3. $\varphi(w) = 0$ *be the equation of the hyperplane* $\mathbf{A}^{\mu}_{2y}$ *in the frame* $\mathbf{E}_y = (y^0, [\mathbf{e}^0])$ *whose origin is at the point* $y \in \varepsilon_y$ *and whose basis is* $[\mathbf{e}^0] \in \mathbf{E}_0$*;*

4. $\tau^{00}_y \in \mathcal{GD}^q_{yt}(\mathcal{R}, 3)$ *be the k–deformator on* $\varepsilon_y \in \sigma^{\mu}_3$*, determined by the relations*

$$\tau^{00}_y h^0_y = h^0_y + \varphi(h^0_y),\ h^0_y \notin \mathbf{V}_{2y} \tag{3.33}$$

$$\tau^{00}_y h^0_y = h^0_y, \qquad h^0_y \in \mathbf{V}_{2y} \tag{3.34}$$

Then the matrix τ^{00}_y *is called a shift along the hyperplane* $\mathbf{A}^{\mu}_{2y}$ *in direction of the vector* $w \in \mathbf{V}_2$ *(Dieudonne 1969 and 1974).*

Comments

1. The shift $\tau^{00}_y \in \mathcal{GD}^q_{yt}(\mathcal{R}, 3)$ (as the matrix $u^{00}_y \in \mathcal{GD}^q_{yt}(\mathcal{R}, 3)$) (see (3.28) and (3.29)) leaves any vector of the plane $\mathbf{A}^{\mu}_{2y}$ fixed, and displaces all remaining vectors along the hyperplane by adding the vector $\varphi(h^0_y)w^0$ where the number $\varphi(h^0_y)$ is defined by the vector $h^0_y = x^{00} - y^{00}$, $x \in \varepsilon_y$.
2. The shift $\tau^{00}_y \in \mathcal{GD}^q_{yt}(\mathcal{R}, 3)$ (as the matrix $u^{00}_y \in \mathcal{GD}^q_{yt}(\mathcal{R}, 3)$) represents the medium k –deformator only in the case when other transformations are absent.
3. The shift $\tau^{00}_y \in \mathcal{GD}^q_{yt}(\mathcal{R}, 3)$ (as the k–deformator $u^{00}_y \in \mathcal{GD}^q_{yt}(\mathcal{R}, 3)$) (see (3.28)) is determined on the vicinity $\varepsilon_y \in \sigma^{\mu}_3$ of the point y only, but not on the whole medium $\mathbf{D}^{\mu}_3$.
4. The set of shifts $\tau^{00}_y \in \mathcal{GD}^q_{yt}(\mathcal{R}, 3)$ is not a group.
5. The frame $\mathbf{E}_y = (y^0, [\mathbf{e}^0])$ whose origin is at the point $y \in \varepsilon_y$ is not an inertial one; there is only the basis $[\mathbf{e}^0]$ being inertial.

Proposition 3.7 *Let: 1.* $\tau^{00}_y \in \mathcal{GD}^q_{yt}(\mathcal{R}, 3)$ *be the shift of the medium* $\varepsilon_y \in \sigma^{\mu}_3$ *along the hyperplane* $\mathbf{A}^{\mu}_{2y}$ *in direction of the vector* $w \in \mathbf{V}_2$*;*

2. $w = [\mathbf{e}^0]w^0 = w^0_1 e^0_1 + w^0_2 e^0_2 + w^0_3 e^0_3$, $w^0 = \mathrm{col}\{w^0_1,\ w^0_2,\ w^0_3\}$.

Then the shift τ^{00}_y *is of the form*

$$\tau^{00}_y = E + \Delta\tau^{00}_y, \quad \Delta\tau^{00}_y = \begin{bmatrix} \varphi(e^0_1)w^0_1 & \varphi(e^0_2)w^0_1 & \varphi(e^0_3)w^0_1 \\ \varphi(e^0_1)w^0_2 & \varphi(e^0_2)w^0_2 & \varphi(e^0_3)w^0_2 \\ \varphi(e^0_1)w^0_3 & \varphi(e^0_2)w^0_3 & \varphi(e^0_3)w^0_3 \end{bmatrix} \tag{3.35}$$

where $\Delta\tau^{00}_y$ *is an infinitesimal matrix in the definition of the k–deformator at the point* $w^0 = 0 \in \mathbf{Q}_3$ *(see (2.9)).*

Proof $\tau^{00}_y e^0_1 = 1e^0_1 + (e^0_1)w^0$, $\tau^{00}_y e^0_2 = 1e^0_2 + (e^0_2)w^0$, $\tau^{00}_y e^0_3 = 1e^0_3 + (e^0_3)w^0$.

Proposition 3.8 *Let: 1.* $w = w^f_1 f^0_1 + w^f_2 f^0_2$ *be the decomposition w.r.t. the basis* $(f_1,\ f_2)$ *in* $\mathbf{V}_2$, w^f_1, $w^f_2 \in \mathbf{R}_1$*;*

2. $[\mathbf{f}] = (h^0_y,\ f^0_1,\ f^0_2)$ *be a non–orthonormal and non–inertial basis of the number representative of* $\mathbf{V}_3 \supset \mathbf{V}_2$*;*

3. $\alpha^f = w_1^f \varphi(h_y^0)$, $\beta^f = w_2^f \varphi(h_y^0)$.

Then the shift τ_y^{00} *in the basis* $[f]$ *is Cavalieri transformation of the I type and it is of the form (Ambartzumjan et al. 1989, Dieudonne 1969 and 1974)*

$$\tau_y^{0f} = E + \Delta\tau_y^{0f}, \quad \Delta\tau_y^{0f} = \begin{bmatrix} 0 & 0 & 0 \\ \alpha^f & 0 & 0 \\ \beta^f & 0 & 0 \end{bmatrix} \tag{3.36}$$

Proof $\tau_y^{00} h_y^0 = 1h_y^0 + w_1^f \varphi(h_y^0) f_1^0 + w_2^f \varphi(h_y^0) f_2^0$, $\tau_y^{00} = 0h_y^0 + 1f_1^0 + 0f_2^0$, $\tau_y^{00} = 0h_y^0 + 0f_1^0 + 1f_2^0$.

Comments

1. $\det \tau_y^{00} = 1$.
2. In the two–dimensional case all the definitions and propositions remain true by substituting 3–dimensional objects with 2–dimensional ones.
3. It follows from (2.11), (3.32) and (3.36) that the extensions (compressions) u_y^{00} are medium k–deformators.

It remains to prove that extensions (compressions) and shifts τ_y^{00} form the set of generatrices of $\mathcal{GD}_{yt}^q(\mathcal{R}, 3)$.

Proposition 3.9 *Let: 1.* $D_y^{00} \in \mathcal{GD}_{yt}^q(\mathcal{R}, 3)$ *be a medium k–deformator on* $\varepsilon_y \in$ *in the basis* $[\mathbf{e}^0]$*;*

2. $\Delta D_y^{00} = D_y^{00} - E$ *be the infinitesimal matrix at some point* $q \in \mathbf{Q}_n$*;*
3. *the k–deformator* D_y^{00} *be such that for its increment* D_y^{00} *w.r.t. the unit* $E \in \mathcal{GD}_{yt}^q(\mathcal{R}, 3)$ *we have*

$$\begin{aligned} \operatorname{rang}(D_y^{00} - E) &= \operatorname{rang} \Delta D_y^{00} = 1 \\ \operatorname{rang}(D_y^{00} - E) &= \operatorname{rang} \Delta D_y^{00} = 2 \\ \operatorname{rang}(D_y^{00} - E) &= \operatorname{rang} \Delta D_y^{00} = 3 \end{aligned} \tag{3.37}$$

4. $\tau_{yi}^{00} (i = 1, 2, 3)$, u_y^{00} *be the shifts and the extensions (compressions) (see (3.28), (3.33)).*

Then: 1. relative to (3.37) (Dieudonne 1969):

$$\begin{aligned} D_y^{00} &\equiv \tau_y^{00} \text{ or } D_y^{00} \equiv u_y^{00} \\ D_y^{00} &\equiv \tau_{y2}^{00}\tau_{y1}^{00} \text{ or } D_y^{00} \equiv \tau_y^{00} u_y^{00} \\ D_y^{00} &\equiv \tau_{y3}^{00}\tau_{y2}^{00}\tau_{y1}^{00} \text{ or } D_y^{00} \equiv \tau_{y2}^{00}\tau_{y1}^{00} u_y^{00} \end{aligned} \tag{3.38}$$

2. *in the opposite case* D_y^{00} *is homothety that coincides with the central dilator (3.30)*

$$D_y^{00} \equiv d_y^{00}(q_y^0) \tag{3.39}$$

Proof Let prove the first relation from (3.38) (for the rest see (Dieudonne 1972)). Let $\text{rang}(D_y^{00} - E) = 1$. Then $\ker(D_y^{00} - E) = \mathbf{A}_{2y}^{\mu}$ ($\dim \mathbf{A}_{2y}^{\mu} = 2$). Hence for each vector $h_y = x^0 - y^0 \in \mathbf{V}_2 \subset \mathbf{A}_{2y}^{\mu}$, $x, y \in \varepsilon_y$, $h_y \neq 0$ from the equality $(D_y^{00} - E)h_y^0 = 0$ we obtain $D_y^{00} h_y^0 = h_y^0$. Therefore, the k–deformator D_y^{00} leaves the elements of $\mathbf{A}2y^{\mu}$ fixed and, using definitions (3.29) and (3.34) we obtain $D_y^{00} = \tau_y^{00}$ or $D_y^{00} = u_y^{00}$.

Comments

1. For any dimension of the mechanics Universe affine vector space, the representation of k–deformator as a superposition of smaller number of generatrices (extensions, shifts, dilators) is impossible.
2. The k–deformators u_y^{00} and τ_y^{00} defined by (3.28) and (3.33) are mathematical models of the simplest medium transformations. They are not of such kind in superpositions (3.27) and (3.38).
3. If the real (physical) medium deformation differs from the simplest one, the linear mathematical model of medium deformation is the k–deformator $D_y^{00} \in \mathcal{GD}_{yt}^q(\mathcal{R}, 3)$, only. Decompositions (3.38) are identities and their right parts give no additional information about medium deformation w.r.t. the information that is included in the k–deformator D_y^{00}.
4. For an initially non–deformed medium in the vicinity of $t = 0$ (when D_y^{00} is small), the medium deformation matrix of form (3.22) can be used in relations (3.32) and (3.36).

3.4. Transvective–dilation decomposition of k–deformators

3.4.1. *Transvections, dilations, dilators and homotheties*

The complicated form of matrices (3.28) and (3.33) in an arbitrary basis makes difficult to use of k–deformators in the further constructions but the formation of the bases guaranteeing the simplicity of k–deformators (3.32) and (3.35), needs additional calculations for each vector $h_y \in \mathbf{V}_{3y}$. The form of matrices (3.32) and (3.35) prompts the way out of situation. The essence is in the following definitions and propositions.

Definition 3.7 *Let: 1. E_{ij} be 3×3–dimensional matrix whose (i, j)–element is 1 and all the rest are 0 ($i, j = \overline{1,3}$);*

2. $q_{ij}^{y0} \in \mathbf{Q}_1$, $i, j = \overline{1,3}$, be scalar parameters calculated at a point $y \in \varepsilon_y$ in the basis $[\mathbf{e}^0]$.

Then: 1. the k–deformator

$$\tau_{ij}^{y0}(q_{ij}^{y0}) = E + \Delta\tau_{ij}^{y0}(q_{ij}^{y0}),\ \Delta\tau_{ij}^{y0}(q_{ij}^{y0}) = q_{ij}^{y0} E_{ij} \tag{3.40}$$

where $\Delta\tau_{ij}^{y0}(q_{ij}^{y0})$ is an infinitesimal matrix at the point $q_{ij}^{y0} = 0 \in \mathbf{Q}_1$, is called a transvection (Gantmacher 1964, Dieudonne 1969 and 1974, Cartan 1967, Skornyakov 1980, Suprunenko 1972).

2. the number $q_{ij}^{y0} \in \mathbf{Q}_1$ *is called an argument (parameter) of transvection.*

Comments

1. The geometric interpretation of transvection is obvious. The k–deformator $\tau_{ij}^{y0}(q_{ij}^{y0})$ shifts each vector $h_y^0 = \mathrm{col}\{h_1^0,\, h_2^0,\, h_3^0\}$ (given in the basis $[\mathbf{e}^0]$ (see (2.6))) along the coordinate axis e_i^0 by the distance $q_{ij}^{y0}h_j^0$. This shows that the shift of the vector h_y^0 w.r.t. its i–th coordinate is proportional to its j–th coordinate with a coefficient q_{ij}^{y0} of proportionality. It means that the medium deformation by the transvection $\tau_{ij}^{y0}(q_{ij}^{y0})$ (*i.e.*, the shift) does not reduce to an independent increase of i–th coordinate of medium points at a constant.
2. The algebraic sense of k–deformator (3.40) is essential for further constructions. Transvections $\tau_{ij}^{y0}(q_{ij}^{y0})$ (regardless of dimensions) realize one of the simple transformations of k–deformator of an arbitrary kind: the multiplication $\tau_{ij}^{y0}(q_{ij}^{y0})D_y^{00}$ leads to adding i–th row of the matrix D_y^{00} with its j–th row, multiplied by q_{ij}^{y0}; while the multiplication $D_y^{00}\tau_{ij}^{y0}(q_{ij}^{y0})$ leads to adding j –th column of D_y^{00} with the i–th column multiplied by q_{ij}^{y0} (Konoplev *et al.* 2001).
3. When the medium is deforming from an initial non–deformed position, the transvection parameter q_{ij}^{y0} is approximately equal to (i,j)–element $z_j^i = \partial z_{yi}^0/\partial y_j^0$ of the deformation matrix $\Delta_y^0 \equiv dz_y^{00}/dy^{00}$ at the point $y \in \varepsilon_y$ according to relation (3.26).
4. The set of transvections is not a group.

Definition 3.8 *Let: 1.* E_{ij} *be* 3×3*–matrix whose* $(i,\, i)$*–element is* 1 *and all the rest are* 0*;*

2. $q_i^{y0} \in \mathbf{Q}_1$, $i = \overline{1,3}$, *be scalar parameters calculated at the point* $y \in \mathbf{D}_3^\mu$.

Then: 1. the matrix of the form

$$\begin{aligned} d_i^{y0}(q_i^{y0}) &= E + \Delta d_i^{y0}(q_i^{y0}) \\ \Delta d_i^{y0}(q_i^{y0}) &= (q_i^{y0}-1)E_{ii},\ q_i^{y0} > 0 \end{aligned} \tag{3.41}$$

where $\Delta d_i^{y0}(q_i^{y0})$ *is an infinitesimal matrix at the point* $q_i^{y0} = 1$ *is called a dilation* $\mathbf{A}_3^\mu$ *(expansion when* $q_i^{y0} > 1$*; a compression in the case* $q_i^{y0} < 1$*) directed along the coordinate axis* e_i^0*;*

2. the number q_i^{y0} *is called an argument (parameter) of the dilation.*

Comments

1. The matrix $d_i^{y0}(q_i^{y0})$ is k–deformator (*i.e.*, a mathematical model of medium transformation) only in the case when other kinds of medium deformation are absent.
2. The geometric and physical interpretation of the dilation are obvious. The operator $d_i^{y0}(q_i^{y0})$ increases (when $q_i^{y0} > 1$) or decreases (in the case $q_i^{y0} < 1$) i–th coordinate of each vector $h_y^0 = \mathrm{col}\{h_1^0,\, h_2^0,\, h_3^0\}$ (given in the basis $[\mathbf{e}^0]$) in q_i^{y0} times. In the both cases, medium points are translated parallel to

one of the coordinate axes, but the dilation action reduces to an independent change of i–th coordinate and this change is multiplicative, not additive. The k–deformator $d_i^{y0}(q_i^{y0})$ extends (if $q_i^{y0} > 1$) or compresses (if $q_i^{y0} < 1$) the medium in direction of the coordinate axis e_i^0.

3. Let us pay attention to the algebraic sense of the kinematic deformator (3.41). The dilation $d_i^{y0}(q_i^{y0})$ realizes one of the elementary transformations of k–deformator D_y^{00} of an arbitrary kind and of an arbitrary dimension: the multiplication $d_i^{y0}(q_i^{y0})D_y^{00}$ leads to multiplying i–th row of D_y^{00} by the number q_i^{y0}, the multiplication $D_y^{00}\ d_i^{y0}(q_i^{y0})$ leads to multiplying j–th column of D_y^{00} by the number q_i^{y0} (Konoplev 1994 and 1995, Konoplev *et al.* 2001).
4. When deforming the medium from a non–deformed initial position, according to (3.26), the number $q_i^{y0} - 1$ approximately equals to $(i,\, i)$–element $z_i^i = \partial z_{yi}^0/\partial y_i^0$ of the deformation matrix $\Delta_y^0 \equiv dz_y^{00}/dy^{00}$ at the point $y \in \varepsilon_y$.

The following statement gives a relation between the dilator group $\mathcal{D}^q(\mathcal{R},3)$ and the dilation group $\mathcal{D}_i^q(\mathcal{R},3)$.

Proposition 3.10 *Let: 1.* $\mathcal{D}^q(\mathcal{R},3)$ *be the group of dilators;*

2. $\mathcal{D}_i^q(\mathcal{R},3), i = \overline{1,3}$, *be the group of dilations in direction of i–th coordinate axis;*

3. P *be the operator of permutations in the product.*

Then the following is true:

$$\mathcal{D}^q(\mathcal{R},3) = P\prod_i \mathcal{D}_i^q(\mathcal{R},3) \tag{3.42}$$

After defining the concept of transvections (3.40), of dilations (3.41), and after studying their basic properties, we will consider the relations between transvections and dilations with generatrices (3.28) and (3.23) of the k–deformator group $\mathcal{GD}_{yt}^q(\mathcal{R},3)$.

Proposition 3.11 *Let: 1.* u_y^{0f} *be the extension (compression) from (to) the corresponding affine plane* $\mathbf{A}_{2y}^\mu$ *in direction of the corresponding vector* $w \in \mathbf{V}_2$ *in the basis* $[\mathbf{e}^f]$ *(see (3.28));*

2. τ_y^{0f} *be the shift along the corresponding affine plane* $\mathbf{A}_{2y}^\mu$ *in direction of the corresponding vector* $w \in \mathbf{V}_2$ *(see (3.33)) in the basis* $[\mathbf{e}^f]$*;*

3. $\tau_{ij}^{yf}(q_{ij}^{yf})$ *and* $d_i^{yf}(q_i^{yf})$ *be transvection (3.40) and dilation (3.41) in the basis* $[\mathbf{e}^f]$.

Then: 1. the k–deformator u_y^{0f} *(in the basis) (see (3.32)) is the multiplication (in any order) of two transvections and one dilation of the kind (depending on the chosen basis* $[\mathbf{e}^f]$*) for example:*

$$\begin{aligned} u_y^{0f} &\equiv \tau_{21}^{yf}(q_{21}^{yf})\tau_{31}^{yf}(q_{31}^{yf})d_1^{yf}(q_1^{yf}) \equiv \tau_{12}^{yf}(q_{12}^{yf})\tau_{32}^{yf}(q_{32}^{yf})d_2^{yf}(q_2^{yf}) \\ &\equiv \tau_{13}^{yf}(q_{13}^{yf})\tau_{23}^{yf}(q_{23}^{yf})d_3^{yf}(q_3^{yf}) \end{aligned} \tag{3.43}$$

2. the k–deformator τ_y^{0f} (see (3.35))(in the basis $[\mathbf{e}^f]$) is the multiplication (in any order) of two transvections (depending on the chosen basis $[\mathbf{e}^f]$), for example:

$$\tau_y^f \equiv \tau_{21}^{yf}(q_{21}^{yf})\tau_{31}^{yf}(q_{31}^{yf}) \equiv \tau_{12}^{yf}(q_{12}^{yf})\tau_{32}^{yf}(q_{32}^{yf}) \equiv \tau_{13}^{yf}(q_{13}^{yf})\tau_{23}^{yf}(q_{23}^{yf}) \tag{3.44}$$

Comments

1. The obtained results do not simplify the solving of the problem for decompositions of k–deformator on elementary factors since the difficulties of the calculation of $[\mathbf{e}^f]$ remain the same ones.
2. The obtained results are basic when resolving the problem of a substitution of a not so large number $(1 \div 3)$ of generatrices of a complex type (3.38) with a large number of generatrices of a simple kind (a transvection (3.40) and a dilation (3.41)). The idea consists of searching an algorithm for multiplicative decompositions of k–deformator with the simplest factors in the beforehand chosen and constant (w.r.t. the solution period) orthonormal basis, for example an inertial one.

Below, the solution of this problem is presented.

3.4.2. *Transvection–dilation decompositions of k–deformator group by Cavalieri group*

Proposition 3.12 *Let: 1. $\mathbf{E}_0$ be an arbitrary (an inertial, for example) frame;*

2. $\tau_{21}^{y0}(q_{21}^{y0})$, $\tau_{31}^{y0}(q_{31}^{y0})$, $\tau_{13}^{y0}(q_{13}^{y0})$, $\tau_{23}^{y0}(q_{23}^{y0})$ be transvections, i.e., elementary shifts of the vector space in direction of basis $[\mathbf{e}^0]$ orths with indexes (corresponding to the first indexes) being proportional to the coordinates of shifted vector with indexes corresponding to the second indexes.

Then: 1. k–deformators that are products of two transvections of the type

$$\tau_{21}^{y0}(q_{21}^{y0})\tau_{31}^{y0}(q_{31}^{y0}) \quad \text{and} \quad \tau_{13}^{y0}(q_{13}^{y0})\tau_{23}^{y0}(q_{23}^{y0}) \tag{3.45}$$

realize a simultaneous shift of the medium parallel to the coordinate planes $(e_1^0,\, o^0,\, e_3^0)$ and $(e_1^0,\, o^0,\, e_2^0)$, respectively;

2. k–deformators (3.45) leave the mentioned above coordinate planes fixed (by the elements);

3. the matrices–factors in (3.45) are commutative (it means that the addition of one and the same row (column) to different rows (columns) does not depend on the operation order).

Proof is the simple verification, for example:

$\tau_{21}^{y0}(q_{21}^{y0})\tau_{31}^{y0}(q_{31}^{y0})h^0 = \tau_{21}^{y0}(q_{21}^{y0})\tau_{31}^{y0}(q_{31}^{y0})(h_1^0,\, h_2^0,\, h_3^0)^T = (h_1^0,\, h_2^0 + q_{21}^{y0}h_1^0,\, h_3^0 + q_{31}^{y0}h_1^0)^T.$

Comments

1. In the integral geometry, matrices (3.45) (without their decomposition in the form of factors of type – transvections) are called Cavalieri transformations (of $\mathbf{R}_3$) of I type (Ambartzumjan *et al.* 1989, Dieudonne 1969). Transvections $\tau_{12}^{y0}(q_{12}^{y0})$ and $\tau_{32}^{y0}(q_{32}^{y0})$ are called Cavalieri transformations of II type.
2. The sets of the mentioned kinematic deformators with regard to the previous proposition (p. 3)

$$\begin{aligned}\mathcal{GK}_L^1(\mathcal{R},3) &= \{\tau_{21}^{y0}(q_{21}^{y0})\tau_{31}^{y0}(q_{31}^{y0})\} \equiv \{\tau_{31}^{y0}(q_{31}^{y0})\tau_{21}^{y0}(q_{21}^{y0})\} && (3.46)\\ \mathcal{GK}_U^1(\mathcal{R},3) &= \{\tau_{13}^{y0}(q_{13}^{y0})\tau_{23}^{y0}(q_{23}^{y0})\} \equiv \{\tau_{23}^{y0}(q_{23}^{y0})\tau_{13}^{y0}(q_{13}^{y0})\} && (3.47)\\ \mathcal{GK}_L^2(\mathcal{R},3) &= \{\tau_{32}^{y0}(q_{32}^{y0})\},\ \mathcal{GK}_U^2(\mathcal{R},3) = \{\tau_{12}^{y0}(q_{12}^{y0})\} && (3.48)\end{aligned}$$

are Cavalieri groups of I and II type (the upper index), the lower and upper ones (the lower index).
3. In the case of acting on medium only by a single transvection (see (3.40)), the last one may be an adequate model of deformation (shift) of the physical medium. If k–deformator $D_y^{00} \in \mathcal{GD}_{yt}^q(\mathcal{R},3)$ is presented as a superposition (a product) of several transvections, then the last ones are simultaneously acting components of the k–deformator, but they are not mathematical models of medium deformation.

Proposition 3.13 *Let: 1.* $\mathcal{GD}_{yt}^q(\mathcal{R},3)$ *be the group of k–deformators of the medium* ε_y*;*

2. $\mathcal{GK}_L^1(\mathcal{R},3), \mathcal{GK}_U^1(\mathcal{R},3), \mathcal{GK}_L^2(\mathcal{R},3)$*, and* $\mathcal{GK}_U^2(\mathcal{R},3)$ *be lower and upper Cavalieri groups of the I and II types (see (3.46), (3.47), (3.48));*

3. $\mathcal{D}^q(\mathcal{R},3)$ *be the dilator group (3.42).*

Then: 1. There exist 6 *different transvection–dilation decompositions of the k–deformators group* $\mathcal{GD}_{yt}^q(\mathcal{R},3)$ *of a locally linear continuous medium by using Cavalieri group:*

$$\begin{aligned}\mathcal{GD}_{yt}^q(\mathcal{R},3) &= \mathcal{GK}_L^1(\mathcal{R},3)\,\mathcal{GK}_L^2(\mathcal{R},3)\,\mathcal{GK}_U^1(\mathcal{R},3)\,\mathcal{GK}_U^2(\mathcal{R},3)\,\mathcal{D}^q(\mathcal{R},3) && (3.49)\\ &= \mathcal{GK}_L^1(\mathcal{R},3)\,\mathcal{GK}_L^2(\mathcal{R},3)\,\mathcal{D}^q(\mathcal{R},3)\,\mathcal{GK}_U^1(\mathcal{R},3)\,\mathcal{GK}_U^2(\mathcal{R},3)\\ &= \mathcal{D}^q(\mathcal{R},3)\,\mathcal{GK}_L^1(\mathcal{R},3)\,\mathcal{GK}_L^2(\mathcal{R},3)\,\mathcal{GK}_U^1(\mathcal{R},3)\,\mathcal{GK}_U^2(\mathcal{R},3)\\ &= \mathcal{GK}_U^1(\mathcal{R},3)\,\mathcal{GK}_U^2(\mathcal{R},3)\,\mathcal{GK}_L^1(\mathcal{R},3)\,\mathcal{GK}_L^2(\mathcal{R},3)\,\mathcal{D}^q(\mathcal{R},3)\\ &= \mathcal{GK}_U^1(\mathcal{R},3)\,\mathcal{GK}_U^2(\mathcal{R},3)\,\mathcal{D}^q(\mathcal{R},3)\,\mathcal{GK}_L^1(\mathcal{R},3)\,\mathcal{GK}_L^2(\mathcal{R},3)\\ &= \mathcal{D}^q(\mathcal{R},3)\,\mathcal{GK}_U^1(\mathcal{R},3)\,\mathcal{GK}_U^2(\mathcal{R},3)\,\mathcal{GK}_L^1(\mathcal{R},3)\,\mathcal{GK}_L^2(\mathcal{R},3)\end{aligned}$$

2. There exist 324 *different transvection–dilation decompositions of each k–deformator* $D_y^{00} \in \mathcal{GD}_{yt}^q(\mathcal{R},3)$ *such that the factors of decompositions are transvections and dilations of type (3.41).*

Proof is in the same time the description of an algorithm for a practical construction of any of the mentioned above decompositions.

1. Let us show, for example, that

$$\begin{aligned}
D_y^{00} &\equiv \tau_{yH}^0 \tau_{yB}^0 d_y^0, D_y^{00} = \{D_{ij}^0,\, i,\, j = \overline{1,3}\} \\
\tau_{yH}^0 &= \tau_{21}^{y0}(q_{21}^{y0})\tau_{31}^{y0}(q_{31}^{y0})\tau_{32}^{y0}(q_{32}^{y0}) \\
\tau_{yB}^0 &= \tau_{13}^{y0}(q_{13}^{y0})\tau_{23}^{y0}(q_{23}^{y0})\tau_{12}^{y0}(q_{12}^{y0}) \\
d_y^0 &= d_1^{y0}(q_1^{y0})d_2^{y0}(q_2^{y0})d_3^{y0}(q_3^{y0}) \\
q_{21}^{y0} &= D_{21}^0/D_{11}^0,\ q_{31}^{y0} = D_{31}^0/D_{11}^0,\ q_{32}^{y0} = \alpha_{32}^0/\beta_{22}^0 \\
q_{13}^{y0} &= D_{13}^0/\gamma_{33}^0,\ q_{23}^{y0} = \beta_{23}^0/\gamma_{33}^0,\ q_{12}^{y0} = D_{12}^0/\beta_{22}^0 \\
q_1^{y0} &= D_{11}^0,\ q_2^{y0} = \beta_{22}^0,\ q_3^{y0} = \gamma_{33}^0 \\
\alpha_{32}^0 &= D_{32}^0 - d_{12}^0 D_{31}^0/D_{11}^0,\ \alpha_{33}^0 = D_{33}^0 - d_{13}^0 - d_{31}^0/D_{11}^0 \\
\beta_{22}^0 &= D_{22}^0 - d_{12}^0 D_{21}^0/D_{11}^0,\ \beta_{23}^0 = D_{23}^0 - d_{13}^0 D_{21}^0/D_{11}^0 \\
\gamma_{33}^0 &= \alpha_{33}^0 - \beta_{23}^0 \alpha_{32}^0/\beta_{22}^0
\end{aligned} \tag{3.50}$$

Here $D_{11}^0 \neq 0$.

Let k–deformator $D_y^{00} = \{D_{ij}^0,\, i,\, j = \overline{1,\,3}\}$ be calculated in result of solving the dynamics equations (see Chapter 4) and kinematics equations (3.6) of the medium. Let us realize the following matrix operations (usually made by computer) which represent Gauss algorithm

$$\begin{aligned}
\tau_{21}^{y0}(-q_{21}^{y0})D_y^{00} &= \begin{bmatrix} D_{11}^0 & D_{12}^0 & D_{13}^0 \\ 0 & \beta_{22}^0 & \beta_{23}^0 \\ D_{31}^0 & D_{32}^0 & D_{33}^0 \end{bmatrix} \\
\tau_{31}^{y0}(-q_{31}^{y0})\tau_{21}^{y0}(-q_{21}^{y0})D_y^{00} &= \begin{bmatrix} D_{11}^0 & D_{12}^0 & D_{13}^0 \\ 0 & \beta_{22}^0 & \beta_{23}^0 \\ 0 & \alpha_{32}^0 & \alpha_{33}^0 \end{bmatrix} \\
\tau_{32}^{y0}(-q_{32}^{y0})\tau_{31}^{y0}(-q_{31}^{y0})\tau_{21}^{y0}(-q_{21}^{y0})D_y^{00} &= \begin{bmatrix} D_{11}^0 & D_{12}^0 & D_{13}^0 \\ 0 & \beta_{22}^0 & \beta_{23}^0 \\ 0 & 0 & \gamma_{33}^0 \end{bmatrix}
\end{aligned}$$

Continuing Gauss process, the to-be-proved result is obtained

$$\tau_{12}^{y0}(-q_{12}^{y0})\tau_{23}^{y0}(-q_{23}^{y0})\tau_{13}^{y0}(-q_{13}^{y0})\tau_{32}^{y0}(-q_{32}^{y0})\tau_{31}^{y0}(-q_{31}^{y0})\tau_{21}^{y0}(-q_{21}^{y0})D_d^{00} = \\ m_1^{y0}(-q_1^{y0})m_2^{y0}(-q_2^{y0})m_3^{y0}(-q_3^{y0})$$

and, therefore, regarding $\tau_{ij}^{-1}(q_{ij}) = \tau_{13}^{y0}(q_{13}^{y0}) = \tau_{ij}(-q_{ij})$ (see (3.41)) the following is true

$$D_y^{00} = \tau_{21}^{y0}(q_{21}^{y0})\tau_{31}^{y0}(q_{31}^{y0})\tau_{32}^{y0}(q_{32}^{y0})\tau_{13}^{y0}(q_{13}^{y0})\tau_{23}^{y0}(q_{23}^{y0})\tau_{12}^{y0}(q_{12}^{y0})m_y^{00} \tag{3.51}$$

Here m_y^{00} is the diagonal matrix with elements q_i^{y0} on the main diagonal. The signs of these elements are not known. But, by definition, D_y^{00} is k–deformator, and hence,

$$\begin{aligned}
A^{-1}D_y^{00} &= m_y^{00} \\
A &= \tau_{21}^{y0}(q_{21}^{y0})\tau_{31}^{y0}(q_{31}^{y0})\tau_{32}^{y0}(q_{32}^{y0})\tau_{13}^{y0}(q_{13}^{y0})\tau_{23}^{y0}(q_{23}^{y0})\tau_{12}^{y0}(q_{12}^{y0})
\end{aligned} \tag{3.52}$$

the last matrix being k–deformator, too (as a finite product of k–deformators). The diagonal matrix m_y^{00} is k–deformator if and only if all elements on its main diagonal are positive (see (3.41)), *i.e.*, $m_y^{00} = d_y^{00}$.

In such a way, in the considered case the k–deformator D_y^{00} in action of an arbitrary vector $h_y^0 = x^{00} - y^{00}$ in ε_y–vicinity of the medium point extends (compresses) its coordinates from/to the coordinate planes in direction of their normals. The k–deformator realizes 6 shifts directed along the coordinate axes with numbers 1, 2, 1, 3, 3, 2 by different values depending on the second index and on the transvection coefficients; besides all actions become simultaneously, not one after another. If the mentioned successive deformation would be physically realizable, then an observer situated in the frame accompanying the medium deformation, would have the possibility to 'see' simultaneously all the dilations and the shifts.

2. The calculation of transvection–dilation decompositions of the k–deformator D_y^{00} is obvious: in each of 6 variants of decompositions (3.49), each element of the groups $\mathcal{GK}_L^1(\mathcal{R},3)\,\mathcal{GK}_L^2(\mathcal{R},3)$ and $\mathcal{GK}_U^1(\mathcal{R},3)\,\mathcal{GK}_U^2(\mathcal{R},3)$ gives 3 variants (due to the permutability of factors in (3.45)) and that of the group $\mathcal{D}^q(\mathcal{R},3)$ gives $3! = 6$ variants. Thus $6 \times 3 \times 3 \times 6 = 324$ variants.

Comments

1. All elements of dilations (3.50) are products of the element D_{11}^0 and of the functions of the rest non–dimensional elements of the deformator D_y^{00}, while the arguments of all transvections in (3.49) are functions of non–dimensional elements of the k–deformator D_y^{00} (*i.e.*, these elements are referred to the element $D_{11}^0 \neq 0$). Thus, relations (3.50) transform into

$$\begin{array}{rcl} q_1^{y0} & = & D_{11}^0,\ q_2^{y0} = D_{11}^0\beta_{22}^0,\ q_3^{y0} = D_{11}^0\gamma_{33}^0,\ q_{21}^{y0} = D_{21}^0 \\ q_{31}^{y0} & = & D_{31}^0,\ q_{32}^{y0} = \alpha_{32}^0/\beta_{22}^0,\ q_{13}^{y0} = D_{13}^0/\gamma_{33}^0 \\ q_{23}^{y0} & = & \beta_{23}^0/\gamma_{33}^0,\ q_{12}^{y0} = D_{12}^0/\beta_{22}^0,\ \alpha_{32}^0 = D_{32}^0 - d_{12}^0 D_{31}^0 \\ \alpha_{33}^0 & = & D_{11}^0 - d_{11}^0 D_{11}^0,\ \beta_{22}^0 = D_{22}^0 - d_{12}^0\delta_{21}^0 \\ \beta_{23}^0 & = & D_{23}^0 - d_{13}^0 D_{21}^0,\ \gamma_{33}^0 = \alpha_{33}^0 - \beta_{23}^0\alpha_{32}^0 \end{array}$$

The situation conserves the same for all 324 transvections–dilations decompositions of the k–deformator. This is realized by substitution of the element D_{11}^0 by the elements D_{22}^0 or D_{33}^0 in corresponding cases.

2. Each one of 324 decompositions of the k–deformator D_d^{00} includes 9 independent parameters. Thus, all 324 complexes of the mentioned elements are equivalent since all calculated products of transvections and dilations give one and the same k–deformator D_d^{00}.

3. The following relations are obvious:

$$\tau_{21}^{y0}(q_{21}^{y0})\tau_{31}^{y0}(q_{31}^{y0})\tau_{32}^{y0}(q_{32}^{y0}) = E + \Delta\tau_H^{y0}$$

$$\tau_{21}^{y0}(q_{21}^{y0})\tau_{31}^{y0}(q_{31}^{y0})\tau_{32}^{y0}(q_{32}^{y0}) = E + \Delta\tau_B^{y0}$$

$$\Delta\tau_H^{y0} = \begin{bmatrix} 0 & 0 & 0 \\ q_{21}^{y0} & 0 & 0 \\ q_{31}^{y0} & q_{32}^{y0} & 0 \end{bmatrix}, \quad \Delta\tau_B^{y0} = \begin{bmatrix} 0 & q_{12}^{y0} & q_{13}^{y0} \\ 0 & 0 & q_{23}^{y0} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Delta d_y^0 = E + \Delta d_y^0,\ \Delta d_y^0 = \begin{bmatrix} q_1^{y0} - 1 & 0 & 0 \\ 0 & q_2^{y0} - 1 & 0 \\ 0 & 0 & q_3^{y0} - 1 \end{bmatrix}$$

If the obtained representations of the Cavalieri generalized operators and of the dilation (defined by (3.41)) are substituted in decomposition (3.50), the k–deformator is represented in the form

$$\begin{aligned} D_y^{00} &\equiv \Delta\tau_{yH}^0 \Delta\tau_{yB}^0 \Delta d_y^0 \\ &= (E + \Delta\tau_H^{y0})(E + \Delta\tau_B^{y0})(E + \Delta d_y^0) = E + \Delta D_y^{00} \end{aligned} \tag{3.53}$$

where

$$\begin{aligned} \Delta D_y^{00} &= \Delta\tau_H^{y0} + \Delta\tau_B^{y0} + \Delta d_y^0 + \Delta\tau_H^{y0}\Delta d_y^0 + \Delta\tau_H^{y0}\Delta\tau_B^{y0} \\ &\quad + \Delta\tau_B^{y0}\Delta d_y^0 + \Delta\tau_H^{y0}\Delta\tau_B^{y0}\Delta d_y^0 \end{aligned}$$

4. If $||\Delta\tau_H^{y0}\Delta\tau_B^{y0}|| = o(||\Delta\tau_H^{y0}\Delta d_y^0||) = o(||\Delta\tau_H^{y0}\Delta d_y^0||)$ and the medium is initially non–deformed (3.21), then

$$\Delta D_y^{00} \approx (\Delta\tau_H^{y0} + \Delta\tau_B^{y0}) d_y^0 + \Delta d_y^0 \tag{3.54}$$

or

$$\Delta D_y^{00} \approx \begin{bmatrix} q_1^{y0} - 1 & q_{12}^{y0} q_2^{y0} & q_{13}^{y0} q_3^{y0} \\ q_{21}^{y0} q_1^{y0} & q_2^{y0} - 1 & q_{12}^{y0} q_3^{y0} \\ q_{31}^{y0} q_1^{y0} & q_{32}^{y0} q_2^{y0} & q_3^{y0} - 1 \end{bmatrix} \tag{3.55}$$

5. If q_{ij}^{y0} are small, it does not mean that non–diagonal elements D_{ij}^0 of the k–deformator D_y^{00} are small. But if they are small, too, then (3.50) turns into

$$\begin{aligned} q_{21}^{y0} &= D_{21}^0/D_{11}^0,\ q_{31}^{y0} = D_{31}^0/D_{11}^0,\ q_{32}^{y0} = D_{32}^0/D_{11}^0 \\ q_{13}^{y0} &= D_{13}^0/D_{33}^0,\ q_{23}^{y0} = D_{23}^0/D_{33}^0,\ q_{12}^{y0} = D_{12}^0/D_{22}^0 \\ q_1^{y0} &= D_{11}^0,\ q_2^{y0} = D_{22}^0,\ q_3^{y0} = D_{33}^0 \end{aligned} \tag{3.56}$$

and then

$$\begin{aligned} D_{11}^0 &= q_1^{y0},\ D_{12}^0 = q_{12}^{y0} q_2^{y0},\ D_{13}^0 = q_{13}^{y0} q_3^{y0} \\ D_{21}^0 &= q_{21}^{y0} q_1^{y0},\ D_{22}^0 = q_2^{y0},\ D_{23}^0 = q_{23}^{y0} q_3^{y0} \\ D_{31}^0 &= q_{31}^{y0} q_1^{y0},\ D_{32}^0 = q_{32}^{y0} q_2^{y0},\ D_{33}^0 = q_3^{y0} \end{aligned} \tag{3.57}$$

The last relations are similar to (3.55).

According to (3.26), in the considered case (*i.e.*, small deformation without initial deformations) we obtain the following approximate relation between deformations and arguments of shifts and dilations, taking into account relation $\Delta D_{ij}^0 = z_j^i = \Delta_{ij}$ (see (3.22)):

$$\begin{aligned} z_1^1 &= \Delta_{11} \approx q_1^{y0} - 1,\ z_2^1 = \Delta_{12} \approx q_{12}^{y0} q_2^{y0},\ z_3^1 = \Delta_{13} \approx q_{13}^{y0} q_3^{y0} \\ z_1^2 &= \Delta_{21} \approx q_{21}^{y0} q_1^{y0},\ z_2^2 = \Delta_{22} \approx q_2^{y0} - 1,\ z_{32}^2 = \Delta_{23} \approx q_{23}^{y0} q_3^{y0} \\ z_{12}^3 &= \Delta_{31} \approx q_{31}^{y0} q_2^{y0},\ z_2^3 = \Delta_{32} \approx q_{32}^{y0} q_2^{y0},\ z_3^3 = \Delta_{33} \approx q_3^{y0} - 1 \end{aligned}$$

where q_{ij}^{y0} and q_i^{y0} are the same as in (3.56).

The 'geometric' sense of the non–diagonal elements of the small matrix of deformations follows from relation (3.57): these elements are products of transvections arguments (shifts along the basic coordinate axes directions) and corresponding dilations arguments. Otherwise, the mentioned deformations (even if they are small) depend (to an equal degree) on shifts and on dilations. If dilations $\Delta_{ii} \approx 0$, then the deformations z_j^i do not depend practically on them and they coincide with the arguments (parameters) of shifts–transvections (3.40).

The obtained results refer to one of the possible concrete decompositions of k–deformator. If other decompositions of the same k–deformator are used, the 'geometric' sense of its non–diagonal elements will be different. If in an arbitrary phenomenon more than one 'geometric' sense is presented, it means that a real 'geometric' sense of this phenomenon does not exist at all.

The last comment gives a reason of the following definition.

Definition 3.9 *Let: 1. the arguments q_{ij}^{y0} of transvections–shifts in decompositions of k–deformator be small;*

2. the non–diagonal elements of k–deformator be small;

3. the medium be initially non–deformed ($D_y^{00}(0) = E$).

Then: 1. 9 *independent functions D_{ij}^0 ($i \neq j$) and $D_{ii}^0 - 1$ (i.e., the elements of ΔD_y^{00}, being approximately equal to the elements of the deformation matrix $dz_y^{00}/d^{00} = \Delta_y^{00}$ – see (3.22)) are called generalized coordinates of the locally linearly deformable medium at $y \in \varepsilon_y$ in the instant t*

$$q_y = \mathrm{col}\{D_{11}^0 - 1,\, D_{21}^0,\, D_{31}^0,\, D_{12}^0, ...,\, D_{33}^0 - 1\} \tag{3.58}$$

2. the manifold $\mathbf{Q}_9$ whose elements are generalized coordinates is called a manifold (space) of configurations for the locally linearly changeable continuous medium at the point $y \in \varepsilon_y$;

3. the manifold $\mathbf{Q}_{18}$ whose elements are 18×1–columns of the type

$$\mathbf{Q}_{18} = \{x : x = \mathrm{col}\{q_y,\, \dot{q}_y\}\} \tag{3.59}$$

is called a phase space of the medium at the point $y \in \varepsilon_y$.

Comments

1. The words 'at the point $y \in \varepsilon_y$' in the above definition play an important role since the position of y in $\mathbf{E}_0$ is determined by a triple of numbers $y^{00} = \mathrm{col}\{y_1^0,\, y_2^0,\, y_3^0\}$.
2. Condition (3.58) is sufficient for determining the manifold of configurations since condition (3.59) consequences from relation (3.58).
3. Each set among the equivalent 324 sets (of 9 generalized coordinates (3.58)) completely and uniquely determines k–deformator, and, therefore completely and in the unique way determines the position of the radius–vector $h_y^0(t)$ of an arbitrary point of the medium in the instant t if the radius–vector $h_y^0(t_0)$ is known (see (3.7)).

4. In the cases of transvection–dilation decompositions of the k–deformator D_y^{00} the concept of rotation of the medium does not arise.
5. The study of trajectories of the phase point $x(t)$ by the contemporary oscillation theory methods is a possible way to the turbulence process understanding.

Definition 3.10 *The classes of decompositions of the k–deformator groups of type (3.49) (with the dilator group $\mathcal{D}_3^q(\mathcal{R},3)$ in the decomposition middle)*

$$\begin{array}{l} \mathcal{GK}_L^1(\mathcal{R},3)\mathcal{GK}_L^2(\mathcal{R},3)\,\mathcal{D}^q(\mathcal{R},3)\,\mathcal{GK}_U^1(\mathcal{R},3)\mathcal{GK}_U^2(\mathcal{R},3) \\ \mathcal{GK}_U^1(\mathcal{R},3)\mathcal{GK}_U^2(\mathcal{R},3)\,\mathcal{D}^q(\mathcal{R},3)\,\mathcal{GK}_L^1(\mathcal{R},3)\mathcal{GK}_L^2(\mathcal{R},3) \end{array}$$

are called Bruhat decompositions (Dieudonne 1969).

Comment The expression of the considered transvection–dilation decompositions of k–deformator is not an unique one. This is only a part of decompositions using Cavalieri groups.

3.4.3. *Transvection–dilation decompositions of k–deformator group by Cavalieri generalized group*

Proposition 3.14 *The set of matrices*

$$\begin{aligned} \mathcal{GK}_L(\mathcal{R},\,3) &= \{\tau_{yL}^0 : \tau_{yL}^0 = \tau_{21}^{y0}(q_{21}^{y0})\tau_{31}^{y0}(q_{31}^{y0})\tau_{32}^{y0}(q_{32}^{y0}) = \\ \tau_{31}^{y0}(q_{31}^{y0})\tau_{21}^{y0}(q_{21}^{y0})\tau_{32}^{y0}(q_{32}^{y0}) &= \tau_{21}^{y0}(q_{21}^{y0})\tau_{32}^{y0}(q_{32}^{y0})\tau_{31}^{y0}(q_{31}^{y0})\} \\ \mathcal{GK}_U(\mathcal{R},\,3) &= \{\tau_{yU}^0 : \tau_{yU}^0 = \tau_{13}^{y0}(q_{13}^{y0})\tau_{23}^{y0}(q_{23}^{y0})\tau_{12}^{y0}(q_{12}^{y0}) = \\ \tau_{23}^{y0}(q_{23}^{y0})\tau_{13}^{y0}(q_{13}^{y0})\tau_{12}^{y0}(q_{12}^{y0}) &= \tau_{23}^{y0}(q_{23}^{y0})\tau_{12}^{y0}(q_{12}^{y0})\tau_{13}^{y0}(q_{13}^{y0})\} \end{aligned}$$

are subgroups of the groups consisting of triangle (lower and upper) matrices with the multiplicative laws

$$\begin{aligned} \tau_L(a_1,b_1,c_1)\tau_L(a_2,b_2,c_2) &= \tau_{21}(a_1)\tau_{31}(b_1)\tau_{32}(c_1)\tau_{21}(a_2)\tau_{31}(b_2)\tau_{32}(c_2) \\ &= \tau_{21}(a_1+a_2)\tau_{31}(b_1+b_2+c_1a_2)\tau_{32}(c_1+c_2) \\ \tau_U(a_1,b_1,c_1)\tau_U(a_2,\,b_2,\,c_2) &= \tau_{13}(b_1+b_2+c_1a_2)\tau_{23}(c_1+c_2)\tau_{21}(a_1+a_2) \end{aligned}$$

and the inversion laws

$$\begin{aligned} \tau_L^{-1}(a,b,c) &= \tau_{21}(-a)\tau_{31}(ac-b)\tau_{32}(c) \\ \tau_U^{-1}(a,b,c) &= \tau_{13}(ac-b)\tau_{23}(-c)\tau_{12}(-a) \end{aligned}$$

Proof consists of a simple verification.

Definition 3.11 *The groups $\mathcal{GK}_L(\mathcal{R},3) = \mathcal{GK}_L^1(\mathcal{R},3)\,\mathcal{GK}_L^2(\mathcal{R},3), \mathcal{GK}_U(\mathcal{R},3) = \mathcal{GK}_U^1(\mathcal{R},3)\mathcal{GK}_U^2(\mathcal{R},3)$ are called Cavalieri generalized groups (lower and upper).*

Proposition 3.15 *There exist* 6 *representations of the group $\mathcal{GD}_{yt}^q(\mathcal{R},3)$ that are based on Cavalieri generalized groups*

$$\begin{aligned} \mathcal{GD}_{yt}^q(\mathcal{R},\,3) &= \mathcal{GK}_L(\mathcal{R},3)\mathcal{GK}_U(\mathcal{R},3)\mathcal{D}^q(\mathcal{R},3) = \\ \mathcal{GK}_L(\mathcal{R},3)\mathcal{D}^q(\mathcal{R},3)\,\mathcal{GK}_U(\mathcal{R},3) &= \mathcal{D}^q(\mathcal{R},3)\mathcal{GK}_L(\mathcal{R},3)\mathcal{GK}_U(\mathcal{R},3) = \\ \mathcal{GK}_U(\mathcal{R},3)\mathcal{GK}_L(\mathcal{R},3)\mathcal{D}^q(\mathcal{R},3) &= \mathcal{GK}_U(\mathcal{R},3)\mathcal{D}^q(\mathcal{R},3)\,\mathcal{GK}_L(\mathcal{R},3) = \\ &= \mathcal{D}^q(\mathcal{R},3)\mathcal{GK}_U(\mathcal{R},3)\mathcal{GK}_L(\mathcal{R},3) \qquad (3.60) \end{aligned}$$

Proof repeats the proof of P 3.13 due to the possibility of substituting transvections (3.45) with the same first and second indexes.

Definition 3.12 *Decompositions of the k–deformator group of the type*

$$\mathcal{GD}^q_{yt}(\mathcal{R},3) = \mathcal{GK}_U(\mathcal{R},3)\mathcal{D}^q(\mathcal{R},3)\mathcal{GK}_L(\mathcal{R},3) \tag{3.61}$$

are called Bruhat decompositions, as in D 3.9 (Dieudonne 1969).

Comments

1. k–deformators from $\mathcal{GK}_L(\mathcal{R},3)$ and $\mathcal{GK}_U(\mathcal{R},3)$, acting individually, realize the medium shift being parallel to the coordinate planes (e^0_2, o^0, e^0_3) and (e^0_1, o^0, e^0_3) (as in P 3.12), respectively.
2. k–deformator (3.61) rests the above coordinate planes fixed at all (but not by elements) as against the case (3.45).
3. In k–deformator (3.61), factors–transvections with one and the same first and second indexes are commutative as it is seen in the definition.

Proposition 3.16 *Any matrix A is k–deformator ($A = D^{00}_y$) if and only if it has transvection–dilation decompositions.*

Proof If the matrix A is represented as a product of transvective k–deformator ($\tau = E + \Delta\tau$) and of k–deformator–dilator ($d = E + \Delta d$), then A is of the form $A = E + \Delta A$ (where $\Delta\tau \to 0$, $\Delta d \to 0$, $\Delta A \to 0$ at the corresponding points of manifolds (2.4)), and therefore it is the k–deformator.

Vice versa: if the matrix A is k–deformator, then it has one of transvection–dilation decompositions (according to algorithm (3.50)).

3.5. Polar decomposition of k–deformator group

Among all multiplicative transvection–dilation decompositions of k–deformator (about 600 variants of decompositions) no one includes rotation factors. Naturally, the question arises whether such decompositions exist. The answer is positive and it is included herein.

3.5.1. *Right–hand polar decomposition of k–deformator group*

Proposition 3.17 *Let: 1. $\mathcal{GS}^q_{yt}(\mathcal{R},3) \subset \mathcal{GD}^q_{yt}(\mathcal{R},3)$ be the subgroup consisting of symmetric k–deformators*

$$\mathcal{GS}^q_{yt}(\mathcal{R},3) = \{s^{00}_s : s^{00}_s \in \mathcal{GD}^q_{yt}(\mathcal{R},3),\ s^{00}_s = s^{00,T}_s\} \tag{3.62}$$

2. $[\mathbf{e}^s]$ be the non–orthonormal and non–inertial basis accompanying the deformation whose mathematical model is the symmetric matrix s^{00}_s that is the matrix of passage from the initial basis (for example the inertial one $[\mathbf{e}^0]$) (that corresponds to the upper internal index) to the basis $[\mathbf{e}^s]$ (corresponding to the lower index), the matrix being calculated in the initial basis $[\mathbf{e}^0]$ (corresponding to the upper external index)

$$[\mathbf{e}^s] = [\mathbf{e}^0]s^{00}_s \tag{3.63}$$

3. *$\mathcal{SO}^q_{yt}(\mathcal{R},3) \subset \mathcal{GD}^q_{yt}(\mathcal{R},3)$ be the subgroup of rotations of the k–deformator group*

$$\mathcal{SO}^q_{yt}(\mathcal{R},3) = \{c^{ss}_d : c^{ss}_d \in \mathcal{GD}^q_{yt}(\mathcal{R},3),\; c^{ss}_d c^{ss,\,T}_d = E,\; \det c^{ss}_d = 1\} \tag{3.64}$$

4. *$[\mathbf{e}^d]$ be the non–orthonormal basis accompanying the medium deformation whose mathematical model is the matrix c^{ss}_d describing the rotation of the initial non–orthonormal basis $[\mathbf{e}^s]$ (see (2.8)) in the basis $[\mathbf{e}^d]$, the matrix being calculated in the initial basis $[\mathbf{e}^s]$:*

$$[\mathbf{e}^d] = [\mathbf{e}^s]c^{ss}_d \tag{3.65}$$

5. *$s^{00}_s c^{ss}_d \in \mathcal{GD}^q_{yt}(\mathcal{R},3)$ be the k–deformator transforming the initial basis $[\mathbf{e}^0]$ into the basis $[\mathbf{e}^d]$ accompanying the medium deformation (see (2.8))*

$$[\mathbf{e}^d] = [\mathbf{e}^s]c^{ss}_d = [\mathbf{e}^0]s^{00}_s c^{ss}_d \tag{3.66}$$

Then: 1. the following right–hand polar multiplicative decomposition of the k–deformator group exists

$$\mathcal{GD}^q_{yt}(\mathcal{R},\,3) = \mathcal{GS}^q_{yt}(\mathcal{R},\,3)\,SO^q_{yt}(\mathcal{R},3) \tag{3.67}$$

2. *the expression of decomposition (3.67) in the basis $[\mathbf{e}^0]$ is of the form*

$$D^{00}_y \equiv D^{00}_d \equiv s^{00}_s c^{ss}_d,\; s^{00}_s \in \mathcal{GS}^q_{yt}(\mathcal{R},3),\; c^{ss}_d \in \mathcal{SO}^q_{yt}(\mathcal{R},3) \tag{3.68}$$

3. *the factors in the right–hand side of relation (3.68) completely and uniquely determine the k–deformator $D^{00}_y \equiv D^{00}_d$ due to the equalities (Belkov et al. 1997, Dieudonne 1969, Kolmogorov et al. 1957)*

$$s^{00}_s \equiv (D^{00}_d D^{00,\,T}_d)^{0.5}, c^{ss}_d \equiv (s^{00}_s)^{-1}D^{00}_d \equiv (D^{00}_d D^{00,\,T}_d)^{-0.5}D^{00}_d \tag{3.69}$$

Proof Let $D^{00}_d \equiv AB$ and $A^2 \equiv D^{00}_d D^{00,\,T}_d$. Consequently $A \in \mathcal{GS}^q_{yt}(\mathcal{R},3)$, *i.e.*, $A \equiv s^{00}_s$. We will show that in this case B is the wanted matrix. Really, 1. $B \equiv A^{-1}D^{00}_d \to BB^T \equiv A^{-1}D^{00}_d(A^{-1}D^{00}_d)^T = (s^{00}_s)^{-1}D^{00}_d D^{00,\,T}_d(s^{00,\,T}_s)^{-1} = (s^{00}_s)^{-1}A^2(s^{00,\,T}_s)^{-1} = (s^{00}_s)^{-1}s^{00}_s s^{00}_s(s^{00,\,T}_s)^{-1} = E$.
2. $\det(BB^T) = \det E \to \det^2 B = 1 \to \mid \det B \mid = 1$. Since D^{00}_d is k–deformator, then $\det D^{00}_d > 0$ and the matrix s^{00}_s in decomposition (3.68) could be equivalent to dilator (3.30) only, and, therefore $\det s^{00}_s > 0$ and $\det B = 1 \to B \in \mathcal{SO}^q_{yt}(\mathcal{R},3) \to B \equiv c^{ss}_d$.

Example Let the k–deformator $D^{00}_y = \begin{bmatrix} 0.4 & 0.28 \\ 0.12 & 0.584 \end{bmatrix}$ act on the plane. $D^{00}_y = \tau_{21}(0.3)d_1(1.4)d_1(1.5)\tau_{12}(0.7)$ is the transvection–dilation decomposition of the k–deformator, *i.e.*, this k–deformator is a continuous superposition of the shift of the radius–vector of an arbitrary point of the medium w.r.t. the first coordinate, an extension of the result obtained w.r.t. both coordinates and a shift w.r.t. the second coordinate. Thus we have $h^0_y = D^{00}_y h^d_y = \text{col}\{0.68; 0.704\}$ when $h^d_y = \text{col}\{1; 1\}$. Despite action of the extension dilation in the above resolution the resulted vector length is reduced, *i.e.*, $\parallel h^d_y \parallel = 1.412$ and $\parallel h^0_y \parallel = 0.9788$.

The right–hand polar decomposition of the given k–deformator is $D_y^{00} \equiv s_s^{00} c_d^{ss}$ where $s_s^{00} = \begin{bmatrix} 0.4398 & 0.2122 \\ 0.2122 & 0.5572 \end{bmatrix}$, $c_d^{ss} = \begin{bmatrix} 0.987 & 0.1605 \\ -0.1605 & 0.987 \end{bmatrix}$. The first matrix is a compressive dilator written in the non–canonical basis, the second one is the matrix of rotation with the angle $\alpha = -9.23^0$. Thus, one and the same k–deformator could be represented as a superposition of shifts and extensions of the medium or of rotation and compressions of this medium.

Proposition 3.18 *Let: 1.* $s_s^{00} \in \mathcal{GS}_{yt}^q(\mathcal{R}, 3)$ *be the symmetric component of k–deformator* $D_y^{00} \equiv D_d^{00}$ *in the right–hand polar decomposition (3.68);*

2. $[\mathbf{e}_r^\lambda]$ *be the orthonormal basis from the normed eigenvectors of the matrix* s_s^{00} *(index r marks the using of right–hand decomposition);*

3. $\lambda_i^{\lambda r}$, $i = \overline{1,3}$, *be the eigenvalues of the matrix* s_s^{00}*;*

4. $c_{\lambda r}^0 \in \mathcal{O}(\mathcal{R}, 3)$ *be the matrix of orthogonal transformation* ($\mid \det c_{\lambda r}^0 \mid = 1$) *of the initial basis (for example the inertial one* $[\mathbf{e}^0]$*) to the basis* $[\mathbf{e}_r^\lambda]$

$$[\mathbf{e}_r^\lambda] = [\mathbf{e}^0] c_{\lambda r}^0 \tag{3.70}$$

Then: 1. the symmetric component s_s^{00} *in (3.69) in the basis* $[\mathbf{e}_r^\lambda]$ *is a dilator with the dilation arguments* $d_{si}^{0\lambda}(\lambda_i^{\lambda r})$, $i = \overline{1,3}$, *which are the eigenvalues of the matrix* s_s^{00}

$$s_s^{0\lambda} = c_{\lambda r}^{00,\,T} s_s^{00} c_{\lambda r}^{00} = d_s^{0\lambda} = \mathrm{diag}\{\lambda_1^{\lambda r},\ \lambda_2^{\lambda r},\ \lambda_3^{\lambda r}\} \tag{3.71}$$

2. the matrix $c_{\lambda r}^{00} \in \mathcal{O}(\mathcal{R}, 3)$ *(see (3.71)) consists of the coordinate columns of the normed eigenvectors of* s_s^{00} *in the basis* $[\mathbf{e}^0]$*;*

3. due to (3.71) the symmetric matrix $s_s^{00} \in \mathcal{GS}_{yt}^q(\mathcal{R}, 3)$ *is of the form*

$$s_s^{00} = c_{\lambda r}^{00} d_s^{0\lambda} c_{\lambda r}^{00,\,T} \tag{3.72}$$

Proof follows from algebra theorems about orthogonal equivalence of the self–conjugate operators (Gantmacher 1964).

Definition 3.13 *The basis* $[\mathbf{e}_r^\lambda]$ *(see (3.70)) is called a canonical basis of the right–hand polar decomposition of k–deformator* D_d^{00}.

Comment According to (3.72) decomposition (3.68) is of the form

$$D_d^{00} = s_s^{00} c_d^{dd} = c_{\lambda r}^{00} d_s^{0\lambda} c_{\lambda r}^{00,\,T} c_d^{dd} \tag{3.73}$$

$$h_r^d = c_{\lambda r}^{00} d_s^{0\lambda} c_{\lambda r}^{00,\,T} c_d^{dd} h_r^0 \tag{3.74}$$

Example If we consider the matrix $s_s^{00} = \begin{bmatrix} 0.4398 & 0.2122 \\ 0.2122 & 0.5572 \end{bmatrix}$ from the previous example, then $d_s^{0\lambda} = \begin{bmatrix} 0.2783 & 0 \\ 0 & 0.7186 \end{bmatrix}$.

Proposition 3.19 *Let: 1. c_d^{ss} be the rotation matrix (3.68);*
2 $c_i(q_i^y)$, $i=\overline{1,3}$, be the simplest rotation matrices (5.6);
3. q_i be the simplest rotation parameter, i.e., the rotation angle (5.16).
Then: a few expressions of the matrix c_d^{ss} can be written

$$\begin{aligned} c_d^{ss} &= c_1(q_1^y)c_2(q_2^y)c_3(q_3^y), \quad c_d^{ss} = c_2(q_2^y)c_1(q_1^y)c_3(q_3^y) \\ c_d^{ss} &= c_3(q_3^y)c_1(q_1^y)c_2(q_2^y), \quad c_d^{ss} = c_3(q_3^y)c_1(q_1^y)c_3(q_3^y) \end{aligned}$$

Comments

1. Decompositions (5.6) rest true for rotations matrices $c_{\lambda r}^{00}$ (see (3.74)) if $\det c_{\lambda r}^{00} = 1$. Thus, if the identical right–hand polar decomposition (3.68) is used for k–deformator D_y^{00} as consecutive realization of the simplest rotations (if $\det c_{\lambda r}^{00} = 1$) and dilations, then there are 9 transformations (3 basic rotations (see P 3.19), 3 analogous rotations in the matrix and 3 dilations). But the polar decomposition will remain 'enlarged' since each simplest rotation matrix can be represented as product of transvections and dilations (see section 3.4).
2. Using the definition of simplest rotations (5.6) and one of the representations of 3–dimensional rotation (see P 3.19), the following right–hand polar decomposition of k–deformator can be written

$$D_y^{00} = E + \Delta D_y^{00} \tag{3.75}$$

$$\Delta D_y^{00} = \begin{bmatrix} d_1 - 1 & \Delta_{12}^s & \Delta_{13}^s \\ \Delta_{21}^s & d_2 - 1 & \Delta_{23}^s \\ \Delta_{31}^s & \Delta_{32}^s & d_3 - 1 \end{bmatrix} + o(\Delta^2)$$

where $o(\Delta^2)$ are components of $\Delta c_i \Delta c_i$ and $\Delta c_i \Delta d_i$ order and higher one.
3. Due to relation (3.75) the deformations matrix of a right–hand polar decomposition is approximately equal to

$$\Delta_y^{00} = dz^{00}/dy^{00} = \begin{bmatrix} d_1 - 1 & \Delta_{12}^s & \Delta_{13}^s \\ \Delta_{21}^s & d_2 - 1 & \Delta_{23}^s \\ \Delta_{31}^s & \Delta_{32}^s & d_3 - 1 \end{bmatrix}$$

4. If $c_{\lambda r}^{00}$ is an orthogonal transformation ($\det c_{\lambda r}^{00} = -1$), then (as the matrix $\Delta c_{\lambda r}^{00} \Delta c_{\lambda r}^{00,T}$ may be not infinitesimal small at the point $y \in \varepsilon_y$) the deformations matrix is

$$\Delta D_y^{00} = \Delta c_d^{ss} + c_{\lambda r}^{00} \Delta d_r^{0\lambda} \Delta c_{\lambda r}^{00,T} + c_{\lambda r}^{00} \Delta d_r^{0\lambda} \Delta c_{\lambda r}^{00,T} \Delta c_d^{ss} \tag{3.76}$$

If $c_{\lambda r}^{00}$ is a rotation ($\det c_{\lambda r}^{00} = -1$), then, *e.g.*, in 2–dimensional case, disregarding the components of $\Delta c \Delta d \Delta c$ and $\Delta c \Delta c \Delta d \Delta c$ order (with different indexes), we obtain

$$\Delta D_y^{00} = \Delta c_d^{ss} + \Delta d_r^{0\lambda} + \Delta c_{\lambda r}^{00} \Delta d_r^{0\lambda,T} + \Delta d_r^{0\lambda}(\Delta c_{\lambda r}^{00} + \Delta c_{\lambda r}^{00,T}) \tag{3.77}$$

where all Δc (with corresponding indexes) are calculated easily using relations (5.8), (5.9), and (5.10).

Comparing (3.55), (3.76) and (3.77) one sees that deformation matrices (if k–deformator has different identical decompositions) could be geometrically treated in different ways, and they model no real motion.

5. From (3.77) follows that if angles of all rotations and dilations arguments are small in right–hand polar decompositions, then the deformation matrix elements depend on deformations–dilations in the canonical basis, on deformations–rotations Δc_d^{ss} (see 3.64)) and on their crossed products.
6. We will note that in the case of right–hand polar decomposition of k–deformator:

   6.1. The deformation is not a sum of independent rotation and dilation (as it is assumed) but a product between them, and consequently the elements of the deformation matrix are functions of the elements of the both matrices mentioned above.

   6.2. Deformation matrices depend on the small angles Δc_d^{ss} of rotations and arguments of deformations–dilations $\Delta d_r^{0\lambda}$ in the canonical basis, and on their crossed influence of rotation angles $\Delta c_{\lambda r}^{00}$ on the canonical basis (including improper orthogonal transformations which are no k–deformators at all – D 2.4).

   6.3. In the case of the polar decomposition of k–deformator, the deformation matrix does not include shifts.
7. The transformation of the symmetric matrix s_s^{00} into a diagonal one (with the help of eigenvalues and eigenvectors) is not unique. There exist non–orthogonal bases where this matrix is a diagonal, too (Gantmacher 1964).

3.5.2. *Kinematics differential equations on groups $\mathcal{GS}_{yt}^q(\mathcal{R},3)$ and $\mathcal{SO}_{yt}^q(\mathcal{R},3)$*

In section 3.1 we obtained the kinematics equations on the k–deformators group $\mathcal{GS}_{yt}^q(\mathcal{R},3)$ having available the solution D_y^{00} whose rotation and symmetric components of the k–deformator D_y^{00} may be calculated with using relation (3.69). The solution of the problem is related with the necessity of laborious calculations of the matrix $(D_y^{00}D_y^{00,T})^{0.5}$ (see (3.69)). The right–hand polar decomposition (3.68) of the k–deformator D_y^{00} makes possible to obtain one more algorithm for a calculation of the D_y^{00} matrix components. Here we will obtain the matrix differential equations (similar to (3.1)) whose solutions are the components mentioned above.

Proposition 3.20 *Let: 1. $D_y^{00} \equiv D_d^{00} \equiv s_s^{00}c_d^{ss}$, $s_s^{00} \in \mathcal{GS}_{yt}^q(\mathcal{R},3)$, and $c_d^{ss} \in \mathcal{SO}_{yt}^q(\mathcal{R},3)$ be the right–hand polar decomposition of k–deformator (see (2.10));*
2. the decomposition of the matrix dv_y^{00}/dy^{00} (see (3.1)) as the sum skew–symmetric and symmetric components be of the form (see section 3.6)

$$\begin{aligned} dv_y^{00}/dy^{00} &= [dv_y^{00}/dy^{00}] + < dv_y^{00}/dy^{00} > \qquad (3.78)\\ [v_y^{00}/dy^{00}] &= \frac{1}{2}\{dv_y^{00}/dy^{00} + (dv_y^{00}/dy^{00})^T\}\\ < v_y^{00}/dy^{00} > &= \frac{1}{2}\{dv_y^{00}/dy^{00} - (dv_y^{00}/dy^{00})^T\} \end{aligned}$$

Then the components s_s^{00} and c_d^{ss} of the k–deformator D_d^{00} are solutions of the following system of kinematics equations on the groups $\mathcal{GS}_{yt}^q(\mathcal{R},3)$ and $\mathcal{SO}_{yt}^q(\mathcal{R},3)$

$$s_s^{00\cdot} = [dv_y^{00}/dy^{00}]s_s^{00},\ c_d^{ss\cdot} =< dv_y^{0s}/dy^{0s} > c_d^{ss} \qquad (3.79)$$

$$< dv_y^{0s}/dy^{0s} >= s_s^{00,-1} < dv_y^{00}/dy^{00} > s_s^{00} \qquad (3.80)$$

Proof Using (3.1), we obtain $D_d^{00\cdot} D_d^{00,-1} = (s_s^{00} c_d^{ss})^{\cdot} (s_s^{00} c_d^{ss})^{-1} = (s_s^{00\cdot} c_d^{ss} + s_s^{00} c_d^{ss\cdot})\, c_d^{ss,-1} s_s^{00,-1} = s_s^{00\cdot} c_d^{ss} c_d^{ss,-1} s_s^{00,-1} + s_s^{00} c_d^{ss\cdot} c_d^{ss,-1} s_s^{00,-1} = s_s^{00\cdot} s_s^{00,-1} + s_s^{00} c_d^{ss\cdot} c_d^{ss,-1} s_s^{00,-1}$.

Taking into account relations (3.70) and (3.1) we reach to the equalities $s_s^{00\cdot} s_s^{00,-1} = [dv_y^{00}/dy^{00}]$, $s_s^{00} c_d^{ss\cdot} c_d^{ss,-1} s_s^{00,-1} = < dv_y^{00}/dy^{00} >$, and then the necessary result follows.

3.5.3. *Left–hand polar decomposition of k–deformator group*

Proposition 3.21 *Let* $\mathcal{GS}_{yt}^q(\mathcal{R},3)$ *and* $\mathcal{SO}_{yt}^q(\mathcal{R},3)$ *be the groups from (3.67),* $c_s^{00} \in \mathcal{SO}_{yt}^q(\mathcal{R},3)$ *and* $s_d^{ss} \in \mathcal{GS}_{yt}^q(\mathcal{R},3)$.

Then: 1. the following left–hand polar multiplicative decomposition of the k–deformators group is true (Gantmacher 1964, Skornyakov 1980, Suprunenko 1972)

$$\mathcal{GD}_{yt}^q(\mathcal{R},\,3) = \mathcal{SO}_{yt}^q(\mathcal{R},\,3)\,\mathcal{GS}_{yt}^q(\mathcal{R},\,3) \tag{3.81}$$

2. the decomposition (3.81) in the basis $[\mathbf{e}^0]$ *is of the form*

$$D_d^{00} = c_s^{00} s_d^{ss},\ c_s^{00} \in \mathcal{SO}_{yt}^q(\mathcal{R},\,3),\ s_d^{ss} \in \mathcal{GS}_{yt}^q(\mathcal{R},\,3) \tag{3.82}$$

3. the factors of decomposition (3.82) are determined completely and uniquely by the kinematic deformator D_d^{00} *itself*

$$s_d^{ss} \equiv (D_y^{00} D_y^{00,\,T})^{0.5},\ c_s^{00} \equiv D_y^{00} (D_y^{00,\,T} D_y^{00})^{0.5} \tag{3.83}$$

Proof Let $D_d^{00} \equiv BA$ and $A^2 \equiv D_d^{00,\,T} D_d^{00}$. Consequently $A \in \mathcal{GS}_{yt}^q(\mathcal{R},3)$, *i.e.*, $A \equiv s_d^{ss}$. We will show that in this case B is the wanted matrix. Really, $B \equiv D_d^{00} A^{-1} \to B^T B \equiv (D_d^{00} A^{-1})^T D_d^{00} A^{-1} = (s_d^{ss})^{-1} D_d^{00} D_d^{00,\,T} (s_d^{ss,\,T})^{-1} = (s_d^{ss})^{-1} A^2 (s_d^{ss,\,T})^{-1} = s_d^{ss,-1} s_d^{ss} s_s^{00} (s_d^{ss,\,T})^{-1} = E \to B \in \mathcal{SO}_{yt}^q(\mathcal{R},3) \to B \equiv c_s^{00}$.

Example Let the same k–deformator $D_y^{00} = \begin{bmatrix} 0.4 & 0.28 \\ 0.12 & 0.584 \end{bmatrix}$ act on the plane. Transvection–dilation decomposition of the k–deformator is of the form $D_y^{00} = \tau_{21}(0.3) d_1(1.4) d_1(1.5) \tau_{12}(0.7)$. In result, we have $h_y^0 = D_y^{00} h_y^d = \mathrm{col}\{0.68; 0.704\}$ when $h_y^d = \mathrm{col}\{1;1\}$. Despite action of the extension dilation in the above resolution the resulted vector length is reduced, *i.e.*, $\parallel h_y^d \parallel = 1.412$ and $\parallel h_y^0 \parallel = 0.9788$. The left–hand polar decomposition of the given k–deformator is $D_y^{00} \equiv c_c^{00} s_d^{cc}$ where $s_d^{cc} = \begin{bmatrix} 0.3756 & 0.1826 \\ 0.1826 & 0.6214 \end{bmatrix}$, $c_c^{00} = \begin{bmatrix} 0.987 & 0.1605 \\ -0.1605 & 0.987 \end{bmatrix}$. The first matrix is a compressive dilator written in the non–canonical basis (rotated by the matrix c_c^{00}), the second one is the matrix of rotation with an angle $\alpha = -9.23^o$. Thus, one and the same k–deformator could be represented as a superposition of shifts and extensions of the medium or of a rotation and compressions of this medium.

Thus, as in the previous example, the medium is acted by superposition of the rotation and the dilation with the same eigenvalues but in the opposite order.

These actions on the medium are different in principle: first, for example the vector $h_y^d = \mathrm{col}\{1;1\}$ is compressed, and then rotated. Otherwise, the operators of dilation and rotation, in the both polar decompositions of the k–deformator, are the same but they act on different vectors in such a way that the results coincide.

Proposition 3.22 *Let: 1. $s_d^{ss} \in \mathcal{GS}_{yt}^q(\mathcal{R},3)$ be the symmetric component of k–deformator D_d^{00} (see (3.82));*

2. $[\mathbf{e}_l^\lambda]$ be an orthogonal basis from the normed eigenvectors of s_d^{ss} (l means left);

3. $\lambda_i^{\lambda l}$, $i = \overline{1,3}$, be the eigenvalues of s_d^{ss}.

Then: 1. the symmetric component s_d^{ss} of k–deformator D_d^{00} in the basis $[\mathbf{e}_l^\lambda]$ is a dilator with dilation arguments $d_{di}^{sl} = \lambda_i^{\lambda l}$, $i = \overline{1,3}$,

$$s_d^{\lambda l} = c_{\lambda l}^{ss,\,T} s_d^{ss} c_{\lambda l}^{ss} = \mathrm{diag}\{\lambda_1^{\lambda l},\, \lambda_2^{\lambda l},\, \lambda_3^{\lambda l}\} \tag{3.84}$$

2. the orthogonal matrix $c_{\lambda l}^{ss}$ describing the transformation of the initial basis $[\mathbf{e}^s]$ to the basis $[\mathbf{e}_l^\lambda]$ consists of the coordinate columns of the normed eigenvectors of s_d^{ss} in the basis $[\mathbf{e}^s]$

$$[\mathbf{e}_l^\lambda] = [\mathbf{e}^s] c_{\lambda l}^{ss} \tag{3.85}$$

Proof follows from the general algebra theorems about the orthogonal equivalence of self–conjugated operators.

Definition 3.14 *Basis (3.85) is called a canonical basis of the left–hand polar decomposition of the k–deformator.*

Comments

1. Due to (3.85) the left–hand polar decomposition of k–deformator has the form

$$D_d^{00} \equiv c_s^{00} s_d^{ss} = c_s^{00} c_{\lambda l}^{ss} d_y^{\lambda l} c_{\lambda l}^{ss,\,T} \tag{3.86}$$

2. The canonical bases $[\mathbf{e}_r^\lambda]$ and $[\mathbf{e}_l^\lambda]$ (see (3.70), (3.85)) (where the symmetric component of k–deformator is a dilator) coincide with each other and they are called traditionally principle ones.

Proposition 3.23 *Let: 1. $D_d^{00} = c_s^{00} s_d^{ss}$, $c_s^{00} \in \mathcal{SO}_{yt}^q(\mathcal{R},3)$, $s_d^{ss} \in \mathcal{GS}_{yt}^q(\mathcal{R},3)$ be the left–hand polar decomposition of k–deformator (3.82);*

2. the identical decomposition of the matrix dv_y^{00}/dy^{00} (see (3.85)) of the skew–symmetric and symmetric components is defined by relation (3.78).

Then the components c_s^{00} and s_d^{ss} of k–deformator D_d^{00} are solutions of the following kinematics differential equations on the groups $\mathcal{GS}_{yt}^q(\mathcal{R},3)$ and $\mathcal{SO}_{yt}^q(\mathcal{R},3)$

$$\begin{aligned} s_d^{ss\bullet} &= [dv_y^{0s}/dy^{0s}] s_d^{ss},\ c_s^{00\bullet} = < dv_y^{0s}/dy^{0s} > c_s^{00} \\ &< dv_y^{0s}/dy^{0s} > = c_s^{00,\,T} < dv_y^{00}/dy^{00} > c_s^{00} \end{aligned} \tag{3.87}$$

Proof Using equation (3.1) we obtain $D_d^{00\cdot}(D_d^{00})^{-1} = (c_s^{00} s_d^{ss})^{\cdot}(c_s^{00} s_d^{ss})^{-1} =$
$(c_s^{00\cdot} s_d^{ss} + c_s^{00} s_d^{ss\cdot})\, s_d^{ss,-1} c_s^{00,-1} = c_s^{00\cdot} s_d^{ss} s_d^{ss,-1} c_s^{00,-1} + c_s^{00} s_d^{ss\cdot} s_d^{ss,-1} c_s^{00,-1} =$
$c_s^{00\cdot} c_s^{00,-1} + c_s^{00} s_d^{ss\cdot} s_d^{ss,-1} c_s^{00,-1}$.
Taking into account (3.5) and (3.87), we arrive at the equalities $c_s^{00\cdot} c_s^{00,-1} =$ $< dv_y^{00}/dy^{00} >$, $c_s^{00} s_d^{ss\cdot} s_d^{ss,-1} c_s^{00,-1} = [dv_y^{00}/dy^{00}]$, and then the necessary result follows.

Comment The velocity $s_d^{ss\cdot}$ of changing the symmetric component of k–deformator D_d^{00} in the case of the left–hand polar decomposition is determined not only by the matrix $[dv_y^{00}/dy^{00}]$ (as Cauchy and Helmholtz have considered (see (3.5))) but by a rotation component of k–deformator D_d^{00} according to equality (3.87) that shows no analogy between the mentioned symmetric component of k–deformator D_d^{00} and the symmetric deformator of an unfrozen particle of the medium (similarly to the case of the right–hand decomposition (3.79)).

3.5.4. *Non–circular (potential) deformation of medium*

Definition 3.15 *Let: 1.* $\mathbf{A}_{2y}^{\mu}$ *be an affine vector plane containing the point* $y \in \varepsilon_y$*;*

2. $\gamma : [a, b] \to \mathbf{R}_2$ *be a fixed in* $\mathbf{E}_0$ *continuous, piecewise smooth closed path,* $\gamma(a) = \gamma(b)$*, with support in* ε_y *(fixed in* $\mathbf{E}_0$*);*
3. $z^0(\gamma)$ *be a radius–vector of an arbitrary point* $z \in \mathbf{E}_0$ *which belongs to the support of* γ*;*
4. v_x^{00} *be the field of medium velocities in* ε_y *(3.4) w.r.t.* $\mathbf{E}_0$ *in the basis* $[\mathbf{e}^0]$*;*

$$\omega = v_x^{00} \cdot dz^{00} \tag{3.88}$$

be 1*–form generated by the vectors* v_x^{00} *and* dz^{00} *(Cartan 1967).*

Then the line integral of the form ω *along the path* γ

$$c_v^0 = \int_{\gamma} \omega = \int_{\gamma(a)}^{\gamma(b)} v_x^{00} \cdot dz^{00} \tag{3.89}$$

is called a circulation of the velocity vector v_x^{00} *along the path.*

Proposition 3.24 *Let: 1.* $\varepsilon_y \in \sigma_3^{\mu}$, $y^0 \in \mathbf{E}_0$, $y^{00} \in \mathbf{R}_3$*;*

2. $\delta_r(y)$ *be the open sphere with a radius* r *whose center is at the point* $y \in \varepsilon_y \in \sigma_3^{\mu}$, $\overline{\delta}_r(y) = \delta_r(y) \cup \partial\delta_r \subset \varepsilon_y$ *be its closure in* $\mathbf{D}_3^{\mu}$, $r = \| z^{00} - y^{00} \|$, $z \in \delta_r$*;* $\partial\delta_r$ *be the boundary of* $\delta_r(y)$*;*
3. $\overline{\delta}_r^{\,i}(y)$ *be the section of the closed sphere* $\overline{\delta}_r(y)$ *with the coordinate plane having a normal vector* $e_i^0 \in [\mathbf{e}^0]$ *(i.e., the closed circle with a radius* r *with a boundary* $\partial\overline{\delta}_r^{\,i}$*),* $i = 1, 2, 3$*;*
4. v_y^{00} *be the coordinate column in the basis* $[\mathbf{e}^0]$ *of the velocity vector of the medium point w.r.t. the frame* $\mathbf{E}_0$*.*

Then: 1. the circulation c^0_{vi} of the velocity vector v^{00}_x along the path support is proportional to the area of the closed circle $\overline{\delta}^i_r$

$$c^0_{vi} = c^0_{vi}(y)\pi r^2 \tag{3.90}$$

2. the proportion coefficients in (3.90) are

$$\begin{aligned} c^0_{v1}(y) &= \partial v^0_3/\partial v^0_2 - \partial v^0_2/\partial v^0_3 \\ c^0_{v2}(y) &= \partial v^0_1/\partial v^0_3 - \partial v^0_3/\partial v^0_1 \\ c^0_{v3}(y) &= \partial v^0_2/\partial v^0_1 - \partial v^0_1/\partial v^0_2 \end{aligned} \tag{3.91}$$

Proof Let $i = 3$, $z^{00} = \mathrm{col}\{r\cos\gamma,\, r\sin\gamma,\, 0\}$, $dz^{00} = \mathrm{col}\{-r\sin\gamma,\, r\cos\gamma,\, 0\}d\gamma$, $c^0_{v3}(y) = \int\limits_{\gamma(a)}^{\gamma(b)} v^{00}_z \cdot dz^{00} = \int\limits_{\gamma(a)}^{\gamma(b)} (dv^{00}_y/dy^{00})z^{00} \cdot dz^{00} = \int\limits_{\gamma(a)}^{\gamma(b)} (\mathrm{grad}\, v^0_{y1} \cdot z^{00})dz^0_1 + \int\limits_{\gamma(a)}^{\gamma(b)} (\mathrm{grad}\, v^0_{y2} \cdot z^{00})dz^0_2 = \pi r^2(\partial v^0_2/\partial v^0_1 - \partial v^0_1/\partial v^0_2)$, where due to $\int \sin^2 x dx = -1/4\sin 2x + 1/2x$, $\int \cos^2 x dx = 1/4\sin 2x + 1/2x$, $\int \sin x \cos x dx = -1/4\cos 2x$.

Definition 3.16 *Let 1–form $\omega = v^{00}_x dz^{00}$ (see (3.88)) be exact at the point $y \in \sigma^\mu_3$ (Cartan 1967), i.e., there exists a vicinity ε_y of the point y where the form ω has the primitive $\varphi(y) \in C^1(\varepsilon_y)$, $d\varphi(y) = \omega$ (Cartan 1967)*

$$\omega = v^{00}_x \cdot dz^{00} = \mathrm{grad}\,\varphi(y) \cdot dz^{00} \tag{3.92}$$

Then: 1. the primitive $\varphi(y)$ of the form ω is called a local potential of the medium deformation (at the point $y \in \mathbf{D}^\mu_3$);

2. the deformation of the medium $\mathbf{D}^\mu_3$ is called locally potential (at the point $y \in \mathbf{D}^\mu_3$).

Definition 3.17 *Let 1–form $\omega = v^{00}_x dz^{00}$ (see (3.92)) be closed (locally exact) on $\mathbf{D}^\mu_3$, i.e., for an arbitrary point $y \in \mathbf{D}^\mu_3$ there exists ε–vicinity ε_y where the form ω has the primitive $\varphi(y) \in \mathbf{C}^1(\varepsilon_y)$*

$$\omega(y) = v^{00}_x \cdot dz^{00} = \mathrm{grad}\,\varphi(y) \cdot dz^{00} \tag{3.93}$$

Then: 1. the primitive $\varphi(y)$ of the form ω is called a potential of the medium deformation;

2. the medium deformation is called potential.

Comment If the medium deformation is potential, it is locally potential, too. The opposite is wrong.

Proposition 3.25 *The medium deformation is potential (locally potential) if and only if it is non–circular (locally non–circular), i.e., (see (3.91))*

$$\begin{aligned} c^0_{v1}(y) &= \partial v^0_3/\partial v^0_2 - \partial v^0_2/\partial v^0_3 = 0 \\ c^0_{v2}(y) &= \partial v^0_1/\partial v^0_3 - \partial v^0_3/\partial v^0_1 = 0 \\ c^0_{v3}(y) &= \partial v^0_2/\partial v^0_1 - \partial v^0_1/\partial v^0_2 = 0 \end{aligned} \tag{3.94}$$

Proof **Necessity** Indeed, let the medium deformation be potential $\omega = v_x^{00} \cdot dz^{00} = \operatorname{grad}\varphi(y) \cdot dz^{00}$ (see (3.93)). Then for the circulation we obtain

$$c_v^0 = \int\limits_\gamma \omega = \int\limits_\gamma \operatorname{grad}\varphi(y) \cdot dz^{00} = 0 \tag{3.95}$$

and the result follows.

Sufficiency Condition (3.94) is sufficient the form $\omega = v_x^{00} \cdot dz^{00}$ to be closed, therefore for the existence of the medium velocity potential in the vicinity of any point.

Proposition 3.26 *To be the medium deformation non–circular (potential), it is necessary and sufficient that one of the following equivalent conditions to be fulfilled:*

1. *in the polar decomposition of k–deformator D_d^{00} (left–hand one (3.82) and right–hand one (3.68)) rotation components (c_s^0 and c_s^d) are absent:*

$$c_s^0 \equiv c_s^d \equiv E \tag{3.96}$$

2. *k–deformator D_d^{00} is the following symmetric matrix*

$$D_d^{00} \equiv s_c^{00} \equiv s_d^{ss} \equiv s_d^{00} \to D_d^{00} \in \mathcal{GS}_{yt}^q(\mathcal{R}, 3) \tag{3.97}$$

Proof **Necessity** Let the medium deformation be non–circular (potential) (see (3.92) and (3.93)). Then the corresponding form (see (3.92) and (3.93)) is closed, *i.e.*, it is the total differential of the potential. In this case integral (3.95), *i.e.*, the circulation along any closed path with necessary properties, equals 0. It means that the circulation coefficients are equal to 0, and hence the skew–symmetric matrices (3.78) in equations (3.79) and (3.87) generated by these coefficients are equal to 0, too. Thus, the result follows to within the constant inertial basis choice.

Sufficiency may be proved in the analogous way in the opposite order.

Proposition 3.27 *Let the medium deformation be non–circular (potential) (see (3.94) or (3.95)).*

Then: 1. the velocity matrix of medium deformation is symmetric:

$$\Delta_y^{0\cdot} = \Delta_y^{0\cdot,\,T} \to dv_y^{00}/dy^{00} = (dv_y^{00}/dy^{00})^T \tag{3.98}$$

2. *the medium deformation matrix is symmetric:*

$$\Delta_y^0 = \Delta_y^{0,\,T} = [dz_y^{00}/dy^{00}] \tag{3.99}$$

3. *the deformation matrix can be written in the form*

$$dz_y^{00}/dy^{00} \equiv [dz_y^{00}/dy^{00}] \equiv \frac{1}{2}[dz_y^{00}/dy^{00} + (dz_y^{00}/dy^{00})^T] \tag{3.100}$$

or in the coordinate form

$$z_i^i \equiv \varepsilon_{ii},\, z_j^i \equiv z_i^j \equiv \frac{1}{2}\gamma_{ij},\, \gamma_{ij} = \gamma_{ji} = z_j^i + z_i^j \tag{3.101}$$

Proof Using relation (3.78) we obtain $\Delta_y^{0\bullet} = dv_y^{00}/dy^{00} \equiv< dv_y^{00}/dy^{00} > +[dv_y^{00}/dy^{00}] \equiv [dv_y^{00}/dy^{00}]$, $\Delta_y^0 = \int \chi_{[0,\,t]} dv_y^{00}/dy^{00} \mu_t(dt) = [dz_y^{00}/dy^{00}]$.

Comments

1. One must make difference between the rotations that are components in one of possible decompositions of *d*–deformator and the rotations of a locally linearly changeable continuous medium. The last ones are present only in the case of a body.
2. The left–hand polar decomposition (3.82) and right–hand one (3.68) of *k*–deformator in the case of non–circular (potential) medium deformation coincide: $D_d^{00} \equiv s_c^{00} \equiv s_d^{ss} \equiv s_d^{00}$.
3. The kinematics equation of non–circular (potential) deformation is of form (3.1) (see (3.79), (3.87))

$$D_y^{00\bullet} = [dv_y^{00}/dy^{00}]D_y^{00} \tag{3.102}$$

3.6. Additive decompositions of medium points velocities

3.6.1. *Transvective–dilation decomposition*

Proposition 3.28 *1. When the medium deformation takes place, then there exist about* 600 *additive transvective–dilation decompositions of the increment velocity* $\Delta v_x^{00} = v_x^{00} - v_y^{00}$ *of an arbitrary point* $x \in \varepsilon_y$ *in the form of the simplest summands obtained with the help of multiplicative decompositions of the k–deformator* $D_d^{00} \in \mathcal{GS}_{yt}^q(\mathcal{R}, 3)$ *(see (3.49));*

2. The pointed decomposition is a linear combination of the velocities $q_{ij}^{y0\bullet}$ *and* $q_i^{y0\bullet}$ *of transvections and dilations parameters with the coefficients* v_{ij}^{y0} *and* v_{yj}^{y0} *whose computing algorithms depend on the decomposition of k–deformator* D_d^{00}*, and, for example, for decomposition (3.50)*

$$D_d^{00} = \tau_{21}^{y0}(q_{21}^{y0})\tau_{31}^{y0}(q_{31}^{y0})\tau_{32}^{y0}(q_{32}^{y0})\tau_{13}^{y0}(q_{13}^{y0})\tau_{23}^{y0}(q_{23}^{y0})\tau_{12}^{y0}(q_{12}^{y0})$$

these coefficients are of the form

$$\begin{aligned} v_{21}^{y0} &= E_{21}\tau_{21}^{y0}(-q_{21}^{y0})h_y^0 \\ v_{31}^{y0} &= \tau_{21}^{y0}E_{31}\tau_{31}^{y0}(-q_{31}^{y0})\tau_{21}^{y0}(-q_{21}^{y0})h_y^0 \\ &\cdots\cdots \\ v_{y1}^{y0} &= \tau_{y1}^{y0}E_{11}\tau_y^0 h_y^0(1/q_1^{y0}),\ \cdots,\ v_{y3}^{y0} = \tau_{y3}^{y0}E_{33}\tau_y^0 h_y^0(1/q_3^{y0}) \end{aligned}$$

Proof is in the same time the description of the algorithm for the practical construction of any of the above decompositions. Differentiating decomposition (3.50) w.r.t. the time and substituting it in (3.4), we obtain

$$\begin{aligned} \Delta v_x^{00} = v_x^{00} - v_y^{00} = D_d^{00\bullet}(D_d^{00})^{-1}h_y^0 = v_{21}^{y0}q_{21}^{y0\bullet} + v_{31}^{y0}q_{31}^{y0\bullet} + \cdots \\ +v_{12}^{y0}q_{12}^{y0\bullet} + v_{y1}^{y0}q_1^{y0\bullet} + \cdots + v_{y3}^{y0}q_3^{y0\bullet} = \textstyle\sum_{ij} v_{ij}^{y0}q_{ij}^{y0\bullet} + \sum_i v_{yi}^{y0}q_i^{y0\bullet} \end{aligned} \tag{3.103}$$

where summing in the first sum is on (i,j)–indexes of decomposition (3.50) while in the second one is on $i=1,2,3$.

Comment Using Bruhat transvection–dilation decompositions of k–deformator D_d^{00} of the type (3.61), we have relation (3.103) in the form

$$\Delta v_x^{00} = v_x^{00} - v_y^{00} = D_d^{00\bullet}(D_d^{00})^{-1}h_y^0 = \sum_{ij}(v_{ij}^{y0} - v_{ij}^{x0})q_{ij}^{y0\bullet} + \sum_i v_{yi}^{y0}q_i^{y0\bullet}$$

as $q_{ij}^{y0\bullet} = q_{ji}^{y0\bullet}$. Note that in the first sum there are 3 summands.

3.6.2. *Polar decompositions*

Proposition 3.29 *Let the decomposition of the left–hand rotation component c_s^{00} of k–deformator $D_d^{00} \in \mathcal{GS}_{yt}^q(\mathcal{R},3)$ in simplest factors–rotations be of form (see (3.82))*

$$\theta_3^{y0} = c_1(\theta_1^{y0})c_2(\theta_2^{y0})c_3(\theta_3^{y0}) \tag{3.104}$$

where θ_i^{y0} are angles of the simplest rotations in the rotation component θ_3^{y0} of k–deformator $D_d^{00} \in \mathcal{GS}_{yt}^q(\mathcal{R},3)$.

Then: 1. the decomposition of the velocity increment $\Delta v_x^{00} = v_x^{00} - v_y^{00}$ in ε_y is of the form

$$\begin{aligned}\Delta v_x^{00} &= \sum_i w_i^{y0}\theta_i^{y0\bullet} + \sum_i c_s^{00}v_{yi}^{ys}c_s^{00,\,T}q_i^{ys\bullet} + \\ &\quad \sum_{ij} c_s^{00}(v_{ij}^{ys} - v_{ij}^{xs})c_s^{00,\,T}q_{ij}^{ys\bullet}\end{aligned} \tag{3.105}$$

2. the coefficients in decomposition (3.105) are

$$\begin{aligned}w_1^{y0} &= \; < e_1^0 > h_y^0,\; w_2^{y0} = c_1(\theta_1^{y0}) < e_2^0 > c_1^T(\theta_1^{y0})h_y^0 \\ w_3^{y0} &= c_1(\theta_1^{y0})c_2(\theta_2^{y0}) < e_3^0 > c_2^T(\theta_2^{y0})c_1^T(\theta_1^{y0})h_y^0\end{aligned} \tag{3.106}$$

Proof $D_d^{00\bullet}(D_d^{00})^{-1}h_y^0 = (c_s^{00}s_d^{ss})\cdot(c_s^{00}s_d^{ss})^{-1} = c_s^{00\bullet}c_s^{00,\,T} + c_s^{00}s_d^{ss\bullet}s_d^{ss,-1}c_s^{00,\,T}$. Using (5.7) and (5.31), we finish the proof.

3.6.3. *Geometrical incompressibility of medium*

Definition 3.18 *A locally linearly changeable medium is said to be geometrically incompressible at the point $y \in \varepsilon_y$ if the k–deformator group $\mathcal{GD}_{yt}^q(\mathcal{R},3)$ on ε_y coincides with its special subgroup*

$$\mathcal{GD}_{yt}^q(\mathcal{R},\,3) \equiv \mathcal{SD}_{yt}^q(\mathcal{R},\,3) = \{D_y^{00} : \det D_y^{00} = 1\} \tag{3.107}$$

Proposition 3.30 *To be a medium geometrically incompressible, it is sufficient to be fulfilled one of the following two conditions:*

1. *the parameters of dilation (see (3.41)) are equal to* 1

$$q_i^{y0} = 1 \tag{3.108}$$

2. *the determinant of dilation (see (3.42)) is equal to* 1

$$\det d_y^{00} = \Pi_i q_i^{y0} = 1 \tag{3.109}$$

Proof Let $\det D_y^{00} = 1$. It follows that $\det d_y^{00} = 1$ since in every decomposition of k–deformator (see (3.49), (3.60), section 3.5), the determinants of the rest factors (transvections, rotations and orthogonal transformations products) are equal to 1. If $\det d_y^{00} = 1$, then $\det d_y^{00} = \Pi_i q_i^{y0} = 1$, in particular $q_i^{y0} = 1$, $i = \overline{1,3}$.

Proposition 3.31 *Let the medium ε_y be geometrically incompressible. Then:*

1. *Lebesgue measure on σ–algebra σ_3^μ is invariant w.r.t. the k–deformators group $\mathcal{SD}_{yt}^q(\mathcal{R}, 3)$;*
2. *the divergence of the medium velocity vector is equal to* 0

$$\operatorname{div}_0 v_y^{00} = 0 \tag{3.110}$$

Chapter 4

Elements of dynamics of locally changeable continuous medium

4.1. Mechanical state equations of continuous medium

Remind that (see Chapter 3):

1. $dz_y^{00}/dy^{00} \equiv \{\partial z_i^0/\partial y_j^0\} \equiv \{z_j^i\}$, $i, j = 1, 2, 3$, is the derivative of the displacement vector z_y^0 of a locally changeable medium at a point y along the radius–vector y^0 of this point in the basis $[\mathbf{e}^0]$ of an inertial frame in an instant $t \in \mathbf{T}_+$ (see (3.13), (3.16)) (the matrix of medium deformations at a point $y \in \varepsilon_y$ under condition (3.22));

2. $(dz_y^{00}/dy^{00})^{\cdot} = \{z_j^i\}^{\cdot} = \Delta_y^{0\cdot}(t) = dv_y^{00}/dy^{00}$ is the velocity of change of dz_y^{00}/dy^{00} coinciding with 3×3–matrix of deformation velocities and the derivative $dv_y^{00}/dy^{00} = \{v_j^i\}$ of a locally changeable medium at the point $y \in \varepsilon_y$ in the basis $[\mathbf{e}^0]$ in the instant $t \in \mathbf{T}_+$ (see (3.13), (3.16));

3. Z_y^0 is 9×1–column consisting of the columns of $dz_y^{00}/dy^{00} \equiv \{z_j^i\}$ (columns of the matrix of medium deformations under condition (3.15))

$$Z_y^0 = \mathrm{col}\{z_1^1, z_1^2, z_1^3, z_2^1, ..., z_3^3\} \tag{4.1}$$

4. V_y^0 is 9×1–column consisting of j–columns $v_{jy}^0 = \mathrm{col}\{v_j^1, v_j^2, v_j^3\}$ of the matrix $dv_y^{00}/dy^{00} = \left(v_{1y}^0, v_{2y}^0, v_{3y}^0\right)$ (see (3.16)):

$$V_y^0 = \mathrm{col}\{v_1^1, v_1^2, v_1^3, v_2^1, ..., v_3^3\} \tag{4.2}$$

5. (T_y^0) is 9×1–column consisting of i–columns $T_{iy}^0 = \mathrm{col}\{T_{i1}^0, T_{i2}^0, T_{i3}^0\}$ of d–deformator $T_y^0 = \left(T_{1y}^0, T_{2y}^0, T_{3y}^0\right)$ at the point y in the basis $[\mathbf{e}^0]$ of the inertial frame $\mathbf{E}_0$ in the instant $t \in T_+$:

$$T_y^0 = \begin{bmatrix} T_{11}^0 & T_{21}^0 & T_{31}^0 \\ T_{12}^0 & T_{22}^0 & T_{32}^0 \\ T_{13}^0 & T_{23}^0 & T_{33}^0 \end{bmatrix} \tag{4.3}$$

$$T_y^0 = \mathrm{col}\{T_{11}^0, T_{12}^0, T_{13}^0, T_{13}^0, ..., T_{33}^0\} \tag{4.4}$$

Definition 4.1 *Let: 1.* $\chi^0_y(Z^0_y, Z^{0\bullet}_y, p_{yt}, \theta_{yt}) = \mathrm{col}\{v^0_y(p_{yt}), \varphi^0_y(Z^0_y, Z^{0\bullet}_y, \theta_{yt})\}$ *be the column of scalar functions depending on Pascal pressure* $p_y = p_{yt}$ *(column* $v^0_y(p_{yt})$*) only or on the elements of the column* (Z^0_y)*, on their velocities* $(Z^{0\bullet}_y)$*, on coordinates of the point y in the basis* $[\mathbf{e}^0]$ *of the inertial frame and on the temperature* θ_{yt} *(the column* $\varphi^0_y(Z^0_y, Z^{0\bullet}_y, \theta_{yt})$*);*

2. $F^0_y\left((T^0_y), Z^0_y, Z^{0\bullet}_y, \chi^0_y\right)$ *be* 9×1*–column of scalar functions of the elements of the columns* $(T^0_y), Z^0_y$ *and* $Z^{0\bullet}_y$*, of the elements of* $\chi^0_y(Z^0_y, Z^{0\bullet}_y, p_{yt}, \theta_{yt})$ *and of the coordinates of the point y in the basis* $[\mathbf{e}^0]$ *of an inertial frame, the function fulfilling the following requirements:*

2.1. the form of the function F^0_y *do not change with transferring to another frame* $\mathbf{E}_f = (o_f, [\mathbf{e}^f])$

$$F^f_y\left((T^f_y), Z^f_y, Z^{f\bullet}_y, \chi^f_y\right) \tag{4.5}$$

2.2. the elements of $\chi^0_y\left(Z^0_y, Z^{0\bullet}_y, p_{yt}, \theta_{yt}\right)$ *do not change with transferring to a new inertial frame* $\mathbf{E}_f = (o_f, [\mathbf{e}^f])$

$$\chi^0_y\left(Z^0_y, Z^{0\bullet}_y, p_{yt}, \theta_{yt}\right) = \chi^f_y(Z^f_y, Z^{f\bullet}_y, p_{yt}, \theta_{yt}) = \chi_y(Z^0_y, Z^{0\bullet}_y, p_{yt}, \theta_{yt}) \tag{4.6}$$

2.3. all functions from F^0_y *be differentiable w.r.t. all arguments;*

2.4. the equation

$$F^0_y\left((T^0_y), Z^0_y, Z^{0\bullet}_y, \chi^0_y\right) = 0 \tag{4.7}$$

be uniquely solved w.r.t. the elements of each of the columns (T^0_y), Z^0_y, $Z^{0\bullet}_y$*, at any point* $y \in \varepsilon_y$ *in an instant* $t \in \mathbf{T}_+$*:*

$$(T^0_y) = F^0_1(Z^0_y, Z^{0\bullet}_y, \chi^0_y),\ Z^0_y = F^0_2\left((T^0_y), Z^{0\bullet}_y, \chi^0_y\right) \tag{4.8}$$

$$Z^{0\bullet}_y = F^0_3\left((T^0_y), Z^0_y, \chi^0_y\right) \tag{4.9}$$

2.5. under condition of existence and uniqueness of the differential equation (4.9) be solvable.

Then: 1. equation (4.7) and the explicit functions being equivalent to them (see (4.8) and (4.9)) are called correct equations of the mechanical state of a locally changeable continuous medium;

2. the set of media that have the same equations of mechanical state is called a media class, generated by these equations;

3. the set of media having the same mechanical state equations of a particular form is called a media subclass generated by these equations.

Comments

1. Physically, requirements (7.4) means the possibility of creating one group of real medium parameters (for example, tensions) that guarantee a unique physical realization of another group of parameters (for example, deformations or their velocities).

2. The medium mechanical state equations (4.7) are not related with the form of writing (scalar, matrix, *etc.*) at all, and therefore, they are not related with the concepts as 'tangents', 'normals', 'tensions tensor', 'deformation tensor', 'symmetry', *etc.* (Kochin *et al.* 1964, Lur'e 1970, Sedov 1972). The state equations (4.7) represent the relation of tensions, deformations and their velocities with the temperature, Pascal pressure and the rheological coefficients of the medium, that responds to requirements p.p. 2.1–2.2 of D 4.1.
3. Here the medium isotropy notion, traditionally considered (Kochin *et al.* 1964, Lur'e 1970, Sedov 1972), does not arise as an independent property since this is not a physical medium property but Galilean mechanics Universe one (see section 2.4.1) and it is a consequence of Galilean mechanics definition as a generalized Galilean group invariant (see D 2.20). Isotropic locally deformable media could be studied by means of Galilean mechanics only. That is why the term 'isotropic' is not be used further but this property of the medium will be always supposed.

Definition 4.2 *Let in the previous definition at least one of conditions (4.8) and (4.9) be not satisfied, i.e., equations (4.7) do not have a solution or have infinite number of solutions w.r.t. some group of variables.*

Then: 1. equations (4.7) are called incorrect equations of the medium mechanical state;

2. the medium having incorrect equations is called an incorrect medium.

Definition 4.3 *Let the form of function (4.5) and the coefficients* φ_y^0 *(see D 4.1-1) being a part of this function do not depend on the point* y*:*

$$F_y^0\left((T_y^0), Z_y^0, Z_y^{0\cdot}, v_y^0, \varphi_y^0\right) = F^0\left((T_y^0), Z_y^0, Z_y^{0\cdot}, v_y^0, \varphi^0\right) \tag{4.10}$$

Then: 1. the mechanical state equations (4.7) are called homogeneous;

2. the medium having homogeneous equations of a mechanical state is called homogeneous.

Comments

1. It is not clear if it is possible that incorrect equations of mechanical state has the right to exist (Tihonov *et al.* 1979). It will be shown later that the mechanical state equations of Navier–Stokes and Lame are incorrect.
2. The medium homogeneity should not be confused with the property of homogeneity of Galilean mechanics Universe (see D 2.20, P 2.6).

Definition 4.4 *Mechanical state equations which relate not columns elements but corresponding matrices and which conserve the rest properties:*

$$F_y^0\left(T_y^0, dz_y^{00}/dy^{00}, (dz_y^{00}/dy^{00})^{\cdot}, \chi_y^0\right) = 0 \tag{4.11}$$

or their equivalent matrices functions of matrix and scalar arguments:

$$T_y^0 = F_1^0\left(dz_y^{00}/dy^{00}, (dz_y^{00}/dy^{00})^{\cdot}, \chi_y^0\right) \quad (4.12)$$

$$dz_y^{00}/dy^{00} = F_2^0\left(T_y^0, (dz_y^{00}/dy^{00})^{\cdot}, \chi_y^0\right) \quad (4.13)$$

$$(dz_y^{00}/dy^{00})^{\cdot} = F_3^0\left(T_y^0, dz_y^{00}/dy^{00}, \chi_y^0\right) \quad (4.14)$$

when the matrix differential equation (4.14) has a unique solution, are called medium determining relations.

Comments 1. The requirements about invariance of the writing form of the equations and of including there coefficients w.r.t. the transfer to another inertial frame are imposed because of the Galilean mechanics general definition as an invariant of the generalized Galilean group (D 2.20).

2. The determining relations have always polynomial form (Gantmacher 1964). For example, in the 3–dimensional case and if one of the matrix arguments is present, (4.12) becomes

$$T_y^0 = F_1^0\left(dz_y^{00}/dy^{00}, \chi_y^0\right) = \chi_0^0(I)E + \chi_1^0(I)dz_y^{00}/dy^{00} + \chi_2^0(I)(dz_y^{00}/dy^{00})^2 \quad (4.15)$$

where $\chi_0^0(I)$, $\chi_1^0(I)$, $\chi_2^0(I)$ are scalar functions of the invariant $I = (I_1, I_2, I_3)$ of the matrix du_y^{00}/dy^{00} ($u_y^{00} = v_y^{00}$ or $u_y^{00} = z_y^{00}$). The invariant is traditionally defined as coefficients of the equation (Lur'e 1970)

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \quad (4.16)$$

3. The concept of 'mechanical state equations' is wider than the concept of 'determining relations': if the last ones exist, then the first ones exist, too, the opposite is wrong.

4. The elements of (T_y^0), but not of d–deformator T_y^0, occur in the motion equations (2.65) of a locally changeable medium. Therefore, the representation of the relation between the elements of d–deformator T_y^0 and the entries of dz_y^{00}/dy^{00} and $(du_y^{00}/dy^{00})^{\cdot}$ as determining relations that have only form (4.15) reduces the set of possible mathematical models of medium deformation, and consequently the possible classes of deformable media.

5. Further the words 'in the instant $t \in T$ at the point y' will be omitted but they will be had in mind unless otherwise is specified.

6. Thus, the motion of a locally changeable continuous medium will be studied by means of Galilean mechanics if:

   6.1. the form of the medium motion equations (2.65) is invariant w.r.t. the choice of an inertial frame;

   6.2. the form of the medium mechanical state equations (4.7) is invariant w.r.t. the choice of an inertial frame;

   6.3. the values of the coefficients χ^0 which are part of equations (4.6) are invariant w.r.t. the choice of an inertial frame;

   6.4. the solution of equation (4.14) w.r.t. dz_y^{00}/dy^{00} is unique.

Definition 4.5 *Let: 1. the motion equations of a locally changeable medium be of form (2.65)*

$$-\rho_y v_y^{00\cdot} + \rho_y g_y^0(\urcorner y, y) + \mathrm{Div}\,{}_0 T_y^0 = 0 \tag{4.17}$$

2. the equations of a locally changeable medium mechanical state be of form (4.7)

$$F_y^0\left((T_y^0), Z_y^0, Z_y^{0\cdot}, \chi_y^0\right) = 0 \tag{4.18}$$

Then: 1. the equation system

$$\begin{aligned} -\rho_y v_y^{00\cdot} + \rho_y g_y^0(\urcorner y, y) + \mathrm{Div}\,{}_0 T_y^0 &= 0 \\ F_y^0\left((T_y^0), Z_y^0, Z_y^{0\cdot}, \chi_y^0\right) &= 0 \end{aligned} \tag{4.19}$$

is called dynamic equations of the medium mechanical state;

2. *the result of solving the equation system (4.19) w.r.t. one of the variables group is called equations of the medium dynamics;*
3. *the equation system (when $v_y^{00} \equiv 0,\ g_{yt}^0(\urcorner y, y) = g_y^0(\urcorner y, y), p_{yt} = p_y, \theta_{yt} = \theta_y$)*

$$\rho_y g_y^0(\urcorner y, y) + \mathrm{Div}\,{}_0 T_y^0 = 0 \tag{4.20}$$

$$F_y^0((T_y^0), Z_y^0, Z_y^{0\cdot}, \chi_y^0) = 0 \tag{4.21}$$

is called static equations of the medium mechanical state;

4. *the result of solving the equation system (4.20) and (4.21) w.r.t. one of the variables group is called equations of medium statics;*
5. *solutions of the mechanical state static equations (4.21) under condition (4.14) w.r.t. the column*

$$Z_y^0 = F_2^0\left((T_y^0), \chi_y^0\right) \tag{4.22}$$

are called equations of the medium tense state.

Comments

1. The medium motion equations (4.17) are direct consequence (because of (2.27)) of the primary property of Galilean mechanics Universe (see (2.24), (2.62)) and in this sense they are 'nature laws' under accepted axioms conditions. The mechanical state equation (4.7) are the mathematical formulations of hypotheses about the medium properties, only about the properties, but not about the Universe properties. That is why neither mechanical state equations nor medium dynamics equations obtained on this base (as a result of joint resolution of dynamics equations and motions ones) are 'nature laws'. Hence, they are not equivalent for different hypotheses mentioned above.
2. Formulation of the medium dynamics problem (4.19) and the medium static problem (4.20) and (4.21) and their joint number integration without preliminary transformation to one dynamics or statics equation give more possibilities since the operation mentioned above is valid only in the case of resolving mechanical state equations w.r.t. one of the variables group (for example, the tensions column (T_y^0)).

Definition 4.6 *Medium sets (classes, subclasses) that have same dynamics (statics) equations are called dynamically (statically) equivalent.*

4.2. Classes of equivalence of quasi–linear continuous media

Definition 4.7 *Let: 1. the functions* $F_i^0 = F_i^0\left((T_y^0), Z_y^0, Z_y^{0\cdot}, \chi_y^0\right)$ *be linear combinations (not linear functions!)*

$$\begin{aligned}
F_1^0 &= T_{11}^0 - v_1^{0y} - \lambda_1^{0y}(I_1, \theta_{yt}) - \mu_{1111}^{0y}(I, \theta_{yt})u_1^1 - \ldots - \mu_{1133}^{0y}(I, \theta_{yt})u_3^3 \\
F_2^0 &= T_{12}^0 - v_2^{0y} - \lambda_2^{0y}(I_1, \theta_{yt}) - \mu_{1211}^{0y}(I, \theta_{yt})u_1^1 - \ldots - \mu_{1233}^{0y}(I, \theta_{yt})u_3^3 \\
&\ldots \\
F_9^0 &= T_{19}^0 - v_9^{0y} - \lambda_9^{0y}(I_1, \theta_{yt}) - \mu_{3311}^{0y}(I, \theta_{yt})u_1^1 - \ldots - \mu_{3333}^{0y}(I, \theta_{yt})u_3^3
\end{aligned} \quad (4.23)$$

or in the vector form ($u_y^{00} = v_y^{00}$ *or* $u_y^{00} = z_y^{00}$)

$$F_y^0 = (T_y^0) - V_y^0 - \Lambda_y^0(I_1, \theta_{yt}) - M_y^0(I, \theta_{yt})U_y^0 \quad (4.24)$$

where $\lambda_i^{0y}(I_1, \theta_{yt}),\ i = \overline{1,9}$, *are scalar linear functions of the first invariant (see (4.16))*

$$I_1 = \operatorname{div}_0 u_y^{00}$$

of the matrix du_y^{00}/dy^{00} *and of the temperature* θ_{yt}, *depending on the point* y *and on the instant* t*;*

2. 9×1*–columns* V_y^0 *and* $\Lambda_y^0(I_1, \theta_{yt})$ *be of the form*

$$\begin{aligned}
V_y^0 &= -(w) = -w\,\mathrm{col}\{1,0,0,0,1,0,0,0,1\} &(4.25)\\
\Lambda_y^0(I_1, \theta_{yt}) &= \lambda_y\,\mathrm{col}\{1,0,0,0,1,0,0,0,1\} &(4.26)
\end{aligned}$$

where w *and* λ_y *are the known positive scalar functions;*

3. 9×9*–matrix* $M_y^0(I, \theta_{yt})$ *with the elements* $\mu_{lkij}^{0y}(I, \theta_{yt}),\ i,j,l,k = 1,2,3$, *(independent and not depending on* λ_y*) being functions of the point* y*, of the invariants of* du_y^{00}/dy^{00}*, of the medium temperature* θ_{yt} *and due to the contribution of the* (ij)*–elements* u_j^i *of the matrix (column)* U_y^0 *in the elements* T_{lk}^0 *of d–deformator* T_y^0 *be such that:*

3.1. there exists the group $\mathcal{M}_{yt}(\mathcal{R}, 9)$ *whose element is*

$$M_y^0(I, \theta_{yt}) \in \mathcal{M}_{yt}(\mathcal{R}, 9) \quad (4.27)$$

3.2. the rotation group $\mathcal{SO}(\mathcal{R}, 9)$ *(see (4.47)) be centralizer of* $\mathcal{M}_{yt}(\mathcal{R}, 9)$*:*

$$C_f^{0,T} M_y^0(I, \theta_{yt}) C_f^0 = M_y^0(I, \theta_{yt}) \quad (4.28)$$

for any matrices $C_f^0 \in \mathcal{SO}(\mathcal{R}, 9),\ M_y^0(I, \theta_{yt}) \in \mathcal{M}_{yt}(\mathcal{R}, 9)$*;*

4. the equation

$$\Lambda_y^0(I_1, \theta_{yt}) + M_y^0(I, \theta_{yt})U_y^0 = (T_y^0) + (w) \equiv (\tau_y^0) \quad (4.29)$$

where (τ_y^0) *is the tensions column, have the unique solution*

$$U_y^0 = F_3^0(\tau_y^0) \quad (4.30)$$

Then: 1. the equation

$$(T_y^0) = -(w) + \Lambda_y^0(I_1, \theta_{yt}) + M_y^0(I, \theta_{yt})U_y^0 \tag{4.31}$$

is called a quasi–linear equation of a locally changeable medium mechanical state generated by the group $\mathcal{M}_{yt}(\mathcal{R}, 9)$*;*

2. *the class of media that have mechanical state equations of form (4.31) is called* $\mathcal{M}$*–class of equivalence of correct quasi–linear media generated by the group* $\mathcal{M}_{yt}(\mathcal{R}, 9)$*;*
3. *the functions* λ_y *and the matrix* $M_y^0(I, \theta_{yt})$ *elements are called Lame rheological coefficients (of* $\mathcal{M}$*–medium) of first and second types, respectively.*
4. *if* $w = p_{yt}$ *is Pascal pressure,* $U = V$*, then the medium is called a correct quasi–linear viscous fluid (*$\mathcal{M}$*–fluid), the scalar functions* $\lambda_y^0(I_1, \theta_{yt})$*, and the elements of the matrix* $M_y^0(I, \theta_{yt})$ *are called coefficients of viscosity of* $\mathcal{M}$*–fluid of first and second types, respectively;*
5. *if* $w = 0, U = Z$*, then the medium is called a correct quasi–linear elastic material and the scalar functions* λ_y *and the matrix* $M_y^0(I, \theta_{yt})$ *elements are called coefficients of (*$\mathcal{M}$*–material) stiffness of the first and second types, respectively;*

Comments

1. The form of columns (w), $\Lambda_y^0(I_1, \theta_{yt})$ (see (4.25), (4.26)) guarantees the independence of equality (4.31) and of the rheological coefficients $\lambda_y^0(I_1, \theta_{yt})$, $\mu_{lkij}^{0y}(I, \theta_{yt})$ from the inertial basis choice.
2. Relation (4.28) is equivalent to the requirement that the rheological coefficients of second type do not depend on the inertial basis choice.
3. Requirement (4.27) is necessary condition (4.30) to be fulfilled. This condition is sufficient, too, if $\Lambda_y^0(I_1, \theta_{yt}) \equiv 0$.
4. The form of the functions $F(\cdot,\cdot,\cdot)$ is determining in the concept 'quasi–linear'. They include a linear function of $\mathrm{div}_0 u_y^{00}$ as a second term and linear combinations of column U_y^0 elements with coefficients depending on the invariants of du_y^{00}/dy^{00}, on the coordinates of the point y, on the parameter w and on the temperature θ_{yt} as third term. For this reason, the use of linear combinations of the column U_y^0 elements does not entail the linearity of the functions $F(\cdot,\cdot,\cdot)$.
5. Later, the notation 'linear' is used for the medium mechanical state equations whose rheological coefficients do not depend on the invariants of du_y^{00}/dy^{00} that transforms the functions $F(\cdot,\cdot,\cdot)$ of equations (4.24) really in linear ones.
6. Attempts of similar representations of mechanical state equations have been already done, for example in (Sedov 1972). But requirements about a symmetry of the stress 'tensor' and deformation 'tensor' that have been imposed, spoil all the advantages of the equalities. If we rename the independent (by definition) elements of the matrix (of the type $A^{iiii} = 2\mu + \lambda$, $A^{iikk} = \lambda$) (Lur'e 1970, Sedov 1972) (that has neither physical, nor mathematical reason), then these elements become dependent ($A^{iiii} = 2\lambda + A^{iikk}$) and as a result the matrix becomes singular, and the equations become incorrect (see section 4.7).

4.3. Conditions for continuous medium entirety

Definition 4.8 *Let: 1. the medium be a fluid;*

2. the condition for an inertial balance of the medium at the point y be of the form (see (2.47), (2.67))

$$\rho_y^{\cdot} + \rho_y \operatorname{div}_0 v_y^{00} = 0 \tag{4.32}$$

Then equality (4.32) is called a condition of fluid entirety.

Let: 1. the continuous medium be quasi–linear elastic material – see (4.1);

2. the velocity vector v_y^{00} of the elastic material be continuous function of the point y – see (4.2).

Then: 1. with the account of (3.13), the condition for an inertial balance of the elastic material at the point y is of the form

$$\rho_y^{\cdot} + \rho_y \operatorname{div}_0 z_y^{00\cdot} = 0 \tag{4.33}$$

2. equation (4.33) is called an equation of elastic material entirety at the point y.

Proof Due to (3.15) from $(dz_y^{00}/dy^{00})^{\cdot} = dv_y^{00}/dy^{00}$we have $(dz_y^{00}/dy^{00})^{\cdot} = dz_y^{00\cdot}/dy^{00}$ or in the element-wise form $z_j^{i\cdot} = z_{j}^{\cdot i}$. Hence, $\operatorname{div}_0 z_y^{00\cdot} = \partial z_{y1}^{0\cdot}/\partial y_1^0 + \partial z_{y2}^{0\cdot}/\partial y_2^0 + \partial z_{y3}^{0\cdot}/\partial y_3^0 = (\partial z_{y1}^{0\cdot}/\partial y_1^0)^{\cdot} + (\partial z_{y2}^{0\cdot}/\partial y_2^0)^{\cdot} + (\partial z_{y3}^{0\cdot}/\partial y_3^0)^{\cdot} = (\operatorname{div}_0 z_y^{00})^{\cdot}$.

Proposition 4.1 *Let the medium be incompressible*

$$\rho_y = \text{const} \tag{4.34}$$

Then: 1. the condition for fluid entirety coincides with the condition for medium geometrical incompressibility (3.110) and is of the form

$$\operatorname{div}_0 v_y^{00} = 0 \tag{4.35}$$

2. the condition for elastic material entirety is of the form

$$\operatorname{div}_0 z_y^{00} = \text{const} \tag{4.36}$$

3. under conditions (3.21) $z_y^{00}(0) = 0 \rightarrow \operatorname{div}_0 z_y^{00}(0) = 0$; for any t

$$\operatorname{div}_0 z_y^{00}(t) = 0 \tag{4.37}$$

Comments

1. Here, as in the formulation of theory fundamentals, condition (3.15) plays a determining role when the condition is not satisfied (for example, a disturbance of material entirety, a material reinforcing, a presence of isolated inclusions, cracks, *etc.*).

2. In the considered here variant of a mathematical formalism of Galilean mechanics, the concepts of entirety and continuity are not equivalent: the first one means an existence of densities of all scalar and vector measures of mechanics w.r.t. Lebesgue measure, the second one means an invariance of the scalar measure of the medium inertia (an inertial mass) w.r.t. the time (*i.e.*, an inertial balance) if the measure density exists. If the medium is geometrically incompressible at a point, then the medium has the entirety property at this point. The opposite is not true. In this book the continuity, the medium inertial balance, and therefore the medium entirety (2.47) are postulated (Axioms D 2 and D 9).
3. Relations (4.36) and (4.37) mathematically and physically are conditions (rich in content) for the elastic material entirety and they have nothing in common with Saint–Venant equalities (Lur'e 1970) which pretend to the same role. Indeed, these equalities as Cauchy–Helmholtz equalities are identities that relate the arithmetic mean $\gamma_{ij} = (z^i_j + z^j_i)/2$ of the symmetric elements of the matrix dz^{00}_y/dy^{00} with the elements themselves under condition that the second mixed derivatives being equal, *i.e.*, $z^i_{jk} = z^i_{kj}$ (it is the necessary condition of z^{00}_y continuity).

4.4. Elements of dynamics of ideal fluid

4.4.1. *Equations of mechanical state and of dynamics of ideal fluid*

Proposition 4.2 *On* 3*–dimensional affine–vector space, the group* $\mathcal{GA}_{yt}(\mathcal{R},3)$ *(see D 2.4) is a product of the group* $\mathcal{T}_{yt}(\mathcal{R},3)$ *of translations and of the k–deformator group* $\mathcal{GD}_{yt}(\mathcal{R},3)$ *– see (2.10).*

Let us begin with the question about the kind of the group $\mathcal{SO}(\mathcal{R},3)$ of rotations of 9–dimensional vector space of the columns (T^0_y) and V^0_y that is induced in this space by the rotation $c^0_f \in \mathcal{SO}(\mathcal{R},3)$ in the fluid motion space $\mathbf{V}_3$ when a passage to a new inertial frame is made.

Proposition 4.3 *Let: 1.* V^0_y *be* 9×1*–column consisting of* 3×1*–columns* $v^0_{iy} = \mathrm{col}\{v^1_i, v^2_i, v^3_i\}$ *of the medium deformation velocities matrix* dv^{00}_y/dy^{00} *(see (4.2)) at a point* y *where* $v^i_j \equiv dv^0_{yi}/dy^0_j$*:*

$$V^0_y = \mathrm{col}\{v^1_1, v^2_1, v^3_1, v^1_2, v^2_2, v^3_2, v^1_3, v^2_3, v^3_3\} \tag{4.38}$$

2. $[\mathbf{e}^f]$ *be a basis of an arbitrary inertial frame* $\mathbf{E}_f$*,* $c^0_f \in \mathcal{SO}(\mathcal{R},3)$ *be the simplest rotation matrix such that* $[\mathbf{e}^f] = [\mathbf{e}^0]c^0_f$ *(Dieudonne 1969 and 1974, Suprunenko 1972) (see (5.6)) (for determination,* $c^0_f = c_3(\theta_6)$*)*

$$c_3(\theta_6) = E + \sin\theta_6 \left\langle e^0_3\right\rangle + (1-\cos\theta_6)\left\langle e^0_3\right\rangle^2 \tag{4.39}$$

3. C^0_{f3} *be* 9×9*–matrix of transfer to a new basis in* $\mathbf{R}_3$ *that is induced by the rotation* $c^0_f \in \mathcal{SO}(\mathcal{R},3)$ *such that*

$$V^0_y = C^0_{f3} V^f_y \tag{4.40}$$

Then: 1. the matrix C^0_{f3} *has the following block structure*

$$C^0_{f3} = \begin{bmatrix} k_1 & -k_2 & 0 \\ k_2 & k_1 & 0 \\ 0 & 0 & c^0_f \end{bmatrix} \tag{4.41}$$

where k_1, k_2 *are three–dimensional similitudes:* $k_1^T k_1 = \cos^2\theta_6 E$, $k_2^T k_2 = \sin^2\theta_6 E$ *(Dieudonne 1969):*

$$k_1 = \cos\theta_6\, c^0_f, \quad k_2 = \sin\theta_6\, c^0_f \tag{4.42}$$

2. the rotation matrix $C^0_{f3} \in \mathcal{SO}^3(\mathcal{R}, 9)$.

Proof follows from standard formulas for matrices transformations when transfer to a new basis is performed $dv^{00}_y/dy^{00} = c_3(\theta_6) dv^{0f}_y/dy^{0f}\, c_3^T(\theta_6)$ and with verifying after that the relations $C^1_{f3} C^{1,T}_{f3} = E$, $\det C^1_{f3} = 1$.

Comment It can be proved by an analogy that if matrices of the simplest rotations for $\mathbf{R}_3$ are of form (5.6)

$$c_1(\theta_4) = E + \sin\theta_4 \left\langle e^0_1 \right\rangle + (1 - \cos\theta_4) \left\langle e^0_1 \right\rangle^2 \tag{4.43}$$

$$c_2(\theta_5) = E + \sin\theta_5 \left\langle e^0_2 \right\rangle + (1 - \cos\theta_5) \left\langle e^0_2 \right\rangle^2 \tag{4.44}$$

Then for $C^0_{f1} \in \mathcal{SO}(\mathcal{R}, 9)$ and for $C^0_{f2} \in \mathcal{SO}(\mathcal{R}, 9)$, respectively, we have

$$C^0_{f1} = \begin{bmatrix} c^0_f & 0 & 0 \\ 0 & d_1 & -d_2 \\ 0 & d_2 & d_1 \end{bmatrix}, \; C^0_{f2} = \begin{bmatrix} p_1 & 0 & p_2 \\ 0 & c^0_f & 0 \\ -p_2 & 0 & p_1 \end{bmatrix} \tag{4.45}$$

Proposition 4.4 *Let: 1.* $V^0_y = \mathrm{col}\{v^{0y}_1, v^{0y}_2, ..., v^{0y}_9\}$ *be* 9×1*–column of functions* v^{0y}_i, $i = 1, 2, ..., 9$, *depending on the point* y *and on the time* $t \in \mathbf{T}_+$*;*

2. the equations of medium mechanical state (4.7) be of the form (see (4.24) and (4.25))

$$F^0 = (T^0_y) - V^0_y = 0 \tag{4.46}$$

3. the rotation matrix $C^0_f \in \mathcal{SO}(\mathcal{R}, 9) = \mathcal{SO}^1(\mathcal{R}, 9)\mathcal{SO}^2(\mathcal{R}, 9)\mathcal{SO}^3(\mathcal{R}, 9)$ *be of the form*

$$C^0_f = C^0_{f1} C^0_{f2} C^0_{f3} \tag{4.47}$$

Then: 1. the column V^0_y *appears to be*

$$(\omega) \equiv p_{yt}\, \mathrm{col}\{1, 0, 0, 0, 1, 0, 0, 0, 1\} \tag{4.48}$$

where $p_{yt} \in \mathbf{R}_1$*;*

2. the medium is called an ideal fluid ($\mathcal{I}$*–class);*

3. the scalar function $p_{yt} \in \mathbf{R}_1$ *of the coordinates and of the time is called Pascal pressure;*

4. the equations of ideal fluid motion are

$$\rho_y v_y^{00\cdot} = \rho_y g_{yt}^0(\urcorner y, y) - \mathrm{grad}_0 p_{yt} \qquad (4.49)$$

Proof consists of verifying the equality (see (2.65) and (4.48))

$$\mathrm{Div}_0 T_y^0 = -\,\mathrm{grad}_0 p_{yt} \qquad (4.50)$$

Comments

1. For 2–dimensional ideal fluid, the group $\mathcal{SO}(\mathcal{R}, 4)$ of rotations of 4–dimensional vector space consisting of columns (T_y^0) and U_y^0 (see (4.2)), the group induced in this space by the rotation $C_f^0 \in \mathcal{SO}(\mathcal{R}, 2)$ of the fluid motion space $\mathbf{V}_2$ when transforming to the basis of a new inertial frame, the group consists of matrices

$$C_f^0 = \begin{bmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{bmatrix} \qquad (4.51)$$

   where k_1, k_2 are the same as in (4.42) for 2–dimensional case.
2. The equations of the mechanical state of 2–dimensional ideal fluid are of same form (4.49) under condition that the column (v_y^0) have four coordinates (see 4.48)

$$(\omega) \equiv (p_{yt}) = p_{yt}\,\mathrm{col}\{1, 0, 0, 1\} \qquad (4.52)$$

   where $p_{yt} \in \mathbf{R}_1$ is Pascal pressure.

4.4.2. *Equations of thermodynamics for ideal fluid*

Proposition 4.5 *The differential equation (2.52) of thermodynamics of a fluid of $\mathcal{I}$–class is*

$$\rho_y(du_y(\theta_y) + p_y dw_y) = (\mathrm{div}_0 q_y^0 + \rho_y\varphi_y)dt \qquad (4.53)$$

where $w_y = \rho_y^{-1}$ is the specific volume of fluid, henceforth the index t is omitted.

Proof According to (2.52) and taking into account (2.49), (4.46) and (4.32), we have

$$\begin{aligned}
\rho_y u_y^{\cdot} &= (T_y^0)\cdot V_y^0 + \mathrm{div}_0 q_y^0 + \rho_y\varphi_y = -(p_y)^{\cdot} V_y^0 + \mathrm{div}_0 q_y^0 + \rho_y\varphi_y \\
&= -p_y\partial v_{y1}^0/\partial y_1^0 - p_y\partial v_{y2}^0/\partial y_2^0 - p_y\partial v_{y3}^0/\partial y_3^0 + \mathrm{div}_0 q_y^0 + \rho_y\varphi_y \\
&= -p_y\,\mathrm{div}_0 v_y^{00} + \mathrm{div}_0 q_y^0 + \rho_y\varphi_y = p_y\rho_y^{-1}\rho_y^{\cdot} + \mathrm{div}_0 q_y^0 + \rho_y\varphi_y \\
&= -\rho_y p_y(\rho_y^{-1})^{\cdot} + \mathrm{div}_0 q_y^0 + \rho_y\varphi_y \to \rho_y(u_y^{\cdot} + p_y(\rho_y^{-1})^{\cdot}) \\
&= \mathrm{div}_0 q_y^0 + \rho_y\varphi_y
\end{aligned}$$

as $p_y^{\cdot}\rho_y^{-1} = -p_y(\rho_y^{-1})^{\cdot} = -p_y\frac{dw}{dt}$.

Comments

1. The internal energy density differential w.r.t. the measure $m(dy)$ of ideal fluid (heat transfer and heat emission are absent, *i.e.*, $q_y^0 = 0$ and $\varphi_y = 0$) equals to the elementary work of the fluid compression multiplying by Pascal pressure with the sign '−'
$$du_y = -p_y dw_y \tag{4.54}$$
2. The internal energy density w.r.t. the measure $m(dy)$ of incompressible ideal fluid ($dw_y = 0$) is constant under the same conditions
$$u_y = \text{const} \tag{4.55}$$

4.5. Elements of $\mathcal{H}$–class dynamics of quasi–linear continuous media

4.5.1. *Equations of mechanical state and dynamics*

Proposition 4.6 *Let: 1.* $\mathcal{H}(\mathcal{R}, 9)$ *be the group of homotheties of* $\mathbf{R}_9$*,* $h(\mu) \in \mathcal{H}(\mathcal{R}, 9)$ *be* 9×9*–matrix of the homothety,* $h(\mu_y(I, \theta_{yt})) = \mu_y(I, \theta_{yt})E$*;*

2. the medium mechanical state equations at a point y be of the form
$$(T_y^0) = -(w) + \Lambda_y^0(I_1, \theta_{yt}) + h(\mu_y(I, \theta_{yt}))U_y^0 \tag{4.56}$$
where $(w) = w\,\mathrm{col}\{1,0,0,0,1,0,0,0,1\}, \Lambda_y(I_1, \theta_{yt}) = \lambda_y\,\mathrm{col}\{1,0,0,0,1,0,0,0,1\}$*,* $w \in \mathbf{R}_1$*,* $\lambda_y = \lambda_y^0(I_1, \theta_{yt})$ *and* $\mu_y = \mu_y(I, \theta_{yt}) \in \mathbf{R}_1$ *are the known positive scalar functions, relations (4.5), (4.6), (4.28) are true as* $C_f^{0,T} h(\mu_y) = h(\mu_y)$*.*

Then: 1. the medium class is called $\mathcal{H}$*–class and the representatives of the class are called quasi–linear* $\mathcal{H}$*–media;*

2. equations (4.56) of quasi–linear $\mathcal{H}$*–medium mechanical state are equivalent to the quasi–linear part of determining relations (4.15):*
$$T_y^0 = (-w + \lambda_y)E + \mu_y du_y^{00}/dy^{00} \tag{4.57}$$

Proposition 4.7 *Let: 1.* $\mathrm{grad}_0 w$ *be gradient of the function w at the point y in* $\mathbf{E}_0$;

2. $\mathrm{grad}_0\lambda_y$ *and* $\mathrm{grad}_0\mu_y$ *be gradients of the coefficients* $\lambda_y = \lambda_y(I_1, \theta_{yt})$ *and* $\mu_y = \mu_y(I, \theta_{yt})$ *in equations (4.56) at the point y in* $\mathbf{E}_0$*;*

3. $\nabla^2 u_y^{00} = \mathrm{Div}\, du_y^{00}/dy^{00} = \mathrm{col}\{\mathrm{div}_0 u_{y1}^{00}, \mathrm{div}_0 u_{y2}^{00}, \mathrm{div}_0 u_{y3}^{00}\}$ *be* 3×1*–column of Laplacians of the coordinates of* u_y^{00} *(or, that is the same,* 3×1*–column of divergences in* $\mathbf{E}_0$ *of the rows* u_{y1}^{00}*,* u_{y2}^{00}*,* u_{y3}^{00} *of the derivative* du_y^{00}/dy^{00} *at the point y in* $\mathbf{E}_0$*).*

Then $\mathcal{H}$*–medium dynamics equation is of the form*
$$\begin{aligned} \rho_y v_y^{00\cdot} \;=\; & \rho_y g_{yt}^0(\urcorner y, y) - \mathrm{grad}_0\, w + \mathrm{grad}_0\, \lambda_y + du_y^{00}/dy^{00} + \\ & \mathrm{grad}_0\, \mu_y + \mu_y \nabla^2 u_y^{00} \end{aligned} \tag{4.58}$$

Proof is obtained by substituting the mechanical state equations (4.56) in the medium motion equations (2.65). Thus, for example, for the first scalar equation in (4.58), we have

$$\begin{aligned}\mathrm{Div}_0(T^0_{11},T^0_{21},T^0_{31}) = \partial T^0_{11}/\partial y^0_1 + \partial T^0_{21}/\partial y^0_2 + \partial T^0_{31}/\partial y^0_3 = \\ (u^1_1,u^1_2,u^1_3)(\partial\mu^0_y/\partial y^0_1,\partial\mu^0_y/\partial y^0_2,\partial\mu^0_y/\partial y^0_3)^T - \\ \partial w/\partial y^0_1 + \partial\lambda^0_y/\partial y^0_1 + \mu^0_y(u^1_{11}+u^1_{22}+u^1_{33})\end{aligned}$$

Proposition 4.8 *Let 1: the rheological coefficient $\lambda_y(I_1,\theta_{yt})$ have the simplest representation ($I_1 = \mathrm{div}_0\, u^{00}_y$)*

$$\lambda_y = \lambda_y(\theta_{yt})I_1 = \lambda_y(\theta_{yt})\,\mathrm{div}_0\, u^{00}_y \tag{4.59}$$

2. $\mathcal{H}$–medium mechanical state equation is of the form

$$(T^0_y) = -(w) + W_H(y,I,\theta_{yt})U^0_y \tag{4.60}$$

$$W_H(y,I,\theta_{yt}) = \begin{bmatrix} \mu+\lambda & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & \lambda \\ 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & \mu+\lambda & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 \\ \lambda & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & \mu+\lambda \end{bmatrix} \tag{4.61}$$

where $\det W_H(y,I,\theta_{yt}) = \mu^8(\mu+3\lambda) \neq 0$ *as $\lambda = \lambda_y$ and $\mu = \mu_y$; are positive – see P 4.6.*

Then: 1. the set $\mathcal{W}_H(\mathcal{R},9)$ of the matrices $W_H(y,I,\theta_{yt})$ is a group whose inverses $W_H(y,I,\theta_{yt})$ is of form (4.61) where the functions α and β stand instead of the coefficients λ and μ and they are called moduli, besides $C^{0,T}_f W_H(y,I,\theta_{yt})C^0_f = W_H(y,I,\theta_{yt})$;

2. if the medium is viscous fluid, the elements $\alpha(y,I,\theta_{yt})$ and $\beta(y,I,\theta_{yt})$ of the matrix $M^{-1}_H(y,I,\theta_{yt})$ are called modules of the viscosity of first and second types, respectively;

3. if the medium is elastic material, the elements $\alpha(y,I,\theta_{yt})$ and $\beta(y,I,\theta_{yt})$ of the matrix $W^{-1}_H(y,I,\theta_{yt})$ are called elasticity modules of first and second types, respectively; functions $\alpha(y,I,\theta_{yt})$ and $\beta(y,I,\theta_{yt})$ are called rheological $\mathcal{H}$–medium modules of first and second types;

4. the modules $\alpha(y,I,\theta_{yt})$ and $\beta(y,I,\theta_{yt})$ are calculated according to the relations

$$\begin{aligned}\alpha(y,I,\theta_{yt}) &= -\lambda/[\mu(\mu+3\lambda)], \beta(y,I,\theta_{yt}) = 1/\mu \\ \alpha+\beta &= (\mu+2\lambda)/[\mu(\mu+3\lambda)]\end{aligned} \tag{4.62}$$

5. *elements* $w_1 = \alpha + \beta,\ w_2 = \alpha,\ w_3 = \beta$ *of the inverse* $W_H(I, \theta_{yt})$ *are called coefficients of the tensions influence* (τ_y^0) *on the column elements* U_y^0

$$U_y^0 = W_H(I, \theta_{yt})(\tau_y^0),\ (\tau_y^0) = (T_y^0) + (w) \tag{4.63}$$

6. *analogs of Young module* E_{3H}*, of shift module* G_{3H} *and of Poisson coefficient* v_{3H} *for the considered* $\mathcal{H}$*–class, their relation with stiffness coefficients and between them are of the form*

$$\begin{aligned} E_{3H} &= E_{3H}(y, I, \theta_{yt}) = 1/w_1 = \mu(\mu + 3\lambda)/(\mu + 2\lambda) \\ G_{3H} &= G_{3H}(y, I, \theta_{yt}) = 1/w_3 = \mu \\ v_{3H} &= v_{3H}(y, I, \theta_{yt}) = -w_2/w_1 = \lambda/(\mu + 2\lambda),\ E_{3H} = G_{3H}(1 + v_{3H}) \\ \lambda &= v_{3H}G_{3H}/(1 - 2v_{3H}),\ \mu = G_{3H} \end{aligned} \tag{4.64}$$

7. $\mathcal{H}$*–medium dynamics vector equation is*

$$\begin{aligned} \rho_y v_y^{00\bullet} &= \rho_y g_{yt}^0(\urcorner y, y) - \mathrm{grad}_0\, w + \lambda_y\, \mathrm{grad}_0\, \mathrm{div}_0 u_{yt}^{00} + \\ &\quad \mathrm{div}_0 u_y^{00}\, \mathrm{grad}_0\, \lambda_y + du_y^{00}/dy^{00}\, \mathrm{grad}_0\, \mu_y + \mu_y \nabla^2 u_y^{00} \end{aligned} \tag{4.65}$$

Example Let $\mu = 2,\ \lambda = 3$. Then $\beta = 0.5,\ \alpha = -0.1364,\ \beta + \alpha = 0.3636$.

Proposition 4.9 *Let there be two different media with the same distribution of tensions*

$$(T_{ya}^0) + (w) = (T_{yb}^0) + (w) \tag{4.66}$$

Then they belong to $\mathcal{H}$*–class iff the columns* U_{ya}^0 *and* U_{yb}^0 *are linearly related by the matrix* $W_{Hab} \in \mathcal{W}_H(\mathcal{R}, 4)$ *such that*

$$U_{ya}^0 = W_{Hab}(y, I, \theta_{yt})U_{yb}^0,\ W_{Hab}(y, I, \theta_{yt}) = W_{H1a}W_{H1b} \tag{4.67}$$

Proof From relation (4.63) under condition that relation (4.66) is fulfilled, we obtain $W_{H1a}U_{ya}^0 = W_{H1b}U_{yb}^0$ whence relation (4.67) follows.

Comments

1. $\mathcal{M}_H$–medium is dynamically non–equivalent to any medium.
2. As the coefficients μ, λ are positive (due to P 4.7) definition), the modules α and β could be negative.
3. If $\mathcal{H}$–medium is 2–dimensional, we use the following relations

$$\begin{aligned} (T_y^0) &= -(w) + W_H(I, \theta_{yt})U_y^0 \\ W_H(I, \theta_{yt}) &= \begin{bmatrix} \mu + \lambda & 0 & 0 & \lambda \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ \lambda & 0 & 0 & \mu + \lambda \end{bmatrix} \end{aligned} \tag{4.68}$$

where $\det W_H(I, \theta_{yt}) = \mu^3(\mu + 2\lambda) \neq 0$.

4. The set $\mathcal{W}_H(\mathcal{R}, 4)$ of the matrices $W_H(I, \theta_{yt})$ whose structure is defined by relation (4.68) represents a group. Their inverses have the same structure with the moduli

$$\alpha = \alpha(I, \theta_{yt}) = -\lambda/[\mu(\mu + 2\lambda)],\ \beta = \beta(I, \theta_{yt}) = 1/\mu \tag{4.69}$$

and the relation $\alpha + \beta = (\mu + \lambda)/(\mu + 2\lambda)$;

5. Two–dimensional analogs of Young modules E_{2H}, of the shift G_{2H} and v_{2H} of Poisson coefficient for the considered $\mathcal{W}_H$–subclass of $\mathcal{H}$–class are of the form

$$\begin{aligned} E_{2H} &= E_{2H}(y, I, \theta_{yt}) = 1/w_1,\ G_{2H} = G_{2H}(y, I, \theta_{yt}) = 1/w_3 \\ E_{2H} &= \mu(\mu + 2\lambda)/(\mu + \lambda),\ G_{2H} = \mu \\ v_{2H} &= v_{2H}(y, I, \theta_{yt}) = \lambda/(\mu + \lambda),\ v_{2H} = -\alpha E_{2H} \end{aligned} \tag{4.70}$$

The relation between the mentioned above values is

$$\begin{aligned} E_{2H} &= G_{2H}(1 + v_{2H}) \\ \lambda &= G_{2H} v_{2H}(1 - v_{2H}),\ \mu = G_{2H} \end{aligned} \tag{4.71}$$

6. If the coefficients μ and λ (viscosity or stiffness) do not depend on $\mathcal{H}$–medium dimensions (they are correct characteristics), then the modules α and β (viscosity or elasticity) depend on H–medium dimension. That is why the last ones are incorrect rheological characteristics of this medium. It is necessary to have in mind this fact when we pass from 2–dimensional medium study to 3–dimensional medium research, and *vice versa*.
7. The analogs of Young module $E_{3H}(y, I, \theta_{yt})$ and of Poisson coefficient as well as their relation with the analogs of Lame coefficients (4.64) do not coincide with the analog characteristics of 2–dimensional $\mathcal{H}$–media (see (4.70)). Hence, if we formulate problems in terms of Young modules and Poisson coefficients analogs for 2–dimensional media, we should not use the means of 3–dimensional medium mechanics truncated w.r.t. one of the coordinates. What has been said concerns 1–dimensional media, too, but the last media are not discussed in this book.
8. The analog of the shift module (coinciding with the coefficient of viscosity (stiffness) of II type) as well as equality (4.71) are correct characteristics of $\mathcal{H}$–medium.

Proposition 4.10 *Let: 1. the medium rheological coefficients of second type do not depend on the point and on the invariants of* du_y^{00}/dy^{00}*;*

2. the medium temperature do not depend on the point.

Then the medium is homogeneous, the mechanical state equations

$$(T_y^0) = -(w) + W_H(\theta_{yt}) U_y^0 \tag{4.72}$$

and dynamics equations

$$\rho_y v_y^{00\cdot} = \rho_y g_{yt}^0(\urcorner y, y) - \mathrm{grad}_0\, w + \lambda(\theta_t)\, \mathrm{grad}_0\, \mathrm{div}_0 u_y^{00} + \mu(\theta_t) \nabla^2 u_y^{00} \tag{4.73}$$

are linear;

Comment No specific 'temperature' tensions and 'temperature' deformations of elastic material exist. There exist deformations, deformations velocities and tensions, related each other, whose relationship (quasi–linear (4.23) in particular) is determined by the temperature of material rheological coefficients. If deformations are absent (i.e., under condition of a fixed form, a fixed volume: $V_y^0 = 0$) the tensions change, according to law (4.60), is

$$(T_y^0) = W_H(\theta_t) Z_y^0 \tag{4.74}$$

Proposition 4.11 *Suppose that: 1. the motion of $\mathcal{H}$–medium is potential (see (3.98)):*

$$du_y^{00}/dy^{00} = (du_y^{00}/dy^{00})^T \tag{4.75}$$

2. the mixed second derivatives of the coordinates of u_y^{00} w.r.t. the coordinates of y^{00} are continuous, and hence are equal

$$u_{12}^1 = u_{21}^1, ..., u_{23}^3 = u_{32}^3 \tag{4.76}$$

Then with account of (4.76) dynamics equations (4.73) of $\mathcal{M}_H$–medium can be represented in the following equivalent forms

$$\begin{aligned} \rho_y v_y^{00\cdot} &= \rho_y g_{yt}^0(\urcorner y, y) - \operatorname{grad}_0 w + \operatorname{div}_0 u_y^{00} \operatorname{grad}_0 \lambda_y(\theta_{yt}) + \\ &\quad (\lambda_y(\theta_{yt}) + \mu_y(I, \theta_t)) \nabla^2 u_y^{00} \\ \rho_y v_y^{00\cdot} &= \rho_y g_{yt}^0(\urcorner y, y) - \operatorname{grad}_0 w + \operatorname{div}_0 u_y^{00} \operatorname{grad}_0 \lambda_y(\theta_{yt}) + \\ &\quad (\lambda_y(\theta_{yt}) + \mu_y(I, \theta_t)) \operatorname{grad}_0 \operatorname{div}_0 u_y^{00} \end{aligned} \tag{4.77}$$

Proof For the first row of d–deformator we obtain $T_{y1}^0 = [(\lambda - w) + \mu u_1^1$, $(\mu/2)(u_2^1 + u_1^2)$, $(\mu/2)(u_3^1 + u_1^3)]$ where $\lambda = \lambda_y(\theta_{yt})$ and $\mu = \mu_y(I, \theta_{yt})$. Let us calculate the divergence of this row having in mind (4.76): $\operatorname{div}_0 T_{y1}^0 = (\lambda - w)_1 + \mu u_{11}^1 + (\mu/2)u_{22}^1 + (\mu/2)u_{12}^2 + (\mu/2)u_{33}^1 + (\mu/2)u_{13}^3 = (\lambda - w)_1 + (\mu/2)(u_{11}^1 + u_{22}^1 + u_{33}^1) + (\mu/2)(u_{11}^1 + u_{12}^2 + u_{13}^3) = (\lambda - w)_1 + (\mu/2)\nabla^2 u_{y1}^0 + (\mu/2)(u_{11}^1 + u_{21}^2 + u_{31}^3) = (\lambda - w)_1 + (\mu/2)\nabla^2 u_{y1}^0 + (\mu/2)(\operatorname{div}_0 u_y^{00})_1 = (-w)_1 + \lambda(\operatorname{div}_0 u_y^{00})_1 + (\mu/2)\nabla^2 u_{y1}^0 + (\mu/2)(\operatorname{div}_0 u_y^{00})_1 = (-w)_1 + (\lambda + \mu/2)(\operatorname{div}_0 u_y^{00})_1 + (\mu/2)\nabla^2 u_{y1}^0$. Proceeding in the same way with the rest rows of d–deformator and using relation (2.65) we reach the result.

Proposition 4.12 *Suppose that under condition (4.77) $\mathcal{M}_H$–medium is incompressible (see (4.35), (4.37))*

$$\operatorname{div}_0 u_y^{00} = 0 \tag{4.78}$$

Then the dynamics equations (4.77) of the potential motion (a flow current, deformation) of $\mathcal{M}_H$–medium are linear and they are the same as the equations (4.49) of an ideal fluid motion ($\mathcal{H}$ –class of the incompressible media under conditions of non–circular (potential) motions is dynamically equivalent to fluid of $\mathcal{I}$–class):

$$\rho_y v_y^{00\cdot} = \rho_y g_{yt}^0(\urcorner y, y) - \operatorname{grad}_0 w \tag{4.79}$$

Proposition 4.13 *Suppose that for matrix (4.57)*

$$\frac{1}{3}\,\mathrm{trace}\,T^0_y = -w \tag{4.80}$$

Then: 1. in infringement of the requirement of definition D 4.7, the coefficients of viscosity (stiffness) of I and II types become dependent one from another and they have opposite signs, i.e.,

$$\lambda = -\frac{1}{3}\mu \tag{4.81}$$

2. the mechanical state equation (4.60)

$$(T^0_y) = -(w) + M_H(y, I, \theta_{yt})U^0_y \tag{4.82}$$

becomes incorrect: $\det M_H = \mu^2(\mu + 3\lambda) = \mu^2(\mu - \mu) = 0$, *hence* $M_H \notin \mathcal{M}_H(\mathcal{R}, 9)$*;*

3. the medium has not the module $\alpha(y, I, \theta_{yt})$ *of viscosity (stiffness) (see (4.62));*

4. the medium dynamics equations (4.65) become incorrect and have the form

$$\begin{aligned}\rho_y v^{00\cdot}_y \;\; = \;\; & \rho_y g^0_{yt}(\urcorner y, y) - \mathrm{grad}_0\, w - \frac{1}{3}\mu_y(I, \theta_{yt})\,\mathrm{grad}_0\,\mathrm{div}_0\, u^{00}_y + \\ & (du^{00}_y/dy^{00} - \frac{1}{3}\,\mathrm{div}_0\, u^{00}_y E)\,\mathrm{grad}_0\, d\mu_y(I, \theta_{yt}) + \mu_y(I, \theta_{yt})\nabla^2 u^{00}_y\end{aligned} \tag{4.83}$$

Proof Determining relations (4.57) under condition (4.59) are of the form

$$T^0_y = (-w + \lambda\,\mathrm{div}_0\, u^{00}_y)E + \mu du^{00}_y/dy^{00} \tag{4.84}$$

For the trace of matrix (4.84) we obtain

$$T^0_{11} + T^0_{22} + T^0_{33} = -3w^{\cdot} + (3\lambda + \mu)\,\mathrm{div}_0\, u^{00}_y$$

Dividing by 3 and taking into account relation (4.80) we reach relations (4.81) and (4.83).

Comments

1. Relation (4.83) makes clear the level of approximation (4.80): the equality is valid only for non–viscous and non–elastic media or in the case $\mathrm{div}_0\, u^{00}_y = 0$.
2. When the medium is 2–dimensional, the corresponding relations are

$$\frac{1}{2}(T^0_{11} + T^0_{22}) = -w, \;\; \lambda_y = -\frac{1}{2}\mu_y \tag{4.85}$$

4.5.2. *Thermodynamics equation*

Definition 4.9 *Let: 1. the medium be a viscous fluid;*

2. $\Lambda_y(I_1, \theta_{yt})$ be the column of the viscosity coefficients of I type, $\mu_y(I, \theta_{yt})$ be the viscosity coefficient of II type (being coefficients in $\mathcal{H}$–fluid mechanical state equations (4.56);

3. V_y^0 be 9×1–column consisting of 3×1–columns of dv_y^{00}/dy^{00} (see (4.2)).

Then: 1. the inner product (in analog to (2.49))

$$\Phi_\lambda^y(I, \theta_{yt}) = V_y^0 \cdot \Lambda_y(I_1, \theta_{yt}) = \lambda_y(I_1, \theta_{yt}) \operatorname{div}_0 v_y^{00} \tag{4.86}$$

is called a dissipative function of I type of a viscous fluid at a point y;

2. the inner square of the vector V_y^0 with coefficients $\mu_y(I, \theta_{yt})$

$$\Phi_\lambda^y(I, \theta_{yt}) = \mu_y(I, \theta_{yt}) V_y^{02} \tag{4.87}$$

is called a dissipative function of II type of a viscous fluid at a point y.

Proposition 4.14 *Let: 1. $\Phi_\lambda^y(I_1, \theta_{yt})$ and $\Phi_\mu^y(I, \theta_{yt})$ be dissipative functions of I and II types, respectively (4.86), (4.87);*

2. $\rho_y w_y^{\boldsymbol{\cdot}}$ be the velocity of an elementary work realized by Pascal pressure ($w_y = \rho_y^{-1}$ is a fluid specific volume) at a point y;

3. $u_y^{\boldsymbol{\cdot}}(\theta_{yt})$ be the density of the velocity of fluid inner energy change at a point y w.r.t. the measure $m(dy)$.

Then the equation of thermodynamics of a viscous $\mathcal{H}$–fluid is of the form

$$\rho_y(u_y^{\boldsymbol{\cdot}}(\theta_{yt}) + \rho_y w_y^{\boldsymbol{\cdot}}) = \Phi_\lambda^y(I_1, \theta_{yt}) + \Phi_\mu^y(I, \theta_{yt}) + \operatorname{div}_0 v_y^{00} + \operatorname{div}_0 q_y^0 + \rho_y \varphi_y \tag{4.88}$$

Proof is reduced to substituting the fluid mechanical state equations (4.56) in the energy balance of Galilean mechanics Universe with account of definition (2.51) and the proof of equality (2.52).

Comments

1. For an incompressible viscous $\mathcal{H}$–fluid ($\operatorname{div}_0 v_y^{00} = 0$), the dissipation of the inner energy of I type (see (4.86)) is absent and the thermodynamics equation is

$$\rho_y u_y^{\boldsymbol{\cdot}}(\theta_{yt}) = \Phi_\mu^y(I, \theta_{yt}) + \operatorname{div}_0 q_y^0 + \rho_y \varphi_y$$

2. If assumption (4.59) is true, the dissipative function of I type is

$$\Phi_\lambda^y(I_1, \theta_{yt}) = \lambda_y(\theta_{yt}) \operatorname{div}_0^2 v_y^{00} \tag{4.89}$$

3. $\mathcal{H}$–class of 3–dimensional viscous fluids is thermodynamically non–equivalent to anyone of the previously discussed classes of fluids. Hence, for obtaining theoretical results which are adequate to the experiments, it is necessary to add the thermodynamics equation (4.88) to the system of resolving equations.

Definition 4.10 *Let: 1. the medium be an elastic $\mathcal{H}$–material ($u \equiv z$);*

2. $\Lambda_y(I_1, \theta_{yt})$ be the column of the stiffness coefficients of I type, $\mu_y(I, \theta_{yt})$ be the stiffness coefficient of II type;

3. Z_y^{00} be 9×1–column consisting of 3×1–columns of dz_y^{00}/dy^{00} (see (3.17)) (i.e., the medium deformation under condition (3.15)).

Then: 1. the inner product (in analog to (4.86))

$$\Phi_\lambda^y(I_1, \theta_{yt}) = \Lambda_y(I_1, \theta_{yt}) \cdot Z_y^{00\cdot} = \lambda_y(I_1, \theta_{yt}) \operatorname{div}_0 z_y^{00\cdot} \tag{4.90}$$

is called a density of the tensions power of I type on deformations $d(dz_y^{00}/dy^{00}) = d\Delta_y^{00}$ (see (3.22)) of an elastic $\mathcal{H}$–material at a point y w.r.t. $m(dy)$;

2. the inner product of the column $M_y(I, \theta_{yt})Z_y^{00}$ and of the deformations velocities column (i.e., the elements of the matrix $(dz_y^{00}/dy^{00})^\cdot = \Delta_y^{00\cdot}$ (see (3.16)))

$$\Phi_\mu^y(I, \theta_{yt}) = M_y(I, \theta_{yt}) Z_y^{00} \cdot Z_y^{00\cdot} \tag{4.91}$$

is called a density of the tensions power of II type on deformations of an elastic $\mathcal{H}$–material at a point y w.r.t. the measure $m(dy)$ (see (3.22)).

Proposition 4.15 *Let $u_y^\cdot(\theta_{yt})$ be the density of the velocity of $\mathcal{H}$–material inner energy change at a point y w.r.t. the measure $m(dy)$. Then the elastic $\mathcal{H}$–material thermodynamics equation is of the form*

$$\rho_y u_y^\cdot(\theta_{yt}) = \Phi_\lambda^y(I_1, \theta_{yt}) + \Phi_\mu^y(I, \theta_{yt}) + \operatorname{div}_0 z_y^{00\cdot} + \operatorname{div}_0 q_y^0 + \rho_y \varphi_y \tag{4.92}$$

Proof is reduced to substituting the material mechanical state equation (4.56) in the condition of thermodynamical balance of Galilean mechanics Universe with account of definitions (4.90), (4.91) and the proof of equality (2.52).

Comments

1. For an elastic material, deformation tensions do not depend on the velocities. That is why their powers (*i.e.*, the work on deformations per unit of time) do not lead to energy dissipation. The work of such tensions and the corresponding change of material inner energy are invertible.
2. For an incompressible $\mathcal{H}$–material $\operatorname{div}_0 z_y^{00\cdot} = 0$ (see (4.33)), the I type internal energy change (see (4.90)) is absent and the thermodynamics equations are

$$\rho_y u_y^\cdot(\theta_{yt}) = \Phi_\mu^y(I, \theta_{yt}) + \operatorname{div}_0 q_y^0 + \rho_y \varphi_y \tag{4.93}$$

3. $\mathcal{H}$–class of elastic materials is thermodynamically non–equivalent to anyone of the previously discussed classes of materials. Hence, for obtaining theoretical results which are adequate to the experiment, it is necessary to add the thermodynamics equation (4.92) to the system of resolving equations.

4.6. Elements of dynamics of $\mathcal{P}$–classes of quasi–linear continuous media

4.6.1. *Group of rheological coefficient matrices – $\mathcal{P}$–media of II type*

Proposition 4.16 *Let: 1. $\mathcal{P}(\mathcal{R},9)$ be the set of per–symmetrical non–singular 9×9–matrices of the kind*

$$P = \begin{bmatrix} p_1 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & p_2 \\ 0 & p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_3 & 0 & 0 & 0 & 0 & 0 \\ p_2 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & p_2 \\ 0 & 0 & 0 & 0 & 0 & p_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 & 0 \\ p_2 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & p_1 \end{bmatrix} \tag{4.94}$$

$$\det P = p_3^6\,(p_1+p_2)(p_1-p_2)^2) \neq 0,\;\; p_3 \neq 0 \tag{4.95}$$

where the upper index is exponent;

2. there be the relations

$$p_1 - p_2 - p_3 = 0 \tag{4.96}$$

Then: 1. $\mathcal{P}(\mathcal{R},9)$ is a group, the inverse P^{-1} has an analog structure with corresponding elements of the form

$$w_1 = (p_1^2-p_2^2)(p_1^3-3p_1p_2^2+2p_2^3)^{-1} \tag{4.97}$$

$$w_2 = p_2(p_2-p_1)(p_1^3-3p_1p_2^2+2p_2^3)^{-1} \tag{4.98}$$

$$w_3 = p_3^{-1} \tag{4.99}$$

2. the group $\mathcal{P}(\mathcal{R},9)$ is centered by the group $\mathcal{SO}(\mathcal{R},9)$ of rotations in $\mathbf{R}_9$, i.e., for an arbitrary matrices $C_f^0 \in \mathcal{SO}(\mathcal{R},9)$ and $P \in \mathcal{P}(\mathcal{R},9)$ the following is fulfilled:

$$C_f^{0,T}PC_f^0 = P \tag{4.100}$$

3. the group $\mathcal{P}(\mathcal{R},9)$ is a union of three subgroups:

$$\mathcal{P}(\mathcal{R},9) = \mathcal{P}_{12}\cup\mathcal{P}_{13}\cup\mathcal{P}_{23} \tag{4.101}$$

$$\mathcal{P}_{12} = \{P_y^o : p_3 = p_1 - p_2\} \tag{4.102}$$

$$\mathcal{P}_{13} = \{P_y^o : p_2 = p_1 - p_3\} \tag{4.103}$$

$$\mathcal{P}_{23} = \{P_y^o : p_1 = p_2 + p_3\} \tag{4.104}$$

Comment From the above proposition, it follows that the matrices of anyone of the mentioned subgroups can be used as II type matrices of rheological coefficients for quasi–linear continuous media.

4.6.2. *$\mathcal{P}_{12}$–class of quasi–linear continuous media*

Proposition 4.17 *Let: 1. $\mathcal{P}_{12}$ be the group of form (4.102);*

2. the next hypothesis be true: the medium mechanical state equations is of the form

$$(T^0_y) = -(w) + \Lambda_y(I_1, \theta_{yt}) + P^{12}_y(I, \theta_{yt})U^0_y \tag{4.105}$$

where $\Lambda_y(I_1, \theta_{yt})$ is the same as in relation (4.56), $P^{12}_y(I, \theta_{yt}) \in \mathcal{P}_{12}$.

Then: 1. the class of quasi–linear media is called $\mathcal{P}_{12}$–class and its elements are called $\mathcal{P}_{12}$–media;

2. determining relations (4.15) for $\mathcal{P}_{12}$–media do not exist;

3. the matrix representation for d–deformator does not exist;

4. $\mathcal{P}_{12}$–media have three independent rheological coefficients;

5. $\mathcal{P}_{12}$–media dynamics equations are

$$\begin{aligned}\rho_y v^{0\cdot}_{y1} &= -(w)_1 + \rho_y g^0_{t1}(\urcorner y, y) + \partial\lambda_y(I_1, \theta_{yt})/\partial y^0_1 + \\ &[\partial p_1/\partial y^0_1, 0, 0, (\partial p_3/\partial y^0_2, 0, \partial p_3/\partial y^0_3, 0, \partial p_2/\partial y^0_1]U^0_y + \\ &p_2(y, I, \theta_{yt})(u^2_{21} + u^3_{31} - u^1_{22} - u^1_{33}) + p_1(y, I, \theta_{yt})\nabla^2 u^0_{y1}\end{aligned} \tag{4.106}$$

$$\begin{aligned}\rho_y v^{0\cdot}_{y2} &= -(w)_2 + \rho_y g^0_{t2}(\urcorner y, y) + \partial\lambda_y(I_1, \theta_{yt})/\partial y^0_2 + \\ &[\partial p_2/\partial y^0_2, \partial p_3/\partial y^0_1, 0, 0, \partial p_1/\partial y^0_2, 0, 0, \partial p_3/\partial y^0_3, \partial p_2/\partial y^0_2]U^0_y \\ &+p_2(y, I, \theta_{yt})(u^1_{12} + u^3_{32} - u^2_{11} - u^2_{23}) + p_1(y, I, \theta_{yt})\nabla^2 u^0_{y2}\end{aligned} \tag{4.107}$$

$$\begin{aligned}\rho_y v^{0\cdot}_{y3} &= -(w)_3 + \rho_y g^0_{t3}(\urcorner y, y) + \partial\lambda_y(I_1, \theta_{yt})/\partial y^0_3 + \\ &[\partial p_2/\partial y^0_3, 0, \partial p_3/\partial y^0_1, 0, (\partial p_2/\partial y^0_3, \partial p_3/\partial y^0_2, 0, 0, \partial p_1/\partial y^0_3]U^0_y \\ &+p_2(y, I, \theta_{yt})(u^1_{13} + u^2_{23} - u^3_{11} - u^3_{22}) + p_1(y, I, \theta_{yt})\nabla^2 u^0_{y3}\end{aligned} \tag{4.108}$$

$$p_3(y, \theta_{yt}) = p_1(y, \theta_{yt}) - p_2(y, \theta_{yt}) \tag{4.109}$$

Proposition 4.18 *Suppose that: 1. the mixed second derivatives of the coordinates of u^{00}_y of $\mathcal{P}_{12}$–medium w.r.t. the vector y^{00} coordinates are continuous at a point y, and consequently are equal (see(4.76));*

2. c^y_1, c^y_2, c^y_3 are coefficients of circulation of u^{00}_y of $\mathcal{P}_{12}$–medium at a point y (see (3.91)).

Then $\mathcal{P}_{12}$–medium dynamics equations are of the form

$$\begin{aligned}\rho_y v^{0\cdot}_{y1} &= -(w)_1 + \rho_y g^0_{t1}(\urcorner y, y) + \partial\lambda_y(I_1, \theta_{yt})/\partial y^0_1 + \\ &[\partial p_1/\partial y^0_1, 0, 0, \partial p_3/\partial y^0_2, \partial p_2/\partial y^0_2, 0, \partial p_3/\partial y^0_3, 0, \partial p_2/\partial y^0_1]U^0_y \\ &+p_2(y, I, \theta_{yt})[\partial c^y_3/\partial y^0_2 - \partial c^y_2/\partial y^0_3] + p_1(y, I, \theta_{yt})\nabla^2 u^0_{y1}\end{aligned} \tag{4.110}$$

$$\begin{aligned}\rho_y v^{0\cdot}_{y2} &= -(w)_2 + \rho_y g^0_{t2}(\urcorner y, y) + \partial\lambda_y(I_1, \theta_{yt})/\partial y^0_2 + \\ &[\partial p_2/\partial y^0_2, \partial p_3/\partial y^0_1, 0, 0, \partial p_1/\partial y^0_2, 0, 0, \partial p_3/\partial y^0_3, \partial p_2/\partial y^0_2]U^0_y \\ &+p_2(y, I, \theta_{yt})[\partial c^y_1/\partial y^0_3 - \partial c^y_3/\partial y^0_1] + p_1(y, I, \theta_{yt})\nabla^2 u^0_{y2}\end{aligned} \tag{4.111}$$

$$
\begin{aligned}
\rho_y v^{0\bullet}_{y3} &= -(w)_3 + \rho_y g^0_{t3}(\urcorner y, y) + \partial\lambda_y(I_1, \theta_{yt})/\partial y^0_3 + \\
&[\partial p_2/\partial y^0_3, 0, \partial p_3/\partial y^0_1, 0, (\partial p_2/\partial y^0_3, \partial p_3/\partial y^0_2, 0, 0, \partial p_3/\partial y^0_3]U^0_y \\
&+ p_2(y, I, \theta_{yt})[\partial c^y_2/\partial y^0_1 - \partial c^y_3/\partial y^0_1] + p_1(y, I, \theta_{yt})\nabla^2 u^0_{y3}
\end{aligned} \tag{4.112}
$$

Proof For example, for equation (4.106) we obtain

$$
\begin{aligned}
u^2_{21} + u^3_{31} - u^1_{22} - u^1_{33} &= u^2_{21} - u^1_{22} + u^3_{31} - u^1_{33} = \\
u^2_{12} - u^1_{22} + u^3_{13} - u^1_{33} &= \partial(u^2_1 - u^1_2)/\partial y^0_2 + \partial(u^3_1 - u^1_3)/\partial y^0_3 \\
&= \partial c^y_3/\partial y^0_2 - \partial c^y_2/\partial y^0_3
\end{aligned} \tag{4.113}
$$

Comment Condition (4.76) is not satisfied if the material is reinforced, if isolated heterogeneous inclusions, a shear of entirety, cracks, *etc.* are present.

Proposition 4.19 *Suppose that under condition (4.76) the rheological coefficient $\Lambda_y(I_1, \theta_{yt})$ has the simplest representation*

$$
\Lambda_y(I_1, \theta_{yt}) = \lambda_y(\theta_{yt})I_1 = \lambda_y(\theta_{yt})\mathrm{div}_0 v^{00}_y \tag{4.114}
$$

Then: 1. $\mathcal{P}_{12}$–medium mechanical state equations remain quasi–linear (since the rheological coefficients of stiffness of II type depend on I) and they are of the form

$$
(T^0_y) = -(w) + W_{12}(y, I, \theta_{yt})U^0_y \tag{4.115}
$$

where the structure of $W_{12}(y, I, \theta_{yt}) \in \mathcal{W}_{12}(\mathcal{R}, 9)$ is analogical to (4.94) with elements

$$
v_1 = p_1 + \lambda,\ v_2 = p_2 + \lambda,\ v_3 = p_1 - p_2
$$

2. the inverse $W^{-1}_{12}(y, I, \theta_{yt}) \in \mathcal{M}_{12}(\mathcal{R}, 9)$ has the same structure with elements (coefficients of influence)

$$
\begin{aligned}
w_1(y, I, \theta_{yt}) &= \beta_1 + \alpha,\ w_2(y, I, \theta_{yt}) = \beta_2 + \alpha \\
w_3(y, I, \theta_{yt}) &= \beta_1 - \beta_2
\end{aligned} \tag{4.116}
$$

3. the rheological modules α, β_1 and β_2 are of the form

$$
\begin{aligned}
\alpha(y, I, \theta_{yt}) &= -2\lambda(p_2 - p_1)/h,\ \beta_1(y, I, \theta_{yt}) = (p^2_1 - p^2_2)/h \\
\beta_2(y, I, \theta_{yt}) &= (p_2 - p_1)(p_2 + 3\lambda)/h \\
h &= 3\lambda(p_1 - p_2)^2 + (p^3_1 - 3p_1p^2_2 + 2p^3_2)
\end{aligned} \tag{4.117}
$$

4. $\mathcal{W}_{12}$–medium dynamics equations are of the form

$$
\begin{aligned}
\rho_y v^{0\cdot}_{y1} &= \rho_y g^0_{t1}(\neg y, y) + \lambda_y(\theta_y)\partial \mathrm{div}_0 u^{00}_y/\partial y^0_1 + \partial\lambda_y(\theta_{yt})/\partial y^0_1 \mathrm{div}_0 u^{00}_y + \\
& [\partial p_1/\partial y^0_1, 0, 0, \partial p_3/\partial y^0_2, \partial p_2/\partial y^0_2, 0, \partial p_3/\partial y^0_3, 0, \partial p_2/\partial y^0_1] U^0_y + \\
& p_2(y, I, \theta_{yt})[\partial c^y_3/\partial y^0_2 - \partial c^y_2/\partial y^0_3] + p_1(y, I, \theta_{yt})\nabla^2 u_{y1} \qquad (4.118)
\end{aligned}
$$

$$
\begin{aligned}
\rho_y v^{0\cdot}_{y2} &= \rho_y g^0_{t2}(\neg y, y) + \lambda_y(\theta_{yt})\partial \mathrm{div}_0 u^{00}_y/\partial y^0_2 + \partial\lambda_y(\theta_{yt})/\partial y^0_2 \mathrm{div}_0 u^{00}_y \\
& + p_1(y, I, \theta_{yt})\nabla^2 u_{y2} + p_2(y, I, \theta_{yt})[\partial c^y_1/\partial y^0_3 - \partial c^y_3/\partial y^0_1] \qquad (4.119) \\
& [\partial p_2/\partial y^0_2, \partial p_3/\partial y^0_1, 0, 0, \partial p_1/\partial y^0_2, 0, 0, \partial p_3/\partial y^0_3, \partial p_2/\partial y^0_2] U^0_y
\end{aligned}
$$

$$
\begin{aligned}
\rho_y v^{0\cdot}_{y3} &= \lambda_y(\theta_{yt})/\partial \mathrm{div}_0 u^{00}_y/\partial y^0_3 + \partial\lambda_y(\theta_{yt})/\partial y^0_3 \mathrm{div}_0 u^{00}_y + \qquad (4.120) \\
& [\partial p_2/\partial y^0_3, 0, \partial p_3/\partial y^0_1, 0, \partial p_2/\partial y^0_3, \partial p_3/\partial y^0_2, 0, 0, \partial p_1/\partial y^0_3] U^0_y + \\
& p_1(y, I, \theta_{yt})\nabla^2 u^0_{y3} + p_2(y, I, \theta_{yt})[\partial c^y_2/\partial y^0_1 - \partial c^y_1/\partial y^0_2] + \\
& \rho_y g^0_{t3}(\neg y, y)
\end{aligned}
$$

Example Let $p_1 = 1,\ p_2 = 2,\ \lambda = 4$, then $\beta_1 = -0.1765,\ \beta_2 = 0.8235,\ \beta_3 = \beta_1 - \beta_2 = -1,\ \alpha = -0.4706$. The elements of the matrix W_{12}^{-1} are $w_1 = \beta_1 + \alpha = -0.6471,\ w_2 = \beta_2 + \alpha = 0.3529,\ w_3 = \beta_3 = -1$.

Comments

1. For 2–dimensional medium the corresponding characteristics are

$$
W_{12}(y, I, \theta_{yt}) = \begin{bmatrix} p_1+\lambda & 0 & 0 & p_2+\lambda \\ 0 & p_1-p_2 & 0 & 0 \\ 0 & 0 & p_1-p_2 & 0 \\ p_2+\lambda & 0 & 0 & p_1+\lambda \end{bmatrix} \qquad (4.121)
$$

$$
\begin{aligned}
\alpha &= \alpha(y, I, \theta_{yt}) = 1/2(p_1 - p_2)/[(p_1+\lambda)^2 - (p_2+\lambda)^2] \qquad (4.122) \\
\beta_1 &= \beta_1(y, I, \theta_{yt}) = 1/2(p_1 - p_2)^{-1} \\
\beta_2 &= \beta_2(y, I, \theta_{yt}) = -1/2(p_1 - p_2)^{-1} \\
\beta_1 - \beta_2 &= 1/(p_1 - p_2),\ p_1 - p_2 \neq 0
\end{aligned}
$$

$$
\begin{aligned}
w_1(y, I, \theta_{yt}) &= \beta_1 + \alpha, \quad w_2(y, I, \theta_{yt}) = \beta_2 + \alpha \qquad (4.123) \\
w_3(y, I, \theta_{yt}) &= \beta_1 - \beta_2
\end{aligned}
$$

Example Let $p_1 = 3$, $p_2 = 2$, $\lambda = 3$. Then $\alpha = 0.045$, $\beta_1 = 0.5$, $\beta_2 = -0.5$.

The analogs of Young modules, of the shift ones and of Poisson coefficients for 2–dimensional and 3–dimensional media coincide (concerning writing form) and are of the form

$$
\begin{aligned}
E_{12} &= E_{12}(y, I, \theta_{yt}) = 1/w_1 = 1/(\beta_1 + \alpha) \qquad (4.124) \\
G_{12} &= G_{12}(y, I, \theta_{yt}) = 1/w_3 = 1/(\beta_1 - \beta_2) = (p_1 - p_2) \qquad (4.125) \\
v_{12} &= v_{12}(y, I, \theta_{yt}) = -w_2/w_1 = -(\beta_2 + \alpha)/(\beta_1 + \alpha) \qquad (4.126)
\end{aligned}
$$

2. 2–dimensional $\mathcal{W}_{12}$–medium dynamics equations in the terms of rheological coefficients are cut–off w.r.t. third coordinate of 3–dimensional $\mathcal{W}_{12}$–medium dynamics equations with the same coefficients. It indicates that these coefficients do not depend on the medium dimension, and thus they are correct characteristics of the medium.
3. The rheological modules and coefficients (of influence) of two– and three–dimensional $\mathcal{W}_{12}$–medium are different in principle, and consequently the representation of the elements of du_y^{00}/dy^{00} (the column U_y^0) in the form of quasi–linear combinations of the tensions T_y^0:

$$U_y^0 = W_{12}^{-1}(y, I, \theta_{yt})[(T_y^0) + (w)] \tag{4.127}$$

in the cases of 2– and 3–dimensional $\mathcal{M}_{12}$–media coincide (concerning the form) but are principally different (concerning the meaning). It means that the rheological modules and the coefficients of influence are incorrect characteristics of $\mathcal{W}_{12}$–media in that sense.
4. The medium rheological modules (4.117) contain the difference between the rheological coefficients and their squares. Thus, it leads to that $\mathcal{M}_{12}$–medium has different behavior for p_1^2 or p_2^2, *i.e.*, the points $p_1 = \pm p_2$ (4.96) of the manifold of rheological coefficients are bifurcation points of $\mathcal{W}_{12}$–medium.
5. If the rheological coefficients of II type do not depend on I, then mechanical state equations (4.115), tense state equations

$$\begin{aligned} (T_y^0) &= -(w) + W_{12}(y, \theta_{yt})U_y && (4.128)\\ U_y^0 &= W_{12}^{-1}(y, \theta_{yt})(\tau_y^0) && (4.129)\end{aligned}$$

as the equations of dynamics and statics of $\mathcal{W}_{12}$–medium are linear, too.

Proposition 4.20 *Suppose that: 1. the mixed second derivatives of the coordinates of u_y^{00} w.r.t. the coordinates of y^{00} are continuous at the point y, and consequently are equal (see (4.76));*

2. the motion (the deformation) of $\mathcal{W}_{12}$–medium is non–circular (potential) (see (3.94), (3.98))

$$du_y^{00}/dy^{00} = (du_y^{00}/dy^{00})^T, \; c_1^y = c_2^y = c_3^y = 0 \tag{4.130}$$

3. $L(I, \theta_{yt})$ is the following matrix

$$\begin{bmatrix} \frac{\partial p_1}{\partial y_1^0} & 0 & 0 & \frac{\partial p_3}{\partial y_2^0} & \frac{\partial p_2}{\partial y_2^0} & 0 & \frac{\partial p_3}{\partial y_3^0} & 0 & \frac{\partial p_2}{\partial y_1^0} \\ \frac{\partial p_2}{\partial y_2^0} & \frac{\partial p_3}{\partial y_1^0} & 0 & 0 & \frac{\partial p_1}{\partial y_2^0} & 0 & 0 & \frac{\partial p_3}{\partial y_3^0} & \frac{\partial p_2}{\partial y_2^0} \\ \frac{\partial p_2}{\partial y_3^0} & 0 & \frac{\partial p_3}{\partial y_1^0} & 0 & \frac{\partial p_2}{\partial y_3^0} & \frac{\partial p_3}{\partial y_2^0} & 0 & 0 & \frac{\partial p_1}{\partial y_3^0} \end{bmatrix} \tag{4.131}$$

Then dynamics equations (4.118), (4.119), (4.120) of $\mathcal{M}_{12}$–medium at the point y are

$$\begin{aligned} \rho_y v_y^{00} &= -\mathrm{grad}_0 w + \rho_y g_{yt}^0(\urcorner y, y) + \mathrm{div}_0 u_y^{00} \mathrm{grad}_0 \lambda_y(\theta_{yt}) + \\ &\quad L_y(I, \theta_{yt})U_y^0 + [\lambda_y(\theta_{yt}) + p_1(y, I, \theta_{yt})]\mathrm{grad}_0 \mathrm{div}_0 u_y^{00} && (4.132)\\ \rho_y v_y^{00\bullet} &= -\mathrm{grad}_0 w + \rho_y g_{yt}^0(\urcorner y, y) + [\lambda_y(\theta_{yt}) + p_1(y, I, \theta_{yt})]\nabla^2 u_y^{00} + \\ &\quad L_y(I, \theta_{yt})U_y^0 + \mathrm{div}_0 u_y^{00} \mathrm{grad}_0 \lambda_y(\theta_{yt}) && (4.133)\end{aligned}$$

Proof is based on using the equality $\nabla^2 u_y^{00} = \mathrm{grad}_0 \mathrm{div}_0 u_y^{00}$ in the points 1 and 2 of the proposition.

Proposition 4.21 *Suppose that in the previous proposition conditions the rheological coefficients and the temperature of $\mathcal{W}_{12}$–medium do not depend on the point y and on I. Then the linear equations of a homogeneous potentially deformable $\mathcal{W}_{12}$–medium are of the form (with account of* $\mathrm{grad}_0\, \mathrm{div}_0\, u_y^{00} = \nabla^2 u_y^{00}$*)*

$$\begin{aligned} \rho_y v_y^{00\cdot} &= -\mathrm{grad}_0 w + \rho_y g_{yt}^0(\urcorner y, y) + [\lambda(\theta_t) + p_1(\theta_t)] \mathrm{grad}_0\, \mathrm{div}_0\, u_y^{00} \\ \rho_y v_y^{00\cdot} &= -\mathrm{grad}_0 w + \rho_y g_{yt}^0(\urcorner y, y) + [\lambda(\theta_t) + p_1(\theta_t)] \nabla^2 u_y^{00} \end{aligned} \tag{4.134}$$

Proposition 4.22 *Suppose that under the previous proposition conditions $\mathcal{H}$–medium is incompressible* ($\mathrm{div}_0\, u_y^{00} = 0$)*. Then $\mathcal{H}$–medium is dynamically equivalent to $\mathcal{H}$–medium ((see 4.79)) and to ideal fluid (see (4.49)):*

$$\rho_y v_y^{00\cdot} = \rho_y g_{yt}^0(\urcorner y, y) - \mathrm{grad}_0 p_{yt} \tag{4.135}$$

Comment The pointed fact is one of the reasons to affirm that in the theoretical research of the potential motion of incompressible viscous fluids by using the ideal fluid dynamics equations, one reaches a suitable coincidence between the modeling results and the experiments result.

Proposition 4.23 *Two different media belong to $\mathcal{H}$–class (in analog with (4.66)) iff the columns U_{ya}^0 and U_{yb}^0 (i.e. the deformations in the case where $u = z$) are linearly related the matrix $W_{12ab}(y, I, \theta_{yt}) \in \mathcal{W}_{12}(\mathcal{R}, 9)$ such that*

$$\tau_{ya}^0 = \tau_{yb}^0 \tag{4.136}$$

under condition that the tensions are distributed in the same way:

$$U_{ya}^0 = W_{12ab}(y, I, \theta_{yt}) U_{yb}^0, \;\; W_{12ab}(y, I, \theta_{yt}) = W_{12a}^{-1} W_{12b} \tag{4.137}$$

Proof Under condition (4.136), from relation (4.128) we obtain $W_{12a} U_{ya}^0 = W_{12b} U_{yb}^0$, whence (4.137) follows.

Proposition 4.24 *$\mathcal{P}_{12}$–medium thermodynamics equations are in the following form ($w = p_{yt}$ or $w = 0$)*

$$\rho_y (u_y^{\cdot}(\theta_{yt}) + w(\theta_y^{-1})^{\cdot}) = \Phi_\lambda^y(I_1, \theta_{yt}) + \Phi_p^y(I, \theta_{yt}) + \mathrm{div}_0\, q_y^0 + \rho_y \varphi_y \tag{4.138}$$

$$\Phi_\lambda^y(I_1, \theta_{yt}) = \Lambda_y(I_1, \theta_{yt}) \cdot V_y^0 = \lambda_1(I_1, \theta_{yt})\, \mathrm{div}_0\, v_y^{00} \tag{4.139}$$

$$\Phi_p^y(I, \theta_{yt}) = P_y^{12}(I, \theta_{yt}) U_y^0 \cdot V_y^0 \tag{4.140}$$

Comment From the dynamical equivalence of $\mathcal{P}_{12}$–media to other classes of material under definite conditions, it does not follow that these media move in the same way. The reason is that $\mathcal{P}_{12}$–class of media is not thermodynamically equivalent to anyone of the medium classes which have been already considered. Solving the problems of dynamics and statics for $\mathcal{P}_{12}$–medium without taking in consideration the thermodynamics equations, one may yield to physically non-adequate results.

4.6.3. *$\mathcal{P}_{13}$–class of quasi–linear continuous media*

Proposition 4.25 *Let: 1. $\mathcal{P}_{13}(\mathcal{R},3)$ be a group of form (4.103);*
2. the medium mechanical state equations be

$$(T_y^0) = -(w) + \Lambda_y(I_1,\theta_{yt}) + P_y^{13}(I,\theta_{yt})U_y^0 \tag{4.141}$$

where $\Lambda_y(I_1,\theta_{yt})$ is the same as in (4.105), $P_y^{13}(I,\theta_{yt}) \in \mathcal{P}_{13}$.

Then: 1. the class of quasi–linear continuous media is called $\mathcal{P}_{13}$–class and its elements are called $\mathcal{P}_{13}$–media;
2. determining relations for $\mathcal{P}_{13}$–medium do not exist;
3. the matrix representation for d–deformator T_y^0 of the medium does not exist;
4. the dynamics equations of $\mathcal{P}_{13}$–medium at the point y are of the form

$$\begin{aligned}\rho_y v_{y1}^{0\bullet} &= -(w)_1 + \rho_y g_{t1}^0(\urcorner y,y) + \partial\lambda_y(I_1,\theta_y)/\partial y_1^0 \\ &\quad +[\partial p_1/\partial y_1^0, 0, 0, \partial p_3/\partial y_2^0, \partial p_2/\partial y_2^0, 0, \partial p_3/\partial y_3^0, 0, \\ &\quad \partial p_2/\partial y_1^0]U_y^{00} + p_2(y,I,\theta_{yt})(u_{22}^1 + u_{33}^1 - u_{21}^1 - u_{31}^3) \\ &\quad +p_1(y,I,\theta_{yt})\partial \operatorname{div}_0 u_y^{00}/\partial y_1^0\end{aligned} \tag{4.142}$$

$$\begin{aligned}\rho_y v_{y2}^{0\bullet} &= -(w)_2 + \rho_y g_{t2}^0(\urcorner y,y) + \partial\lambda_y(I_1,\theta_y)/\partial y_2^0 \\ &\quad +[\partial p_2/\partial y_2^0, \partial p_3/\partial y_1^0, 0, 0, \partial p_1/\partial y_2^0, 0, 0, \partial p_3/\partial y_3^0, \\ &\quad \partial p_2/\partial y_2^0]U_y^{00} + p_2(y,I,\theta_{yt})(u_{11}^2 + u_{33}^2 - u_{12}^1 - u_{32}^3) \\ &\quad +p_1(y,I,\theta_{yt})\partial \operatorname{div}_0 u_y^{00}/\partial y_2^0\end{aligned} \tag{4.143}$$

$$\begin{aligned}\rho_y v_{y3}^{0\bullet} &= -(w)_3 + \rho_y g_{t3}^0(\urcorner y,y) + \partial\lambda_y(I_1,\theta_y)/\partial y_3^0 \\ &\quad +[\partial p_2/\partial y_3^0, 0, \partial p_3/\partial y_1^0, 0, \partial p_2/\partial y_3^0, \partial p_3/\partial y_2^0, 0, 0, \\ &\quad \partial p_1/\partial y_3^0]U_y^{00} + p_2(y,I,\theta_{yt})(u_{11}^3 + u_{22}^3 - u_{13}^1 - u_{23}^2) \\ &\quad +p_1(y,I,\theta_{yt})\partial \operatorname{div}_0 u_y^{00}/\partial y_3^0;\end{aligned} \tag{4.144}$$

$$p_2(y,I,\theta_{yt}) = p_1(y,I,\theta_{yt}) - p_3(y,I,\theta_{yt}) \tag{4.145}$$

Proof is reached by substituting the equations (4.141) of mechanical state of $\mathcal{P}_{13}$–medium in motion equations (2.65).

Proposition 4.26 *Suppose that: 1. the mixed second derivatives of the coordinates of u_y^{00} w.r.t. the coordinates of y^{00} are continuous at the point y, and hence are equal (see (4.76));*
2. c_1^y, c_2^y, c_3^y are the coefficients of circulation of u_y^{00} of $\mathcal{P}_{13}$–medium at the point y (see (3.91)).
Then $\mathcal{P}_{13}$–medium dynamics equations are of the form

$$\begin{aligned}\rho_y v_{y1}^{0\bullet} &= -(w)_1 + \rho_y g_{t1}^0(\urcorner y,y) + \partial\lambda_y(I_1,\theta_y)/\partial y_1^0 \\ &\quad +[\partial p_1/\partial y_1^0, 0, 0, \partial p_3/\partial y_2^0, \partial p_2/\partial y_2^0, 0, \partial p_3/\partial y_3^0, 0, \\ &\quad \partial p_2/\partial y_1^0]U_y^{00} + p_2(y,I,\theta_{yt})[\partial c_2^y/\partial y_3^0 - \partial c_3^y/\partial y_2^0] \\ &\quad +p_1(y,I,\theta_{yt})\partial \operatorname{div}_0 u_y^{00}/\partial y_1^0\end{aligned} \tag{4.146}$$

$$\begin{aligned}\rho_y v^{0\cdot}_{y2} &= -(w)_2 + \rho_y g^0_{t2}(\urcorner y, y) + \partial\lambda_y(I_1,\theta_{yt})/\partial y^0_2 \\ &+[\partial p_2/\partial y^0_2, \partial p_3/\partial y^0_1, 0, 0, \partial p_1/\partial y^0_2, 0, 0, \partial p_3/\partial y^0_3, \\ &\partial p_2/\partial y^0_2]U^{00}_y + p_2(y, I, \theta_{yt})[\partial c^y_3/\partial y^0_1 - \partial c^y_1/\partial y^0_3] \\ &+p_1(y, I, \theta_{yt})\partial \operatorname{div}_0 u^{00}_y/\partial y^0_2\end{aligned} \tag{4.147}$$

$$\begin{aligned}\rho_y v^{0\cdot}_{y3} &= -(w)_3 + \rho_y g^0_{t3}(\urcorner y, y) + \partial\lambda_y(I_1,\theta_{yt})/\partial y^0_3 \\ &+[\partial p_2/\partial y^0_3, 0, \partial p_3/\partial y^0_1, 0, \partial p_2/\partial y^0_3, \partial p_3/\partial y^0_2, 0, 0, \\ &\partial p_1/\partial y^0_3]U^{00}_y + p_2(y, I, \theta_{yt})[\partial c^y_1/\partial y^0_2 - \partial c^y_2/\partial y^0_1] \\ &+p_1(y, I, \theta_{yt})\partial \operatorname{div}_0 u^{00}_y/\partial y^0_3\end{aligned} \tag{4.148}$$

Proof is analog of the proof of (4.106), (4.107), and (4.108).

Proposition 4.27 *Suppose that one more condition is added to the conditions of the previous proposition, i.e., the rheological coefficient $\lambda_y(I_1, \theta_{yt})$ has the following simplest representation*

$$\lambda_y(I_1, \theta_{yt}) = \lambda_y(\theta_{yt})I_1 = \lambda_y(\theta_{yt}) \operatorname{div}_0 u^{00}_y \tag{4.149}$$

Then: 1. the quasi–linear equations of $\mathcal{P}_{13}$–medium mechanical state are of the form

$$(T^0_y) = -(w) + W_{13}(y, I, \theta_{yt})U^0_y \tag{4.150}$$

where the matrix $W_{13}(y, I, \theta_{yt}) \in \mathcal{W}_{13}(\mathcal{R}, 9)$ has the structure of (4.94) with the elements

$$\begin{aligned}v_1(y, I, \theta_{yt}) &= p_1 + \lambda, \quad v_2(y, I, \theta_{yt}) = p_1 - p_3 + \lambda \\ v_3(y, I, \theta_{yt}) &= p_3\end{aligned} \tag{4.151}$$

2. the inverse matrix $W^{-1}_{13}(y, I, \theta_{yt}) \in \mathcal{M}_{13}(\mathcal{R}, 9)$ has the same structure and its corresponding elements are the coefficients of influence

$$\begin{aligned}w_1(y, I, \theta_{yt}) &= \beta_1 + \alpha, \quad w_2(y, I, \theta_{yt}) = \beta_1 - \beta_3 + \alpha \\ w_3(y, I, \theta_{yt}) &= \beta_3\end{aligned} \tag{4.152}$$

3. the rheological modules α, β_1 and β_3 are calculated by the relations

$$\begin{aligned}\alpha &= \alpha(y, I, \theta_{yt}) = 2p_3\lambda/h \\ \beta_1 &= \beta_1(y, I, \theta_{yt}) = p_3(2p_1 - p_3)/h, \quad \beta_3 = \beta_3(y, I, \theta_{yt}) = 1/p_3 \\ h &= (p_1 + \lambda)(p_1 - p_3 + \lambda)^2 + 2(p_1 - p_3 + \lambda)^3\end{aligned} \tag{4.153}$$

4. For 2–dimensional $\mathcal{P}_{13}$–medium the corresponding characteristics are

$$\begin{aligned}
W_{13}(y,I,\theta_{yt}) &= \begin{bmatrix} p_1+\lambda & 0 & 0 & p_1-p_3+\lambda \\ 0 & p_3 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ p_1-p_3+\lambda & 0 & 0 & p_1+\lambda \end{bmatrix} && (4.154)\\
\alpha &= \alpha(y,I,\theta_{yt}) = \lambda/[(p_1+\lambda)^2-(p_1-p_3+\lambda)^2] \\
\beta_1 &= \beta_1(y,I,\theta_{yt}) = p_1/[(p_1+\lambda)^2-(p_1-p_3+\lambda)^2] \\
\beta_3 &= \beta_3(y,I,\theta_{yt}) = 1/p_3 && (4.155)\\
\beta_1-\beta_3 &= -(p_1-p_3+2\lambda)/[(p_1+\lambda)^2-(p_1-p_3+\lambda)^2] \\
w_1(y,I,\theta_{yt}) &= \beta_1+\alpha, \quad w_2(y,I,\theta_{yt}) = \beta_1-\beta_3+\alpha \\
w_3(y,I,\theta_{yt}) &= \beta_3
\end{aligned}$$

5. The analogs of Young module E_{13}, of the shift G_{13} and of Poisson coefficient v_{13} in the case of 2–dimensional and 3–dimensional medium are of the form

$$\begin{aligned}
E_{13}(y,I,\theta_{yt}) &= 1/w_1, \quad v_{13}(y,I,\theta_{yt}) = -w_2/w_1 && (4.156)\\
G_{13}(y,I,\theta_{yt}) &= 1/w_3 = 1/\beta_3 = p_3
\end{aligned}$$

Proposition 4.28 *Let: 1. in addition to the previous proposition conditions, the deformation of $\mathcal{M}_{13}$–medium be potential (non–circular) (see (3.94), (3.98))*

$$\begin{aligned}
du_y^{00}/dy^{00} &= (du_y^{00}/dy^{00})^T && (4.157)\\
c_1^y &= c_2^y = c_3^y = 0
\end{aligned}$$

2. the matrix $L_y(I,\theta_{yt})$ be of form (4.131).

Then $\mathcal{M}_{13}$–medium dynamics equations at the point y are of one of the equivalent forms

$$\begin{aligned}
\rho_y v_y^{00\cdot} &= \rho_y g_{yt}^0(\urcorner y,y) + L_y(I,\theta_{yt})U_y^0 + \mathrm{div}_0\, u_y^{00}\mathrm{grad}_0\lambda_y(\theta_{yt}) - && (4.158)\\
&\quad \mathrm{grad}_0\, w + [\lambda_y(\theta_{yt}) + p_1(y,I,\theta_{yt})]\,\mathrm{grad}_0\,\mathrm{div}_0\, u_y^{00} \\
\rho_y v_y^{00\cdot} &= \rho_y g_{yt}^0(\urcorner y,y) + L_y(I,\theta_{yt})U_y^0 + \mathrm{div}_0\, u_y^{00}\,\mathrm{grad}_0\,\lambda_y(\theta_{yt}) - && (4.159)\\
&\quad \mathrm{grad}_0\, w + [\lambda_y(\theta_{yt}) + p_1(y,I,\theta_{yt})]\nabla^2 u_y^{00}
\end{aligned}$$

Proof is based on the use of the relation $\nabla^2 u_y^{00} = \mathrm{grad}_0\,\mathrm{div}_0\, u_y^{00}$ under conditions (4.76) and (4.157).

Proposition 4.29 *Let in the previous proposition conditions the rheological coefficients and the temperature of $\mathcal{M}_{13}$–medium do not depend on the point y and on I. Then $\mathcal{M}_{13}$–medium is homogeneous and the medium dynamics equations are linear:*

$$\begin{aligned}
\rho_y v_y^{00\cdot} &= -\,\mathrm{grad}_0\, w + (\lambda_y(\theta_t) + p_1(\theta_{yt}))\,\mathrm{grad}_0\,\mathrm{div}_0\, u_y^{00} + && (4.160)\\
&\quad \rho_y g_{yt}^0(\urcorner y,y) \\
\rho_y v_y^{00\cdot} &= -\,\mathrm{grad}_0\, w + (\lambda_y(\theta_t) + p_1(\theta_t))\nabla^2 u_y^{00} + \rho_y g_{yt}^0(\urcorner y,y)
\end{aligned}$$

Consequently, the homogeneous linear potentially deformable $\mathcal{M}_{12}$– and $\mathcal{M}_{13}$–media are dynamically equivalent (see (4.134), (4.160)).

Proposition 4.30 *Suppose that under the previous proposition conditions $\mathcal{M}_{13}$–material is incompressible ($\operatorname{div}_0 u_y^{00} = 0$). Then $\mathcal{M}_{13}$–medium is dynamically equivalent to $\mathcal{H}$–medium ((see 4.79)), to $\mathcal{M}_{12}$–medium ((see 4.135)) and to $\mathcal{I}$–medium ((see 4.49)):*

$$\rho_y v_y^{00\bullet} = \rho_y g_{yt}^0(\urcorner y, y) - \operatorname{grad}_0 w \tag{4.161}$$

Proposition 4.31 *In order that two different media belong to $\mathcal{M}_{13}$–class (respectively (4.136)) it is necessary and sufficient in same distribution of tensions in them*

$$\tau_{ya}^0 = \tau_{yb}^0 \tag{4.162}$$

the columns U_{ya}^0 and U_{yb}^0 (the columns of deformations under condition (4.162)) to be linearly related by the matrix $W_{13ab}(y, I, \theta_{yt}) \in \mathcal{M}_{13}(\mathcal{R}, 9)$ such that

$$U_{ya}^0 = W_{13ab}(y, I, \theta_{yt}) U_{yb}^0,\ \ W_{13ab}(y, I, \theta_{yt}) = W_{13a}^{-1} W_{12b} \tag{4.163}$$

Proof Under condition (4.162), from relation (4.150) we obtain $W_{13a} U_{ya}^0 = W_{13b} U_{yb}^0$, and this gives relation (4.163).

Proposition 4.32 *$\mathcal{P}_{13}$–medium thermodynamics equations are of the form ($w = p_{yt}$ or $w = 0$)*

$$\rho_y(u_y^{\bullet}(\theta_{yt}) + w\,(\rho_y^{-1})^{\bullet}) = \Phi_\lambda^y(I_1, \theta_{yt}) + \Phi_p^y(I, \theta_{yt}) + \operatorname{div}_0 q + \rho_y \varphi_y \tag{4.164}$$

4.6.4. *$\mathcal{P}_{23}$–class of quasi–linear continuous media*

Proposition 4.33 *Let: 1. $\mathcal{P}_{23}(\mathcal{R}, 9)$ be a group of kind (4.104);*
2. the medium mechanical state equations be of the form

$$(T_y^0) = -(w) + \lambda_y(I_1, \theta_{yt}) + P_y^{23}(I, \theta_{yt}) U_y^0 \tag{4.165}$$

where $\lambda_y(I_1, \theta_{yt})$ is the same as in relation (4.141), $P_y^{23}(I, \theta_{yt}) \in \mathcal{P}_{23}(\mathcal{R}, 9)$;
3. $L_y(I, \theta_{yt})$ be 3×9–matrix of form (4.131).

Then: 1. the quasi–linear medium is called $\mathcal{P}_{23}$–medium;
2. determining relations (4.15) for $\mathcal{P}_{23}$–medium do not exist;
3. no matrix representation for d–deformator T_y^0 exists;
4. $\mathcal{P}_{23}$–medium dynamics equations are of the form

$$\begin{aligned} \rho_y v_y^{00\bullet} \quad &= \quad \rho_y g_{yt}^0(\urcorner y, y) - \operatorname{grad}_0 w + \operatorname{grad}_0 \lambda_y(I_1, \theta_{yt}) + \\ &\quad\ L_y(I, \theta_{yt}) U_y^0 + p_2(y, I, \theta_{yt}) \operatorname{grad}_0 \operatorname{div}_0 u_y^{00} + p_3(y, I, \theta_{yt}) \nabla^2 u_y^{00} \end{aligned} \tag{4.166}$$

Proof is based on substituting $\mathcal{P}_{23}$–medium mechanical state equations (4.165) in the motion equations (2.65).

Proposition 4.34 *Suppose that the rheological coefficient $\lambda_y(I_1, \theta_{yt})$ has the simplest representation*

$$\Lambda_y(I_1, \theta_{yt}) = \lambda_y(\theta_{yt}) I_1 = \lambda_y(\theta_{yt}) \operatorname{div}_0 u_y^{00} \tag{4.167}$$

Then: 1. $\mathcal{P}_{23}$–medium mechanical state equations are of the form

$$(T_y^0) = -(w) + W_{23}(y, I, \theta_{yt})U_y^0 \tag{4.168}$$

where the matrix $W_{23}(y, I, \theta_{yt}) \in \mathcal{M}_{23}(\mathcal{R}, 9)$ has structure (4.94) with the (corresponding) elements

$$\begin{aligned} v_1(y, I, \theta_{yt}) &= v_2 + v_3 = p_2 + p_3 + \lambda \\ v_2(y, I, \theta_{yt}) &= p_2 + \lambda, \quad v_3(y, I, \theta_{yt}) = p_3 \end{aligned} \tag{4.169}$$

2. the inverse matrix $W_{23}^{-1}(y, I, \theta_{yt}) \in \mathcal{M}_{23}(\mathcal{R}, 9)$ naturally has the analog structure with the elements being coefficients of influence

$$\begin{aligned} w_1(y, I, \theta_{yt}) &= w_2 + w_3 = \beta_2 + \beta_3 + \alpha \\ w_2(y, I, \theta_{yt}) &= \beta_2 + \alpha, \quad w_3(y, I, \theta_{yt}) = \beta_3 \end{aligned} \tag{4.170}$$

3. the rheological modules α,β_2, and β_3 are calculated by the relations

$$\begin{aligned} \alpha = \alpha(y, I, \theta_{yt}) &= 2p_3\lambda h^{-1} \\ \beta_2 = \beta_2(y, I, \theta_{yt}) &= -p_3(p_2 + 3\lambda)h^{-1}, \quad \beta_3 = \beta_3(y, I, \theta_{yt}) = p_3^{-1} \\ h = (p_2 + p_3 + \lambda)^3 &- 3(p_2 + p_3 + \lambda)(p_2 + \lambda)^2 + 2(p_2 + \lambda)^3 \end{aligned} \tag{4.171}$$

4. $\mathcal{P}_{23}$–medium dynamics equations are of the form

$$\begin{aligned} \rho_y v_y^{00\cdot} = {} & \rho_y g_{yt}^0(\urcorner y, y) - \mathrm{grad}_0 w + \mathrm{div}_0\, u_y^{00} \mathrm{grad}_0 \lambda_y(\theta_{yt}) + \\ & L_y(I, \theta_{yt})U^0 + [p_2(y, I, \theta_{yt}) + \lambda_y(\theta_{yt})]\mathrm{grad}_0\, \mathrm{div}_0\, u_y^{00} + \\ & p_3(y, I, \theta_{yt})\nabla^2 u_y^{00} \end{aligned} \tag{4.172}$$

Example Let $p_2 = 2$, $p_3 = 3$, $\lambda = 4$. Then $\beta_2 = -0.2222$, $\beta_3 = 0.3333$, $\alpha = 0.1270$, $w_1 = 0.2381$, $w_2 = -0.0952$, $w_3 = 0.3333$.

Proposition 4.35 *Suppose that: 1. the medium deformation is non–circular (potential)*

$$du_y^{00}/dy^{00} = (du_y^{00}/dy^{00})^T \tag{4.173}$$

2. the mixed second derivatives of the coordinates of u_y^{00} w.r.t. the coordinates of y^{00} are continuous at the point y, and they consequently are equal (see (4.76)).

Then $\mathcal{P}_{23}$–medium dynamics equations at the point y have the following equivalent forms of writing

$$\begin{aligned} \rho_y v_y^{00\cdot} = {} & \rho_y g_{yt}^0(\urcorner y, y) - \mathrm{grad}_0 w + \mathrm{div}_0\, u_y^{00} \mathrm{grad}_0 \lambda_y(\theta_{yt}) + \\ & L_y(I, \theta_{yt})U_y^0 + [\lambda_y(\theta_{yt}) + p_1(y, I, \theta_{yt}) + \\ & p_3(y, I, \theta_{yt})]\mathrm{grad}_0\, \mathrm{div}_0\, u_y^{00} \end{aligned} \tag{4.174}$$

$$\begin{aligned} \rho_y v_y^{00\cdot} = {} & \rho_y g_{yt}^0(\urcorner y, y) - \mathrm{grad}_0 w + \mathrm{div}_0\, u_y^{00} \mathrm{grad}_0 \lambda_y(\theta_{yt}) + \\ & L_y(I, \theta_{yt})U_y^0 + [\lambda_y(\theta_{yt}) + p_1(y, I, \theta_{yt}) + p_3(y, I, \theta_{yt})]\nabla^2 u_y^{00} \end{aligned} \tag{4.175}$$

Proof is based on using the equality $\nabla^2 u_y^{00} = \mathrm{grad}_0 \mathrm{div}_0\, u_y^{00}$ under conditions (4.76) and (4.173).

Proposition 4.36 *Let the medium rheological coefficients depend on the temperature only, and do not depend on the point y. Then $\mathcal{P}_{23}$–medium is homogeneous, and dynamics equations (4.174), (4.175) are linear:*

$$\begin{aligned} \rho_y v_y^{00\cdot} &= \rho_y g_{yt}^0(\urcorner y, y) - \mathrm{grad}_0 w + (\lambda(\theta_t) + p(\theta_t))\mathrm{grad}_0 \mathrm{div}_0\, u_y^{00} \quad (4.176)\\ \rho_y v_y^{00\cdot} &= \rho_y g_{yt}^0(\urcorner y, y) - \mathrm{grad}_0 w + [\lambda(\theta_t) + p(\theta_t)]\nabla^2 u_y^{00}\\ p(\theta_t) &= p_1(\theta_t) + p_3(\theta_t) \end{aligned}$$

Proposition 4.37 *Suppose that in the previous propositions conditions, $\mathcal{M}_{23}$–medium is incompressible* ($\mathrm{div}_0\, u_y^{00} = \mathrm{const}$)*. Then $\mathcal{M}_{23}$–medium is dynamically equivalent to $\mathcal{I}$–medium (see (4.49)), to $\mathcal{H}$–medium (see (4.79)), to $\mathcal{M}_{12}$–medium (see (4.135)), and to $\mathcal{M}_{13}$–medium (see (4.161))*

$$\rho_y v_y^{00\cdot} = \rho_y g_{yt}^0(\urcorner y, y) - \mathrm{grad}_0 w \quad (4.177)$$

Proposition 4.38 *Two different media belong to $\mathcal{M}_{23}$–class (in analog to (4.162)) if and only if in the case of same distribution of tensions in them*

$$\tau_{ya}^0 = \tau_{yb}^0 \quad (4.178)$$

the columns U_{ya}^0 and U_{yb}^0 are linearly related by the matrix $W_{23ab}(y, I, \theta_{yt}) \in \mathcal{M}_{23}(\mathcal{R}, 9)$ such that

$$U_{ya}^0 = W_{23ab}(y, I, \theta_{yt})U_{yb}^0,\ W_{23ab}(y, I, \theta_{yt}) = W_{23a}^{-1} W_{23b} \quad (4.179)$$

Proof From relation (4.168) under condition (4.178) we obtain $W_{23a}U_{ya}^0 = W_{23b}U_{yb}^0$, and this gives relation (4.179).

Proposition 4.39 *$\mathcal{P}_{23}$–medium thermodynamics equation is of the form* ($w = p_{yt}$ *or* $w = 0$)

$$\rho_y(u_y(\theta_{yt}) + w(\rho_y^{-1})^\cdot) = \Phi_\lambda^y(I_1, \theta_{yt}) + \Phi_p^y(I, \theta_{yt}) + \mathrm{div}_0\, q_y + \rho_y \varphi_y \quad (4.180)$$

4.7. Elements of dynamics for quasi–linear continuous $\mathcal{NSL}$–media

4.7.1. *Equations of mechanical state and dynamics for $\mathcal{NSL}$–media*

From the point of view of the presented in this book theory, Navier–Stocks–Lame media ($\mathcal{NSL}$–media) (Kochin *et al.* 1964, Lur'e 1970, Sedov 1972) are 'unusual'.

Definition 4.11 *The per–symmetric matrix of the form*

$$M_{NSL} = \begin{bmatrix} 2\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \mu & 0 & \mu & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \mu & 0 & 0 & 0 & \mu & 0 & 0\\ 0 & \mu & 0 & \mu & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 2\mu & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \mu & 0 & \mu & 0\\ 0 & 0 & \mu & 0 & 0 & 0 & \mu & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \mu & 0 & \mu & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \quad (4.181)$$

is called Navier–Stocks–Lame matrix (NSL–matrix).

Proposition 4.40 *NSL–matrix is singular*

$$\det M_{NSL} = 0 \tag{4.182}$$

Proof $\det M_{NSL} = 8\mu_y^8(1-1) = 0$.

It follows from the previous proposition that the set of Navier–Stocks–Lame matrices is not a group, and hence they can take part only in the incorrect equations of continuous media mechanical state, if such equations and media exist.

Proposition 4.41 *The set of NSL–matrices is centered by the rotation group* $\mathcal{SO}(\mathcal{R}, 9)$ *(see (4.47)):*

$$C_f^{0,T} M_{NSL} C_f^0 = M_{NSL} \tag{4.183}$$

Proof is reaching by verifying analytically or by calculations (for example, by MatLab) the equality $C_f^{0,T} M_{NSL} C_f^0 = M_{NSL}$, $C_f^0 \in \mathcal{SO}(\mathcal{R}, 9)$.

Comment From equation (4.183) follows that NSL–matrix can be treated as a matrix of rheological coefficients of II type in the incorrect equations of the quasi–linear continuous medium mechanical state (4.24).

Proposition 4.42 *Let: 1.* $M_{NSL}^y(I, \theta_{yt})$ *be NSL–matrix:*

2. $[du_y^{00}/dy^{00}]$ *be the symmetric term in identity (3.5):*

$$[du_y^{00}/dy^{00}] = 0.5\{du_y^{00}/dy^{00} + (du_y^{00}/dy^{00})^T\} \tag{4.184}$$

3. the incorrect equations of quasi–linear medium mechanical state (4.31) be of the form

$$(T_y^0) = -(w) + \Lambda_y(I_1, \theta_{yt}) + M_{NSL}^y(I, \theta_{yt}) U_y^0 \tag{4.185}$$

where $\Lambda_y(I_1, \theta_{yt}) = \lambda_y(I_1, \theta_{yt}) \operatorname{col}\{1, 0, 0, 0, 1, 0, 0, 0, 1\}$, $\lambda_y(I_1, \theta_{yt})$ *is the medium rheological coefficient of I type,* $M_{NSL}^y(I, \theta_{yt})$ *is the matrix of rheological coefficients of II type* $w = p_{yt}$ *(medium – fluid) or* $w = 0$ *(medium – elastic material).*

Then: 1. the class of quasi–linear incorrect continuous media, having relation (4.185) as the mechanical state equation, is called a class of quasi–linear $\mathcal{NSL}$*–media (Navier–Stocks–fluids and Lame–elastic materials);*

2. the functions $\lambda_y(I_1, \theta_{yt})$, $\mu_y(I, \theta_{yt})$ *of a point* y *which are invariants of the matrix* du_y^{00}/dy^{00} *and of the temperature* θ_{yt} *are called* $\mathcal{NSL}$*–medium rheological coefficients of I and II types.*

3. $\mathcal{NSL}$*–medium mechanical state incorrect equations (4.185) are equivalent to the linear part of the determining* $\mathcal{NSL}$*-relations (see (4.15))*

$$T_y^0 = (-(w) + \lambda_y(I_1, \theta_{yt}))E + 2\mu_y(I, \theta_{yt})[du_y^{00}/dy^{00}] \tag{4.186}$$

4. $\mathcal{NSL}$–medium dynamics incorrect equations are of the form

$$\begin{aligned}\rho_y v_{y1}^{00\cdot} &= \rho_y g_{t1}^0(\urcorner y, y) - (w)_1 + \partial\lambda_y(I_1,\theta_{yt})/\partial y_1^0 + \\ &\quad (2\partial\mu_y/\partial y_1^0, \partial\mu_y/\partial y_2^0, \partial\mu_y/\partial y_3^0, \partial\mu_y/\partial y_2^0, 0, 0, \partial\mu_y/\partial y_3^0, 0, 0)U_y^0 \\ &\quad +\mu_y(I,\theta_{yt})(u_{11}^1 + u_{12}^2 + u_{13}^3) + \mu_y(I,\theta_{yt})\nabla^2 u_{y1}^0\end{aligned} \tag{4.187}$$

$$\begin{aligned}\rho_y v_{y2}^{00\cdot} &= \rho_y g_{t2}^0(\urcorner y, y) - (w)_2 + \partial\lambda_y(I_1,\theta_{yt})/\partial y_2^0 + \\ &\quad \mu_y(I,\theta_{yt})(u_{21}^1 + u_{22}^2 + u_{23}^3) + \mu_y(I,\theta_{yt})\nabla^2 u_{y2}^0 + \\ &\quad (0, \partial\mu_y/\partial y_1^0, 0, \partial\mu_y/\partial y_1^0, 2\partial\mu_y/\partial y_2^0, \partial\mu_y/\partial y_3^0, 0, \partial\mu_y/\partial y_3^0, 0)U_y^0\end{aligned} \tag{4.188}$$

$$\begin{aligned}\rho_y v_{y3}^{00\cdot} &= \rho_y g_{t3}^0(\urcorner y, y) - (w)_3 + \partial\lambda_y(I_1,\theta_{yt})/\partial y_3^0 + \\ &\quad \mu_y(I,\theta_{yt})(u_{31}^1 + u_{32}^2 + u_{33}^3) + \mu_y(I,\theta_{yt})\nabla^2 u_{y3}^0 + \\ &\quad (0, 0, \partial\mu_y/\partial y_1^0, 0, 0, \partial\mu_y/\partial y_2^0, \partial\mu_y/\partial y_1^0, \partial\mu_y/\partial y_2^0, 2\partial\mu_y/\partial y_3^0)U_y^0\end{aligned} \tag{4.189}$$

Comments

1. If the matrix of tensions is non–symmetric (the opposite does not follow at all) and $\lambda_y(I_1,\theta_{yt}) = \lambda_y(\theta_{yt})$, then the equation system (4.185) is not solvable w.r.t. the elements of the non–symmetric matrix du_y^{00}/dy^{00}. If the matrix T_y^0 is symmetric, the mentioned system of equations has an infinite number of solutions. If additionally the medium motion is non–circular, equations (4.185) are solvable w.r.t. the unknowns U_y^0 but the inverse of matrix (4.181) has another structure, and consequently equations (4.185) rest incorrect.
2. The rheological coefficients in the equations of two– and three–dimensional $\mathcal{NSL}$–media are ones and the same. It means that the pointed physical characteristics of $\mathcal{NSL}$–media do not depend on medium dimension, and in this sense they are correct characteristics.

Proposition 4.43 *Suppose that: 1. in addition to the previous proposition conditions the mixed second derivatives of the coordinates of u_y^{00} w.r.t. the coordinates of y^{00} are continuous, and hence are equal (see (4.76)):*

$$\begin{aligned}u_{12}^1 &= u_{21}^1, \quad u_{13}^1 = u_{31}^1, \quad u_{12}^2 = u_{21}^2 \\ u_{12}^2 &= u_{21}^2, \quad u_{13}^3 = u_{31}^3, \quad u_{23}^3 = u_{32}^3\end{aligned} \tag{4.190}$$

2. the matrix $L(I,\theta_{yt})$ is of form

$$\begin{bmatrix} \frac{2\partial\mu_y}{\partial y_1^0} & \frac{\partial\mu_y}{\partial y_2^0} & \frac{\partial\mu_y}{\partial y_3^0} & \frac{\partial\mu_y}{\partial y_2^0} & 0 & 0 & \frac{\partial\mu_y}{\partial y_3^0} & 0 & 0 \\ 0 & \frac{\partial\mu_y}{\partial y_1^0} & 0 & \frac{\partial\mu_y}{\partial y_1^0} & 2\frac{\partial\mu_y}{\partial y_2^0} & \frac{\partial\mu_y}{\partial y_3^0} & 0 & \frac{\partial\mu_y}{\partial y_3^0} & 0 \\ 0 & 0 & \frac{\partial\mu_y}{\partial y_1^0} & 0 & 0 & \frac{\partial\mu_y}{\partial y_2^0} & \frac{\partial\mu_y}{\partial y_1^0} & \frac{\partial\mu_y}{\partial y_2^0} & \frac{2\partial\mu_y}{\partial y_3^0} \end{bmatrix} \tag{4.191}$$

Then $\mathcal{NSL}$–medium dynamics equations are of the form

$$\begin{aligned}\rho_y v_y^{0\cdot} &= \rho_y g_{yt}^0(\urcorner y, y) - \mathrm{grad}_0 w + \mathrm{grad}_0 \lambda_y(I_1, \theta_{yt}) + \\ &\quad L_y(I, \theta_{yt}) U_y^0 + \mu_y(I, \theta_{yt}) \mathrm{grad}_0\, \mathrm{div}_0\, u_y^{00} + \mu_y(I, \theta_{yt}) \nabla^2 u_y^{00}\end{aligned} \tag{4.192}$$

Comments

1. No group generating $\mathcal{NSL}$–media exists. It means that either these media occupy a special place among quasi–linear continuous media, or they are not adequate models of real media being incorrect approximation of the earlier considered media. Their appearance is obliged by representing the matrix $[du_y^{00}/dy^{00}]$ as a deformation matrix (for an elastic material under condition (3.21)) or as a deformation velocities matrix. Indeed, this matrix is the symmetric part of the identical decomposition of du_y^{00}/dy^{00} whose elements are either deformation velocities ($du_y^{00}/dy^{00} \equiv dv_y^{00}/dy^{00}$) or the deformations themselves (if we talk about an elastic material under definite conditions (3.21) ($du_y^{00}/dy^{00} \equiv dz_y^{00}/dy^{00}$).
2. Condition (3.15) is the determining relation for obtaining the dynamics equations (4.186)–(4.187) and all $\mathcal{NSL}$–media dynamics equations following after that.

   These equations are broken in the case of reinforcing the medium, of the presence in the medium of isolated heterogeneous inclusions, cracks, *etc.* Thus, neither equations (4.185) nor all following after that dynamics equations could be used for $\mathcal{NSL}$–media deformation research in the above mentioned case directly (without any other assumptions).
3. $\mathcal{NSL}$–medium is dynamically equivalent to (4.166) when $p_2 = p_3$.
4. It is possible that the previous proposition explains frequently met differences between $\mathcal{NSL}$–equations integration results and experiments results: one and the same $\mathcal{NSL}$–equations are always integrated, but the experimental media could be different. If the real (physical) medium is modeled as $\mathcal{NSL}$–medium (see (4.185)), then it is difficult to expect the coincidence between the theoretical and experimental results as $\mathcal{NSL}$–medium is incorrect. If the model of real medium is correct, and it is dynamically equivalent to $\mathcal{NSL}$–medium, then the coincidence between theoretical and experimental results can be satisfactory.

Proposition 4.44 *Suppose that in addition to the previous propositions conditions, the rheological coefficient $\lambda_y(I_1, \theta_{yt})$ for $\mathcal{NSL}$–medium has the simplest linear representation*

$$\lambda_y(I_1, \theta_{yt}) = \lambda_y(\theta_{yt}) I_1 = \lambda_y(\theta_{yt})\, \mathrm{div}_0\, u_y^{00} \tag{4.193}$$

Then: 1. the relations

$$(T_y^0) = -(w) + W_{NSL}(y, I, \theta_{yt}) U_y^0 \tag{4.194}$$

where the matrix $W_{NSL}(y, I, \theta_{yt})$ has structure (4.181) with new coefficients

$$v_{11} = v_{55} = v_{99} = 2\mu + \lambda,\ v_{15} = v_{19} = v_{51} = v_{15} = v_{91} = v_{99} = \lambda$$

and the rest are the previous ones, are incorrect quasi–linear equations of the mechanical state of $\mathcal{M}$–class of $\mathcal{NSL}$–media ($\det W_{NSL}(y, I, \theta_{yt}) = 0$);

2. *$\mathcal{M}_{NSL}$–medium has not the inverse of $W_{NSL}(y, I, \theta_{yt})$, and hence, it has neither rheological modules nor coefficients of influence (i.e., elements of non–existing inverse $W_{NSL}^{-1}(y, I, \theta_{yt})$ and Young modules, shift ones and Poisson coefficient) obtained with their help;*
3. *the relations*

$$T_y^0 = -(w + \lambda_y(I_1, \theta_{yt}) \operatorname{div}_0 u_y^{00})E + 2\mu_y(I, \theta_{yt})[du_y^{00}/dy^{00}] \tag{4.195}$$

are incorrect determining relations for quasi–linear $\mathcal{M}_{NSL}$–medium;

4. *the incorrect dynamics equations for $\mathcal{M}_{NSL}$–class of media are of the form*

$$\begin{aligned} \rho_y v_y^{00\bullet} &= \rho_y g_{yt}^0(\urcorner y, y) - \operatorname{grad}_0 w + \operatorname{div}_0 u_y^{00} \operatorname{grad}_0 \lambda_y(\theta_{yt}) + \\ &L_y(I, \theta_{yt}) U_y^0 + [\lambda_y(\theta_{yt}) + \mu_y(I, \theta_{yt})] \operatorname{grad}_0 \operatorname{div}_0 u_y^{00} + \\ &\mu_y(I, \theta_{yt}) \nabla^2 u_y^{00} \end{aligned} \tag{4.196}$$

Proposition 4.45 *Suppose that in the previous propositions conditions $\mathcal{NSL}$–medium deformation is non–circular (potential) (see (3.94), (3.98)):*

$$du_y^{00}/dy^{00} = (du_y^{00}/dy^{00})^T \tag{4.197}$$

Then: 1. dynamics equations for $\mathcal{M}_{NSL}$–medium have the following equivalent forms of writing (here $\operatorname{grad}_0 \operatorname{div}_0 u_y^{00} = \nabla^2 u_y^{00}$ is fulfilled as relation (4.76) is true):

$$\begin{aligned} \rho_y v_y^{00\bullet} &= [2\mu_y(I, \theta_{yt}) + \lambda_y(\theta_{yt})] \nabla^2 u_y^{00} - \operatorname{grad}_0 w + \rho_y g_{yt}^0(\urcorner y, y) \\ \rho_y v_y^{00\bullet} &= [2\mu_y(I, \theta_{yt}) + \lambda_y(\theta_{yt})] \operatorname{grad}_0 \operatorname{div}_0 u_y^{00} - \operatorname{grad}_0 w + \rho_y g_{yt}^0(\urcorner y, y) \end{aligned} \tag{4.198}$$

2. *the elements of the symmetric matrix du_y^{00}/dy^{00} can be represented in three equivalent forms of writing, including form (3.101):*

$$du_y^{00}/dy^{00} = (du_y^{00}/dy^{00}])^T \tag{4.199}$$

$$u_i^i = \varepsilon_{ii},\ u_j^i = u_i^j = 1/2(u_j^i + u_i^j) = \gamma_{ij}/2 = \gamma_{ji}/2 \tag{4.200}$$

hence, for this form of writing we have

$$\overline{U}_y^0 = \operatorname{col}\{\varepsilon_{11}, \gamma_{21}, \gamma_{31}, \gamma_{12}, \varepsilon_{22}, \gamma_{32}, \gamma_{13}, \gamma_{23}, \varepsilon_{33}\} \tag{4.201}$$

3. *the quasi–linear equations of mechanical state for $\mathcal{NSL}$–medium have the form (that coincides with (4.60)):*

$$(T_y^0) = -(w) + W_{NSL}(y, I, \theta_{yt})\overline{U}_y^0 \tag{4.202}$$

$$W_{NSL} = \begin{bmatrix} 2\mu+\lambda & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & \lambda \\ 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 2\mu+\lambda & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 \\ \lambda & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 2\mu+\lambda \end{bmatrix} \tag{4.203}$$

$$\det W_{NSL}(y, I, \theta_{yt}) = 4\mu^8(2\mu + 3\lambda) \neq 0$$

4. *the set of non–singular matrices $W_{NSL}(y, I, \theta_{yt})$ is not a group (the product of matrices of form (4.202) is not a matrix of the same form) but (because of (4.203)) these matrices have inverse matrices with corresponding coefficients (of influence) $w_1(y, I, \theta_{yt})$, $w_2(y, I, \theta_{yt})$ and $w_3(y, I, \theta_{yt})$;*
5. *in the case of non–circular (potential) deformation an inverse matrix exists, and thus $\mathcal{NSL}$–media have rheological modules of I and II types which coincide with the coefficients (of influence) w_2 and w_3:*

$$\begin{aligned} \alpha(y, I, \theta_{yt}) &= w_2(y, I, \theta_{yt}) = -\lambda/[2\mu(2\mu + 3\lambda)] \\ \beta(y, I, \theta_{yt}) &= w_3(y, I, \theta_{yt}) = 1/\mu \end{aligned} \tag{4.204}$$

but the coefficient w_1 of influence does not coincide with the sum $2\beta + \alpha$ as it is possible to suppose, and it is calculated by the relation

$$w_1(y, I, \theta_{yt}) = (\mu + \lambda)/[\mu(2\mu + 3\lambda)] \tag{4.205}$$

6. *since the inverse matrix exists in the case of non–circular (potential) deformation, $\mathcal{NSL}$–media have Young module, shift module and Poisson coefficient (the index NSL is omitted for the sake of writing brevity):*

$$E_3 = E_3(y, I, \theta_{yt}) = 1/w_1 = \mu(2\mu + 3\lambda)/(\mu + \lambda) \tag{4.206}$$

$$G_3 = G_3(y, I, \theta_{yt}) = 1/w_3 = \mu \tag{4.207}$$

$$v_3 = v_3(y, I, \theta_{yt}) = -w_2/w_1 = \lambda/[2(\mu + \lambda)] \tag{4.208}$$

which are related each other by

$$\lambda_y = 2v_3G_3/(1 - 2v_3), \;\; \mu_y = G_3 \tag{4.209}$$

$$E_3 = 2G_3(1 + v_3) \tag{4.210}$$

Proof Equations (4.198) and (4.202) are obtained by substituting (4.197) in relation (4.194). The satisfaction of the rest relations it is checked by simple calculations.

Comments

1. If the motion is not potential, then the rheological modules of I and II types as well as Young module, shift module and Poisson coefficient of $\mathcal{NSL}$–medium do not exist. In this case the use of the mentioned characteristics leads to errors whose order is determined by the entries of $\frac{1}{2}(du_y^{00}/dy^{00} - (du_y^{00}/dy^{00})^T)$. The pointed out matrix is equal to zero only in the case of medium potential motion (see (4.197)).
2. In the case of potential deformation, $\mathcal{NSL}$–medium rest incorrect (there exists no group that generates this medium) but the matrix of rheological coefficients has an inverse matrix. It permits us to determine rheological modules, and consequently Young module, shift module and Poisson coefficient.
3. It is necessary to point out that $du_y^{00}/dy^{00} \neq (du_y^{00}/dy^{00})^T$ in the common case. Substituting equalities (4.206), (4.207), and (4.208) in equation (4.198) we obtain dynamics equations w.r.t. the 'old' variables with coefficients depending on $E_3(I,\theta_{yt})$, $G_3(I,\theta_{yt})$ and $v_3(I,\theta_{yt})$ obtained by using mechanical state equations (4.202) w.r.t. 'new' variables $\overline{U}_y^0$ (see (4.201)). In terms of 'old' variables, $E_3(I,\theta_{yt})$, $G_3(I,\theta_{yt})$ and $v_3(I,\theta_{yt})$ do not exist at all.
4. For 2–dimensional $\mathcal{NSL}$–media the corresponding characteristics are of the form

$$M_{NSL} = \begin{bmatrix} 2\mu & 0 & 0 & 0 \\ 0 & \mu & \mu & 0 \\ 0 & \mu & \mu & 0 \\ 0 & 0 & 0 & 2\mu \end{bmatrix} \tag{4.211}$$

$$\det M_{NSL} = 0$$

$$W_{NSL}(y,I,\theta_{yt}) = \begin{bmatrix} 2\mu+\lambda & 0 & 0 & \lambda \\ 0 & \mu & \mu & 0 \\ 0 & \mu & \mu & 0 \\ \lambda & 0 & 0 & 2\mu+\lambda \end{bmatrix} \tag{4.212}$$

$$\det W_{NSL} = 0$$

$$W_{NSL}(y,I,\theta_{yt}) = \begin{bmatrix} 2\mu+\lambda & 0 & 0 & \lambda \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ \lambda & 0 & 0 & 2\mu+\lambda \end{bmatrix} \tag{4.213}$$

$$\det W_{NSL}(y,I,\theta_{yt}) = 4\mu^3(\mu+\lambda) \neq 0$$

The inverse matrix has the analog structure with coefficients of influence:

$$w_1^y(I,\theta_{yt}) = (2\mu+\lambda)/[4\mu(\mu+\lambda)] \tag{4.214}$$

$$w_2^y(I,\theta_{yt}) = -\lambda/[4\mu(\mu+\lambda)], \quad w_3^y(I,\theta_{yt}) = 1/\mu$$

$$\alpha(y,I,\theta_{yt}) = w_2^y(I,\theta_{yt}) = -\lambda/[4\mu(\mu+\lambda)] \tag{4.215}$$

$$\beta(y,I,\theta_{yt}) = w_3^y(I,\theta_{yt}) = 1/\mu$$

Young module, the shift module and Poisson coefficient are related with the rheological coefficients and each with other by the relations:

$$E_2^y = E_2^y(I,\theta_{yt}) = 4\mu(\mu+\lambda)/(2\mu+\lambda), \quad G_2^y = G_2^y(I,\theta_{yt}) = \mu \tag{4.216}$$

$$E_2^y = E_2^y(I,\theta_{yt}) = 2G_2^y(1+v_2^y), \quad v_2^y = v_2^y(I,\theta_{yt}) = \lambda/(2\mu+\lambda)$$

The relation between the rheological coefficients and the quantities E_2^y, G_2^y and v_2^y is

$$\lambda_y(I_1, \theta_{yt}) = 2G_2^y v_2^y(1 - v_2^y), \quad \mu_y(I, \theta_{yt}) = G_2^y \tag{4.217}$$

5. Young module $E_3^y(I, \theta_{yt})$, Poisson coefficient $v_3^y(I, \theta_{yt})$ and Lame coefficient $\lambda_y(I_1, \theta_{yt})$ (as their relation, too) do not coincide with the analog characteristics of 2–dimensional $\mathcal{NSL}$–material (see (4.216), (4.217)).

   Hence, if formulating applied problems in the terms of Young module and Poisson coefficient in 2–dimensional $\mathcal{NSL}$–media, one should not use 3–dimensional media mechanics apparatus reduced by one coordinate. The same is true for 1–dimensional materials, the last ones being not objects of this book.
6. The analog of Young module (coinciding with the rheological coefficient of II type) and relations (4.210) are correct characteristics of $\mathcal{NSL}$–medium.
7. The fact that $\mathcal{NSL}$–media are not generated by any group means that either these media occupy an essential place among the quasi–linear continuous media or they do not exist at all in the nature, being incorrect approximations of the considered above media.

Proposition 4.46 *Suppose that the mean value of the elements along the main diagonal of d–deformator is equal to zero:*

$$\frac{1}{3}(T_{11}^0 + T_{22}^0 + T_{33}^0) = 0 \tag{4.218}$$

Then: 1. in infringement of the requirement of definition D 4.7, $\mathcal{NSL}$–medium rheological coefficients of I and II types become dependent and one of them is negative, i.e.,

$$\lambda_y(I, \theta_{yt}) = -\frac{2}{3}\mu_y(I, \theta_{yt}) \tag{4.219}$$

2. mechanical state equations (4.185) and $\mathcal{NSL}$–medium determining relations which are equivalent to (4.185) continue to be incorrect (here $\det W_{NSL} = 0$*);*

3. incorrect dynamics equations (4.196) for $\mathcal{NSL}$–medium are of the form

$$\begin{aligned} \rho_y v_y^{00\bullet} \quad = \quad & \rho_y g_{yt}^0(\urcorner y, y) - \mathrm{grad}_0(w) + \mathrm{div}_0\, u_y^{00}\, \mathrm{grad}_0\, \lambda_y(I, \theta_{yt}) + \\ & L_y(I, \theta_{yt})U_y^0 + 1/3\mu_y(I, \theta_{yt})\, \mathrm{grad}_0\, \mathrm{div}_0\, u_y^{00} + \mu_y(I, \theta_{yt})\nabla^2 u_y^{00} \end{aligned} \tag{4.220}$$

Comments

1. Relation (4.219) contradicts to the requirement about the independence of the viscosity rheological coefficients (by definition D 4.7) and can be used only as an assumption. It is equivalent to the assumption that terms of viscous or elastic nature in diagonal entries of d–deformator T_y^0 are slightingly small.
2. The pointed above assumption leads to a negative rheological coefficient of I type and hence to an artificial underestimation of the influence of viscosity and elasticity on the medium deformation ($\frac{1}{3}\mu < \lambda + \mu$).

Proposition 4.47 *Suppose that in addition to previous proposition conditions (4.76), (4.193), (4.197) $\mathcal{NSL}$–medium is incompressible* ($\operatorname{div}_0 u_y^{00}/dy^{00} = 0$)*. Then $\mathcal{NSL}$–medium is dynamically equivalent to media (4.79), (4.135), (4.161), (4.177):*

$$\rho_y v_y^{00\bullet} = \rho_y g_{yt}^0(\urcorner y, y) - \operatorname{grad}_0 w \tag{4.221}$$

Proposition 4.48 *Let the rheological coefficient of II type for $\mathcal{M}_{NSL}$–medium do not depend on the invariant I of the matrix du_y^{00}/dy^{00}.*

Then: 1. The equations of the mechanical state and of the dynamics for $\mathcal{M}_{NSL}$–medium are linear

$$\begin{aligned}(T_y^0) &= -(w) + W_{NSL}(y, \theta_{yt})U_y^0 && (4.222)\\ \rho_y v_y^{00\bullet} &= \rho_y g_{yt}^0(\urcorner y, y) - \operatorname{grad}_0 w + \operatorname{div}_0 u_y^{00} \operatorname{grad}_0 \lambda_y(\theta_{yt}) + && (4.223)\\ &\quad L_y(I, \theta_{yt})U_y^0 + [\lambda_y(\theta_{yt}) + \mu_y(\theta_{yt})] \operatorname{grad}_0 \operatorname{div}_0 u_y^{00} + \\ &\quad \mu_y(\theta_{yt})\nabla^2 u_y^{00}\end{aligned}$$

2. if all rheological coefficients for $\mathcal{M}_{NSL}$–medium depend only on the temperature (which does not depend on the point), then the equations of the strained state and of dynamics are linear and homogeneous:

$$\begin{aligned}(T_y^0) &= -(w) + W_{NSL}(\theta_{yt})U_y^0 && (4.224)\\ \rho_y v_y^{00\bullet} &= \rho_y g_{yt}^0(\urcorner y, y) - \operatorname{grad}_0 w + \mu(\theta_t)\nabla^2 u_y^{00} + && (4.225)\\ &\quad [\lambda(\theta_t) + \mu_y(\theta_t)] \operatorname{grad}_0 \operatorname{div}_0 u_y^{00}\end{aligned}$$

4.7.2. *Thermodynamics equation*

Definition 4.12 *Let: 1. $\Lambda_y(I, \theta_{yt})$ and $M_{NSL}^y(I, \theta_{yt})$ be the column and the matrix of I and of II type rheological coefficients of $\mathcal{NSL}$–medium (see (4.185));*

2. U_y^0 be 9×1–column consisting of 3×1–columns of u_y^{00}/dy^{00};

3. V_y^0 be 9×1–column consisting of 3×1–columns of v_y^{00}/dy^{00}.

Then: 1. the inner product

$$\Phi_\lambda^y(I_1, \theta_{yt}) = \Lambda_y(I_1, \theta_{yt}) \cdot V_y^0 = \lambda_y(I_1, \theta_{yt}) \operatorname{div}_0 v_y^{00} \tag{4.226}$$

similarly to relation (4.86) are called tensions power density of I type w.r.t. the measure at the point y;

2. the inner product (the bilinear form with a matrix $M_{NSL}^y(I, \theta_{yt})$) of the form

$$\Phi_\mu^y(I, \theta_{yt}) = M_{NSL}^y(I, \theta_{yt})U_y^0 \cdot V_y^0 \tag{4.227}$$

is called tensions power density of II type with respect to the measure at the point y.

Proposition 4.49 *Let: 1. $\Phi_\lambda^y(I_1, \theta_{yt})$ and $\Phi_\mu^y(I, \theta_{yt})$ be the functions of the previous definition;*

2. $u^{\bullet}_y(\theta_{yt})$ *be the density of the velocity of* $\mathcal{NSL}$*–medium inner energy change at the point* y *w.r.t. the measure* $m(dy)$*.*

Then the thermodynamics equation for incorrect quasi–linear $\mathcal{NSL}$*–medium is of the form (*$w = p_{yt}$ *or* $w = 0$*)*

$$\rho_y[u^{\bullet}_y(\theta_{yt}) + w\,(\rho_y^{-1})^{\bullet}] = \Phi^y_\lambda(I_1, \theta_{yt}) + \Phi^y_\mu(I, \theta_{yt}) + \mathrm{div}_0\, q^0_y + \rho_y\varphi_y \quad (4.228)$$

Proof is reduced to substituting mechanical state equations (4.185) for $\mathcal{NSL}$–medium under the condition for thermodynamical balance of Galilean mechanics Universe with account of definitions (4.226), (4.227) and the proof of equality (2.52).

Comments

1. Inner energy change of I type (see (4.226)) is the result of dilation deformation (3.41), and it is absent in the case of incompressible $\mathcal{NSL}$–medium motion. Thermodynamics equation, similarly to (4.88), is of the form

$$\rho_y(u^{\bullet}_y(\theta_{yt}) + w(\rho_y^{-1})^{\bullet}) = \Phi^y_\lambda(I_1, \theta_{yt}) + \mathrm{div}_0\, q^0_y + \rho_y\varphi_y \quad (4.229)$$

2. The class of $\mathcal{NSL}$–media thermodynamically is not equivalent to any of the above discussed media classes. Solving $\mathcal{NSL}$–media dynamics problems without regarding to thermodynamics equations one can arrive at physically non-adequate results.

4.8. Elements of dynamics of $\mathcal{R}$–class of 2–dimensional quasi–linear continuous media

4.8.1. *Mechanical state equations*

Proposition 4.50 *Let: 1.* $\mathcal{GO}_{yt}(\mathcal{R}, 2)$ *be the group of 2–dimensional similitudes (Dieudonne 1969);*

2. $M^y_1,\ M^y_2 \in \mathcal{GO}_{yt}(\mathcal{R}, 2)$ *be* 2×2*–matrices of similitudes of the form*

$$M^y_1 = \begin{bmatrix} \mu^y_1 & -\mu^y_2 \\ \mu^y_2 & \mu^y_1 \end{bmatrix},\ M^y_2 = \begin{bmatrix} \mu^y_3 & -\mu^y_4 \\ \mu^y_4 & \mu^y_3 \end{bmatrix} \quad (4.230)$$

such that

$$(M^y_1)^T M^y_1 = (\mu^{y2}_1 + \mu^{y2}_2)E,\ (M^y_2)^T M^y_2 = (\mu^{y2}_3 + \mu^{y2}_4)E \quad (4.231)$$

3. $\mathcal{GR}_{yt}(\mathcal{R}, 4)$ *be the group on* $\mathbf{R}_4$ *whose elements are* 4×4*–matrices with* 2×2*–matrix blocks as 2–dimensional similitudes (4.230) from* $\mathcal{GO}_{yt}(\mathcal{R}, 2)$

$$M_y(I) = \begin{bmatrix} M^y_1 & -M^y_2 \\ M^y_2 & M^y_1 \end{bmatrix} \quad (4.232)$$

4. $\mathcal{SO}(\mathcal{R}, 4)$ *be the group of matrices (4.51) and be the subgroup of the rotation group of the vector space* $\mathbf{R}_4$*.*

Then: 1. the group $\mathcal{GR}_{yt}(\mathcal{R},4)$ is a centralizer of its own subgroup $SO(\mathcal{R},4)$ of rotations, i.e. for an arbitrary matrices $M_y(I) \in \mathcal{GR}_{yt}(\mathcal{R},4)$ and $C_f^0 \in \mathcal{SO}(\mathcal{R},4)$ (see (4.51))

$$M_y^{-1}C_f^0 M_y = C_f^0 \tag{4.233}$$

2. the rotation group $\mathcal{SO}(\mathcal{R},4)$ is a centralizer of the group $\mathcal{GR}_{yt}(\mathcal{R},4)$, i.e., for any matrices $M_y(I) \in \mathcal{GR}_{yt}(\mathcal{R},4)$ and $C_f^0 \in \mathcal{SO}(\mathcal{R},4)$

$$C_f^{0,T} M_y C_f^0 = M_y,\ \mathcal{SO}(\mathcal{R},4) \subset \mathcal{GR}_{yt}(\mathcal{R},4) \tag{4.234}$$

Proof follows form the fact that the group of 2×2–similitudes (4.230) on $\mathbf{R}_2$ is commutative under condition that $\mathcal{SO}(\mathcal{R},4) \subset \mathcal{GR}_{yt}(\mathcal{R},4)$.

Comments

1. Conditions (4.233) and (4.234) are equivalent to the requirement that the commutant of the matrices C_f^0 and M_y is equal to 0:

$$[C_f^0, M_y] = M_y C_f^0 - c_f^0 M_y = 0 \tag{4.235}$$

2. The 4×4–matrix (4.232) with blocks (4.230) is not a similitude (the matrix $M_y^T M_y$ is not a homothety).

Example Let $M_y = \begin{bmatrix} 1 & -2 & -3 & 4 \\ 2 & 1 & -4 & -3 \\ 3 & -4 & 1 & -2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$ then $\det M_y = 884$, $M_y^{'} M_y =$
$\begin{bmatrix} 30 & 0 & 0 & -4 \\ 0 & 30 & 4 & 0 \\ 0 & 4 & 30 & 0 \\ -4 & 0 & 0 & 30 \end{bmatrix}$, $M_y^{-1} = \begin{bmatrix} 0.052 & 0.0543 & 0.0928 & 0.1403 \\ -0.0543 & 0.052 & -0.1403 & 0.0928 \\ -0.0928 & -0.1403 & 0.052 & 0.0543 \\ 0.1403 & -0.0928 & -0.0543 & 0.052 \end{bmatrix}$

3. Matrices of the group $\mathcal{GR}_{yt}(\mathcal{R},4)$ (represented in element 4×4–form) being invariants of the choice (4.234) of an inertial basis, can be used for forming the third term in the correct equations of mechanical state in the case of 2–dimensional quasi–linear continuous medium.

Definition 4.13 *Let: 1. V_y^0 and $\Lambda_y^0(I_1,\theta_{yt})$ be 4×1–columns of the (4.52) form*

$$\begin{aligned} V_y^0 \equiv -(w) &= -w\,\mathrm{col}\{1,0,0,1\} && (4.236)\\ \Lambda_y^0(I_1,\theta_{yt}) &= \lambda_y(I_1,\theta_{yt})\,\mathrm{col}\{1,0,0,1\} && (4.237)\end{aligned}$$

2. $M_y^0(I,\theta_{yt})$ be 4×4–matrix of the group $\mathcal{GR}_{yt}(\mathcal{R},4)$ (see (4.232));

3. the non–linear equations

$$M_y^0(I,\theta_{yt})U_y^0 + \Lambda_y^0(I_1,\theta_{yt}) = (T_y^0) + (w) \equiv (\tau_y^0) \tag{4.238}$$

have a unique solution w.r.t. the unknown column U_y^0.

Then: 1. the relation

$$(T_y^0) = -(w) + \Lambda_y^0(I_1, \theta_{yt}) + M_y^0(I, \theta_{yt})U_y^0 \tag{4.239}$$

is called equations of the mechanical state of $\mathcal{R}$–class of 2–dimensional correct quasi–linear continuous media (from the word – Resemblance);

2. the elements of the column $\Lambda_y^0(I_1, \theta_{yt})$ and of the matrix $M_y^0(I, \theta_{yt})$ are called rheological coefficients of $\mathcal{R}$–medium of I and II type, respectively.

Proposition 4.51 *Let: 1. $|du_y^{00}/dy^{00}|$ be the matrix*

$$|du_y^{00}/dy^{00}| = \begin{bmatrix} -u_2^1 & u_1^1 \\ -u_2^2 & u_1^2 \end{bmatrix} \tag{4.240}$$

2. $M_y^0(I, \theta_{yt})$, $M_y^0(I, \theta_{yt})$ be the matrices from (4.230).

Then d–deformator T_y^0 has the following matrix form of writing:

$$\begin{aligned} T_y^0 \quad = \quad & h(-w + \lambda_y(I_1, \theta_{yt})) + M_1^y(I, \theta_{yt}) du_y^{00}/dy^{00} + \\ & M_2^y(I, \theta_{yt})|du_y^{00}/dy^{00}| \end{aligned} \tag{4.241}$$

Comments

1. From (4.241) it follows that 2–dimensional $\mathcal{R}$–medium has 5 independent rheological coefficients. It can be explained by the fact that the requirement about the matrix of rheological coefficients of II type w.r.t. 2–dimensional group of rotations (consisting only of matrices of the simplest rotations – see (4.39), (4.43), (4.44)) to be invariant is more weak than the analog requirements about 3–dimensional group of rotations (consisting of three simplest factors).
2. Determining relations (4.15) for $\mathcal{R}$–class do not exist since in the right–hand side of equality (4.241) there exist the matrix (not numerical) factor $M_1^y(I, \theta_{yt})$ in the derivative du_y^{00}/dy^{00} and the term $M_2^y(I, \theta_{yt})\ |du_y^{00}/dy^{00}|$. Equality (4.241) is not a matrix function of the matrix argument (4.15) but a matrix writing of the dependence of d–deformator elements from the matrix du_y^{00}/dy^{00} elements, *i.e.*, a matrix writing of the mechanical state equations (4.239).
3. From (4.241) follows that the derivative du_y^{00}/dy^{00} and d–deformator T_y^0 of $\mathcal{R}$–medium could not be simultaneously symmetric matrices in the most common case. Because of skew–symmetry of the matrices $M_1^y(I, \theta_{yt})$ and $M_2^y(I, \theta_{yt})$, d–deformator T_y^0 cannot be symmetric matrix in principle.

Proposition 4.52 *Let: 1.* $\mathrm{grad}_0\, w$ *be the gradient of w at the point y in $\mathbf{E}_0$;*

2. $\mathrm{grad}_0\, \lambda_y(I_1, \theta_{yt})$ *be the gradient of the rheological coefficient $\lambda_y(I_1, \theta_{yt})$ at the point y in $\mathbf{E}_0$;*

3. $\mathrm{Div}_0\, du_y^{00}/dy^{00} \equiv \mathrm{col}\{\mathrm{Div}_0\, u_{1y}^0, \mathrm{div}_0\, u_{2y}^0\}$ *be 2×1–column of the divergences in $\mathbf{E}_0$ of the rows u_{1y}^0 and u_{2y}^0 of the derivative du_y^{00}/dy^{00} at the point y in $\mathbf{E}_0$:*

$$\mathrm{div}_0\, du_y^{00}/dy^{00} = \nabla^2 u_y^0 \tag{4.242}$$

4. *the mixed second derivatives* $u^1_{12} \equiv \partial^2 u^0_{1y}/\partial y^0_1 \partial y^0_2$, $u^1_{21} \equiv \partial^2 u^0_{1y}/\partial y^0_2 \partial y^0_1$, $u^2_{12} \equiv \partial^2 u^0_{2y}/\partial y^0_1 \partial y^0_2$, $u^2_{21} \equiv \partial^2 u^0_{2y}/\partial y^0_2 \partial y^0_1$ *of the vector* u^{00}_y *coordinates w.r.t. the space coordinates at the point* y *exist and*

$$N^0_y(I, \theta_{yt}) = \mathrm{col}\{u^1_{21} - u^1_{12}, \quad u^2_{12} - u^2_{21}\} \tag{4.243}$$

5. $\left[\ \partial M^y_2/\partial y^0_2 \quad -\partial M^y_2/\partial y^0_1\ \right]$ *be* 2×4*–matrix of the form*

$$\left[\ \partial M^y_2/\partial y^0_2 \quad -\partial M^y_2/\partial y^0_1\ \right] = \left[\begin{array}{cccc} \partial\mu^y_3/\partial y^0_2 & -\partial\mu^y_4 \partial y^0_2 & -\partial\mu^y_3/\partial y^0_1 & \partial\mu^y_4/\partial y^0_1 \\ \partial\mu^y_4/\partial y^0_2 & \partial\mu^y_4/\partial y^0_2 & -\partial\mu^y_4/\partial y^0_1 & -\partial\mu^y_3/\partial y^0_1 \end{array} \right]$$

6. $\left[\partial M^y_1/\partial y^0_1 | \partial M^y_1/\partial y^0_2\right]$ *be* 2×4*–matrix of the form*

$$\left[\ \partial M^y_1/\partial y^0_1 \quad \partial M^y_1/\partial y^0_2\ \right] = \left[\begin{array}{cccc} \partial\mu^y_1/\partial y^0_1 & -\partial\mu^y_2/\partial y^0_1 & \partial\mu^y_1/\partial y^0_2 & -\partial\mu^y_2/\partial y^0_2 \\ \partial\mu^y_2/\partial y^0_1 & \partial\mu^y_1/\partial y^0_1 & -\partial\mu^y_1/\partial y^0_2 & \partial\mu^y_1/\partial y^0_2 \end{array} \right]$$

7. $L_y(I, \theta_{yt})$ *be* 2×4*–matrix which is a sum of the matrices*

$$L_y(I, \theta_{yt}) = \left[\ \partial M^y_1/\partial y^0_1 + \partial M^y_2/\partial y^0_2 \quad \partial M^y_1/\partial y^0_2 - \partial M^y_2/\partial y^0_1\ \right] \tag{4.244}$$

Then the column $\mathrm{Div}_0 T^0_y$ *consisting of the motion equations of 2–dimensional* $\mathcal{R}$*–medium (2.65) is of the form*

$$\begin{aligned} \mathrm{Div}_0 T^0_y \quad = \quad & -\mathrm{grad}_0 w + \mathrm{grad}_0 \lambda_y(I_1, \theta_{yt}) + M^y_1(I, \theta_{yt}) \nabla^2 u^0_y + \\ & M^y_2(I, \theta_{yt}) N^0_y(I, \theta_{yt}) + L_y(I, \theta_{yt}) U^0_y \end{aligned} \tag{4.245}$$

Proof is realized by simple transformations on the column $\mathrm{Div}_0\, T^0_y$ in view of equality (4.241).

Comment From (4.241) and (4.245) follows that $\mathcal{R}$–medium rheological coefficients in $M^y_1(I, \theta_{yt})$ define the contribution of Laplacian $\nabla^2 u^{00}_y$ in the medium motion, while the $\mathcal{R}$–medium rheological coefficients of the matrix $M^y_2(I, \theta_{yt})$ determine the contribution of the effects connected with continuity absence in the medium motion, and consequently the second mixed derivatives are equal ($N^0_y(I) = 0$ in the opposite case).

4.8.2. *Dynamics equations*

Proposition 4.53 *1. The correct dynamics equations of the mechanical state for* $\mathcal{R}$*–class of* 2*–dimensional media (4.19) are of the form*

$$\rho_y v^{00\cdot}_y \quad = \quad \rho_y g^0_{yt}(\urcorner y, y) + \mathrm{Div}_0\, T^0_y \tag{4.246}$$

$$(T^0_y) \quad = \quad -(w) + \Lambda_y(I_1, \theta_{yt}) + M^0_y(I, \theta_{yt}) U^0_y \tag{4.247}$$

2. *Dynamics equations for* $\mathcal{R}$*–class of* 2*–dimensional quasi–linear continuous media are of the form*

$$\begin{aligned} \rho_y v^{00\cdot}_y \quad = \quad & \rho_y g^0_{yt}(\urcorner y, y) - \mathrm{grad}_0 w + \mathrm{grad}_0 \lambda_y(I_1, \theta_{yt}) + \\ & L_y(I, \theta_{yt}) U^0_y + M^y_1(I, \theta_{yt}) \nabla^2 u^0_y + M^y_2(I, \theta_{yt}) N^0_y \end{aligned} \tag{4.248}$$

Proof is based on substituting equality (4.245) in equation (4.246).

Comments

1. In the last equations all $\mathcal{R}$–medium rheological coefficients are functions of the coordinates of y, of the invariant $I=(I_1,I_2,I_3)$ of the matrix du_y^{00}/dy^{00} and of the temperature θ_{yt}.
2. The term $\left[\ \partial M_1^y/\partial y_1^0+\partial M_2^y/\partial y_2^0 \quad \partial M_1^y/\partial y_2^0-\partial M_2^y/\partial y_1^0\ \right]$ takes into account the dependence of the rheological coefficients of II type from the coordinates of y, from the invariant I of the matrix du_y^{00}/dy^{00} and of the temperature θ_{yt}.
3. The term $M_2^y(I,\theta_{yt})N_y^0(I)$ takes into account the possible absence of continuity of the derivatives $\partial^2u_{y1}^0/\partial y_1^0\partial y_2^0\equiv u_{12}^1$, $\partial^2u_{y1}^0/\partial y_2^0\partial y_1^0\equiv u_{21}^1$, $\partial^2u_{y2}^0/\partial y_1^0\partial y_2^0\equiv u_{12}^2$, $\partial^2u_{y2}^0/\partial y_2^0\partial y_1^0\equiv u_{21}^2$ (for example, in the problems of supersonic aero-gas–dynamics, the presence of non–homogeneous inclusions, cracks, material reinforcing, breaks of entirety, *etc.*).
4. S–class includes in itself $\mathcal{H}$–class of media as a private case ($\mu_1=\mu$, $\mu_2=\mu_3=\mu_4=0$).
5. In the framework of the assumed hypotheses and definitions (see (4.239)), equations (4.248) are exact. All equations that follow are obtained as simplification of the exact ones that. They are formulated as assumptions. These assumptions are determined in the Preface as wrong assertions accepted under the arrangement.

Proposition 4.54 *Suppose that: 1. the rheological coefficients $\mu_i^y(I,\theta_{yt})$ of II type do not depend on the invariant I, on the point y, and the temperature does not depend on the point y*

$$\mu_i^y(I,\theta_{yt})=\mu_i(\theta_t),\ \ L_y(I,\theta_{yt})U_y^0=0 \tag{4.249}$$

2. the rheological coefficient $\lambda_y(I_1,\theta_{yt})$ of I type (similarly to (4.59)) has the simplest form

$$\lambda_y(I_1,\theta_{yt})=\lambda(\theta_t)\mathrm{div}_0u_y^{00} \tag{4.250}$$

3. the mixed second derivatives

$$u_{12}^1,u_{21}^1,u_{12}^2,u_{21}^2 \tag{4.251}$$

of the coordinates u_{y1}^0, u_{y2}^0 of the vector u_y^{00} of the medium at the point y are continuous, and consequently are equal (similarly to (4.76)):

$$u_{12}^1=u_{21}^1,u_{12}^2=u_{21}^2\Longleftrightarrow N_y^0(I)=0 \tag{4.252}$$

Then: 1. the mechanical state equations, d–deformator and dynamics linear equations of $\mathcal{M}_R$–class of homogeneous media are of the form

$$\begin{aligned}(T_y^0)&=-(w)+W_R(\theta_t)U_y^0 \qquad (4.253)\\ T_y^0&=h[-w+\lambda(\theta_t)\mathrm{div}_0u_y^{00}]+M_1^y(\theta_t)du_y^{00}/dy^{00}+M_2^y(\theta_t)|du_y^{00}/dy^{00}|\\ \rho_yv_y^{00\cdot}&=\rho_yg_{yt}^0(\urcorner y,y)-\mathrm{grad}_0w+M_1(\theta_t)\nabla^2u_y^{00}+\lambda(\theta_t)\mathrm{grad}_0\mathrm{div}_0u_y^{00}\end{aligned}$$

2. the matrix $W_R(\theta_t)$ is an element of the group $\mathcal{M}_R(\mathcal{R},4)$ and is of the form

$$W_R(\theta_t) = \begin{bmatrix} \mu_1+\lambda & \mu_2 & -\mu_3 & \mu_4+\lambda \\ \mu_2 & \mu_1 & -\mu_4 & -\mu_3 \\ \mu_3 & -\mu_4 & \mu_1 & -\mu_2 \\ \mu_4+\lambda & \mu_3 & \mu_2 & \mu_1+\lambda \end{bmatrix} \tag{4.254}$$

3. the matrix $W_R(\theta_t)$ includes 5 rheological coefficients (1 of I type, and 4 of II type) and thus under condition of above assumptions $\mathcal{M}_R$–medium has 5 rheological modules and 5 coefficients of influence (see D 4.7).

Comments

1. In the considered case, the mechanical state equations and d–deformator expression include 4 rheological coefficients of II type (because of terms including matrix M_2^y), while the motion equations include 2 rheological coefficients only (since $N_y^0 = 0$).
2. Assumption (4.252) and the of medium dynamics equations related with it simplifications could seem principally not acceptable if in the medium there exist supersonic currents, cracks, *etc.*
3. Under conditions (4.250) and (4.252) determining relations (4.15) do not exist for $\mathcal{M}_R$–medium.

Proposition 4.55 *Two different linear homogeneous media ($\mathcal{M}_R$–media) belong to $\mathcal{M}_R$–class (similarly to (4.67) if and only if (in the case of same distribution of tensions in them) the columns U_{ya}^0 and U_{yb}^0 are linearly related by the matrix $W_{Rab}(\theta_t) \in \mathcal{M}_R(\mathcal{R},4)$ such that*

$$U_{ya}^0 = W_{Rab}(\theta_t)U_{yb}^0,\ W_{Rab}(\theta_t) = W_{Ra}^{-1}W_{Rb} \tag{4.255}$$

Proof From relation (4.253) under condition (4.66) we obtain $W_{Ra}U_{ya}^0 = W_{Rb}U_{yb}^0$ and hence we have relation (4.255).

Proposition 4.56 *Suppose that the crossed influence of Laplacians $\nabla^2 u_{y1}^0$ and $\nabla^2 u_{y2}^0$ on the $\mathcal{M}_R$–medium motion w.r.t. second and first coordinates, respectively, is absent*

$$\mu_2(\theta_t) = 0 \Longrightarrow M_1(\theta_t) = h(\mu_1(\theta_t)) = \mu_1(\theta_t)E \tag{4.256}$$

Then: 1. the mechanical state equations, the d–deformator and dynamics equations for $\mathcal{M}_R$–class are of the form (subclasses $\mathcal{H}-$ and $\mathcal{M}_R$–fluids are dynamically equivalent – see (4.73)):

$$\begin{aligned} (T_y^0) &= -(w) + W_R(\theta_t)U_y^0 \\ T_y^0 &= h[-w+\lambda(\theta_t)\mathrm{div}_0 u_y^{00}] + \mu_1(\theta_t)du_y^{00}/dy^{00} + M_2^y(\theta_t)|du_y^{00}/dy^{00}| \\ \rho_y v_y^{00\cdot} &= \rho_y g_{yt}^0(\urcorner y, y) - \mathrm{grad}_0 w + \lambda(\theta_{yt})\mathrm{grad}_0\mathrm{div}_0 u_y^{00} + \mu_1(\theta_t)\nabla^2 u_y^{00} \end{aligned} \tag{4.257}$$

2. the matrix $W_R(\theta_t)$ is of the form

$$W_R(\theta_t) = \begin{bmatrix} \mu_1+\lambda & 0 & -\mu_3 & \mu_4+\lambda \\ 0 & \mu_1 & -\mu_4 & -\mu_3 \\ \mu_3 & -\mu_4 & \mu_1 & 0 \\ \mu_4+\lambda & \mu_3 & 0 & \mu_1+\lambda \end{bmatrix}$$

Comments

1. Under condition of the accepted suppositions, $\mathcal{M}_R$–media dynamics equations coincide with $\mathcal{M}_H$–media dynamics equations (the mentioned media are dynamically equivalent) while the classes of the considered media differ each other since their mechanical state equations have different determining relations (4.60) and (4.253) and the thermodynamics equivalence is absent. The reason of this phenomenon is in the fact that the mechanical state equations include medium tensions while the dynamics equations include sums of the partial derivatives of these tensions; the fluids thermodynamics equations include different matrices of rheological coefficients. The pointed fact can be used one to do false models of a real S–medium motion if this motion is examined without taking into account the mentioned factors.
2. In any further suppositions, the term $M_2^y(\theta_t)|du_y^{00}/dy^{00}|$ in the right–hand side of relation (4.241) remains the same. This fact points out on the following:

   2.1. In any assumptions about the rheological coefficients properties, determining relations (4.15) do not exist for $\mathcal{M}_R$–medium (since the term $M_2^y(\theta_t)$ $|du_y^{00}/dy^{00}|$ is present in the right–hand side of equality (4.241)).

   2.2. Neither mechanical state equations (4.239), nor the matrix representation of d–deformator (4.241) (in any suppositions) determine the form of $\mathcal{M}_R$–medium dynamics equations: the mechanical state equations (4.257) and the matrix representation of d–deformation (4.241) include terms depending on the rheological coefficients μ_3 and μ_4 and determining the contribution of null terms including the columns $N_y^0 = 0$ and $L_y U_y^0 = 0$ in the motion equations. Otherwise, the tensions distribution in the medium depends always on μ_3 and μ_4, but under some conditions the distribution of tensions matrix rows divergences (4.245) does not depend on them.

Proposition 4.57 *Suppose that: 1. $\mathcal{M}_R$–medium motion is non–linear (potential) (see (4.75));*

2. second mixed derivatives of the vector u_y^{00} w.r.t. the point coordinates are continuous and equal (see (4.76));

3. the medium is incompressible (see (4.78)):

$$\mathrm{div}_0 v_y^{00} = 0 \tag{4.258}$$

Then linear dynamics equations (4.248) of 2–dimensional linear homogeneous $\mathcal{M}_R$–medium (similarly to (4.79)) coincide with the ideal fluid equations (4.49):

$$\rho_y v_y^{00\bullet} = \rho_y g_{yt}^0(\urcorner y, y) - \mathrm{grad}_0 w \tag{4.259}$$

Proof of relation (4.259) follows from (4.245) and (4.248) or from (4.253) and (4.257) under conditions (4.75) and (4.76).

4.8.3. *Thermodynamics equation*

Definition 4.14 *Let: 1. $\Lambda_y(I_1, \theta_{yt})$ and $M_y(I, \theta_{yt})$ be the column and the matrix of rheological coefficients of I and II types of S–fluid (see (4.239));*

2. V_y^0 be 4×1–column consisting of 2×1–columns of du_y^{00}/dy^{00} (see (4.239)).

Then: 1. the inner product

$$\Phi_\lambda^y(I_1, \theta_{yt}) = V_y^0 \cdot \Lambda_y(I, \theta_{yt}) = \lambda_y(I_1, \theta_{yt}) \mathrm{div}_0 v_y^{00} \tag{4.260}$$

similarly to (4.86), is called a dissipative function of I type of $\mathcal{R}$–fluid at the point y.

2. the inner product (bilinear form with a matrix $M_y(I, \theta_{yt})$) of form (4.87)

$$\Phi_\mu^y(I, \theta_{yt}) = V_y^0 \cdot M_y(I, \theta_{yt}) V_y^0 \tag{4.261}$$

is called a dissipative function of II type of $\mathcal{R}$–fluid at the point y.

Proposition 4.58 *Let: 1. $\Phi_\lambda^y(I_1, \theta_{yt})$ and $\Phi_\mu^y(I, \theta_{yt})$ be dissipative functions of I and II types (see (4.260) and (4.261));*

2. $w_y^\cdot$ be the velocity of realization of the elementary work of the function w_y ($W_y^\cdot = \rho_y^{-1}$ is the specific volume of fluid) at the point y;

3. $u_y^\cdot(\theta_{yt})$ be the density of the velocity of the fluid inner energy change at the point y w.r.t. the measure $m(dy)$.

Then the thermodynamics equation of 2–dimensional $\mathcal{R}$–fluid is of the form

$$\rho_y(u_y^\cdot(\theta_{yt}) + p_y w_y^\cdot) = \Phi_\lambda^y(I_1, \theta_{yt}) + \Phi_\mu^y(I, \theta_{yt}) + \mathrm{div}_0 q_y^0 + \rho_y \varphi_y \tag{4.262}$$

Proof is based on substituting medium mechanical state equations (4.239) under condition of thermodynamical balance of Galilean mechanics Universe with account of definitions (4.260), (4.261) and the proof of equality (2.52).

Comments

1. The dissipation of the inner energy of I type is present in dilation deformation (3.27), but in the case of an incompressible S–fluid motion it is absent. $\mathcal{R}$–fluid thermodynamics equation, similarly to (4.229), is of the form

$$\rho_y u_y^\cdot(\theta_{yt}) = \Phi_\mu^y(I, \theta_{yt}) + \mathrm{div}_0 q_y^0 + \rho_y \varphi_y \tag{4.263}$$

2. If the mechanical state equation (4.253) is true, then the dissipative function of $\mathcal{R}$–fluid of I type is of the form

$$\Phi_\lambda^y(I_1, \theta_{yt}) = \lambda_y(\theta_{yt}) \mathrm{div}_0^2 v_y^{00} \tag{4.264}$$

Chapter 5

Rigid body mechanics

5.1. A rigid body

Remind that $\mathcal{GM}^q_{yt}(\mathcal{R},3) = \mathcal{T}_{yt}(\mathcal{R},3)\mathcal{SO}^q_{yt}(\mathcal{R},3) \subset \mathcal{GA}^q_{yt}(\mathcal{R},3)$ is the 4–parameter group of motions on $\mathbf{A}^\mu_3$ (see Axiom K 3, (2.13)) where the symbol $\mathcal{M}$ in the group abbreviation reminds 'motion', $\mathcal{SO}^q_{yt}(\mathcal{R},3) \subset \mathcal{GD}^q_{yt}(\mathcal{R},3)$ is the 4–parameter group of rotations in $\mathbf{V}_3$ (see (2.10)).

Definition 5.1 *We shall call a limited closed mechanical system $\varepsilon_y \equiv \sigma^\mu_3$ a rigid body if ε–affine transformation (see D 2.4) which acts on the system is a motion.*

Comments

1. Otherwise, any rigid body is a closed mechanical system (locally linearly changeable continuous medium) (see (2.12)) whose group (see (2.10)) of k–deformators at each point and in each instant coincides with the rotation group

$$\mathcal{GD}^q_{yt}(\mathcal{R},3) \equiv \mathcal{SO}^q_{yt}(\mathcal{R},3) \tag{5.1}$$

2. The 3–dimensional group $\mathcal{SO}^q_{yt}(\mathcal{R},3)$ of rotations of a rigid body is non–commutative (Dieudonne 1969).
3. Any rigid body is an element of the σ–algebra σ^μ_3 but the set of rigid bodies is not a σ–algebra (for example, the complement of an arbitrary rigid body is not a rigid body).

Proposition 5.1 *Some mechanical system $B \in \sigma^\mu_3$ is a rigid body, iff in left–hand side polar decompositions (3.82) of k–deformator of medium and in right–hand side ones (3.68) the symmetric factors (the dilators in the canonical bases) are identical units (unit matrices) of the k–deformator group $\mathcal{GD}^q_{yt}(\mathcal{R},3)$.*

Proof is obvious: in both cases the conditions $s^{00}_d \equiv E$ and $s^{00}_c \equiv E$ lead to $D^{00}_d \equiv c^{00}_d$ that is the definition of a rigid body.

Comment The above formalization of the question is not a unique one. Each of the just formulated aspects can be used as a rigid body definition and the other as necessary and sufficient conditions in order that some mechanical system is a rigid body.

Proposition 5.2 *Let:*

1. *$B \in \sigma_3^\mu$ be a rigid body, $\mathbf{E}_b$ be the frame attached to it;*
2. *c_b^{00} be a rotation matrix, $[\mathbf{e}^b] = [\mathbf{e}^0]c_b^{00}$ (see (3.64));*
3. *$< \omega_b^0 >^0$ be a skew–symmetric matrix generated by the rigid body angular velocity vector w_b^{00}.*

Then the kinematics equation (3.1) of the medium, i.e., the rigid body, is of the form (Konoplev 1984 and 1985, Konoplev et al. 2001)

$$c_b^{00\bullet} =< \omega_b^0 >^0 c_b^{00} \tag{5.2}$$

Proof follows from the kinematics equations on groups (3.79) and (3.87) based on the previous assertion.

Comment It is necessary to understand that the locally linearly changeable medium (see D 2.4) is not a rigid body on which a dilator (not depending on the rotation) acts. Otherwise, the locally linearly changeable medium is not the generalization of a rigid body as it has been assumed since Euler time without any mathematical or physical reasons. Indeed, the converse is true: a rigid body is the partial case of a locally linearly changeable continuous medium where k–deformators do not involve dilators in themselves. The mathematical model of a locally linearly changeable medium motion is k–deformator (see D 2.4), *i.e.*, the solution of equation (3.1), k–deformator having nothing in common relation with the rotation.

The pointed k–deformator has the infinite number of multiplicative decompositions (see Chapter 3) (two polar ones among them), that include the rotation matrix factors and these matrices are not k–deformators. The mathematical model of rigid body rotation is a rotation matrix that is the solution of equation (5.2) which is a private case of equation (3.1). The rigid body and the locally linearly changeable medium have only one common property: all determined on them scalar and vector measures of mechanics are absolutely continuous w.r.t. Lebesgue measure.

5.2. Rigid body kinematics

5.2.1. *Simple and composite motions*

Definition 5.2 *Let:*

1. *$(s)_-$ be a kinematic chain (Konoplev et al. 2001) of a element E_s with a root frame $\mathbf{E}_0$ (the set of a contra-accessibility of the element (s)) (Konoplev 1986c);*
2. *the kinematical chain $(s)_-$ includes two kinematical couples at least.*

Then the motion of the rigid body w.r.t. the root frame $\mathbf{E}_0$ is called a composite one.

Definition 5.3 *Let the kinematic chain $(s)_-$ include two kinematical couples $(s-2; s-1)$ and $(s-1; s)$.*

Then: 1. the motion of the rigid body E_s w.r.t. the root frame $\mathbf{E}_{s-2}$ is called a simple composite motion;

2. the motion of $(s-1)$–body w.r.t. the root frame $\mathbf{E}_{s-2}$ is called a transport one;

3. the motion of s–body w.r.t. $(s-1)$–body is called a relative one.

Definition 5.4 *Let the mathematical chain $(s)_-$ include one kinematical couple. Then the motion of the rigid body E_s w.r.t. the root frame $\mathbf{E}_0$ is called a simple one.*

5.2.2. *Kinematics of simple free and related motions*

Definition 5.5 *Let:*

1. *o_s^{s-1} be the radius–vector of the origin o_s of the frame $\mathbf{E}_s$ related to the body E_s in the frame $\mathbf{E}_{s-1}$ attached to the rigid body E_{s-1} in the kinematical couple $(s-1; s)$;*
2. *$o_s^{s-1;s-1} \equiv o^s$ be the coordinate column of the radius–vector o_s^{s-1} in the basis $[\mathbf{e}^{s-1}]$ in the frame $\mathbf{E}_{s-1}$*

$$o^s = \mathrm{col}\{o_1^s, o_2^s, o_3^s\} \tag{5.3}$$

3. *the condition*

$$\mathbf{E}_{s-1} = \mathbf{E}_s \tag{5.4}$$

be fulfilled in the initial instant of time.

Then the triple of scalar functions (5.3) is called a vector of a shift of the rigid body E_s in the simple free motion w.r.t. the root frame $\mathbf{E}_{s-1}$.

Proposition 5.3 *Let: 1. $\mathcal{SO}^q_{yt}(\mathcal{R},3)$ be the group of rotations of the rigid body E_s in the simple motion w.r.t. the root frame $\mathbf{E}_{s-1}$;*

2. $c_s^{s-1} \equiv c^s \in \mathcal{SO}^q_{yt}(\mathcal{R},3)$ be the matrix of rotation of the basis $[\mathbf{e}^s]$ w.r.t. the basis $[\mathbf{e}^{s-1}]$ calculated in this basis

$$[\mathbf{e}^s] = [\mathbf{e}^{s-1}]c_s^{s-1} \tag{5.5}$$

Then there exist three matrices of the simplest rotations $c_1(\theta_4^s)$, $c_2(\theta_5^s)$, $c_3(\theta_6^s) \in \mathcal{SO}^q_{yt}(\mathcal{R},3)$ such that

$$c_p(\theta^s_{p+3}) = E + \sin\theta^s_{p+3} < e^s_p > +(1-\cos\theta^s_{p+3}) < e^s_p >^2 \tag{5.6}$$

Here and further $\sin\theta^s_{p+3} \equiv \sin\theta^s_{p+3}$, $\cos\theta^s_{p+3} \equiv \cos\theta^s_{p+3}$ are such that (Konoplev et al. 2001)

$$c_s^{s-1} \equiv c^s = c_1(\theta_4^s)c_2(\theta_5^s)c_3(\theta_6^s) \tag{5.7}$$

Proof First, let us simplify the notations. Let the basis $[\mathbf{e}^1]$ be obtained from the basis $[\mathbf{e}^0]$ by a rotation with a matrix c_1^0, *i.e.*, $[\mathbf{e}^1] = [\mathbf{e}^0]c_1^0$. In the same way, the bases $[\mathbf{e}^2]$ and $[\mathbf{e}^3]$ are obtained from $[\mathbf{e}^1]$ and from $[\mathbf{e}^2]$ by rotations c_2^1 and c_3^2, respectively. Thus we may write the following chain of equalities $[\mathbf{e}^3] = [\mathbf{e}^2]c_3^2 = [\mathbf{e}^1]c_2^1c_3^2 = [\mathbf{e}^0]c_1^0c_2^1c_3^2$. Since all three rotations are realized with orthonormal bases, all three matrices are the simplest ones and $c_1^0c_2^1c_3^2 = c_1(\alpha_1)c_2(\alpha_2)c_3(\alpha_3)$.

Comments

1. The number of the simplest matrices in the decompositions of type (5.7) can be as large as wanted, but in the common case it should be three at least.
2. The representation of the rotation matrix c^s as a product of three simplest matrices of type (5.7) is not a unique one. For example, we know such representations as (Lur'e 1961)

$$\begin{aligned} c_s^{s-1} \equiv c^s &= c_1(\theta_4^s)c_2(\theta_5^s)c_3(\theta_6^s) && (5.8)\\ c_s^{s-1} \equiv c^s &= c_1(\theta_4^s)c_2(\theta_5^s)c_3(\theta_6^s) && (5.9)\\ c_s^{s-1} \equiv c^s &= c_3(\theta_6^s)c_1(\theta_4^s)c_3(\theta_6^s) && (5.10)\end{aligned}$$

   that are called 'aeroplane', 'ships' and Euler's, respectively. The existence of the mentioned decompositions binds on the traditions of such fields as an aviation, rocket–space technics, shipbuilding, gyroscope theory, *etc.*
3. Formally, each simplest rotation (5.7) can be represented as a product of transvection (shift) and dilation, for example

$$c_3(\theta_6^s) = \tau_{21}(\tan\theta_6^s)\,\mathrm{diag}\{\cos\theta_6^s, 1/c\theta_6^s, 1\}\tau_{21}^{-1}(\tan\theta_6^s) \tag{5.11}$$

   But shifts and dilations change distances between the mechanical system points that is forbidden by proposition P 5.1.
4. The condition could not be fulfilled if in the kinematical couple $(s-1; s)$ there exist constant shift and rotations except the varying ones.
5. When studying rigid body rotations w.r.t. 'small' angles Euler's decomposition (5.10) is the most suitable. The reason is that, according to k–deformator definition (2.11), the rotation matrix (5.7) is of the form

$$c^s = E + \Delta c^s \approx E + < \begin{pmatrix} \theta_4^s \\ 0 \\ \theta_6^s + \alpha_6^s \end{pmatrix} > \tag{5.12}$$

   which differs from

$$c^s = E + \Delta c^s \approx E + < \theta^s >,\ \theta^s = \mathrm{col}\{\theta_4^s, \theta_5^s, \theta_6^s\} \tag{5.13}$$

   for decompositions (5.8) and (5.9). Troubles with the use of representation (5.12) begin when the sum of two small angles θ_6^s and α_6^s is necessary to be considered small, too, that is not always true.
6. According to definition (3.24), from relation (5.13) follows that the matrix

$$\Delta_d^{00} = < \theta^s > \tag{5.14}$$

   is the medium deformation matrix under condition that this medium is a rigid body, *i.e.*, the deformations of the medium (the rigid body) are the angles of rotation of this medium when it is transformed from the initial position in the given one.

Definition 5.6 *The triple*

$$\theta_s^{s-1} \equiv \theta^s = \mathrm{col}\{\theta_4^s, \theta_5^s, \theta_6^s\} \tag{5.15}$$

of scalar functions is called angles of simplest rotations of the rigid body in the simple free motion w.r.t. the basis $[\mathbf{e}^0]$ of the root frame $\mathbf{E}_0$.

Comments

1. The triple

$$\theta^s = \mathrm{col}\{\theta_4^s, \theta_5^s, \theta_6^s\} \notin \mathbf{R}_3 \tag{5.16}$$

of scalar functions is not a vector it is an element of 3–dimensional manifold (see D 2.2) (Hirsch 1976).

2. If the angles θ_p^s (see (5.15)) are so small that we may use decomposition (5.16), then column (5.16) can be considered at the same exactness level as the vector

$$\theta^s = \mathrm{col}\{\theta_4^s, \theta_5^s, \theta_6^s\} \in \mathbf{R}_3 \tag{5.17}$$

since in this case

$$c^s(\theta^s)c^s(\alpha^s) \cong E + < \theta^s + \alpha^s > \tag{5.18}$$

Definition 5.7 *Let:*

1. *o^s and θ^s be the vector of shift and the angles of the simplest rotations of E_s in the simple motion w.r.t. the basis $[\mathbf{e}^0]$ of the root frame $\mathbf{E}_0$;*
2. *$o^{s\cdot}$ and $\theta^{s\cdot}$ be the derivatives of the vector of shift and the angles of the simplest rotations of the rigid body in the simple motion w.r.t. the basis $[\mathbf{e}^0]$ of the root frame $\mathbf{E}_0$*

$$o^{s\cdot} = \mathrm{col}\{o_1^{s\cdot}, o_2^{s\cdot}, c_3^{s\cdot}\} \tag{5.19}$$

$$\theta^{s\cdot} = \mathrm{col}\{\theta_4^{s\cdot}, \theta_5^{s\cdot}, \theta_6^{s\cdot}\} \tag{5.20}$$

Then: 1. the elements o_p^s and θ_p^s of the columns o^s and θ^s (see (5.3), (5.15)) and the elements $o_p^{s\cdot}$ and $\theta_p^{s\cdot}$ of the columns $o^{s\cdot}$ and $\theta^{s\cdot}$ are called canonical generalized coordinates and velocities of the rigid body E_s in the simple free motion w.r.t. the root frame $\mathbf{E}_0$

$$q^s = \mathrm{col}\{o^s, \theta^s\}, q^{s\cdot} = \mathrm{col}\{o^{s\cdot}, \theta^{s\cdot}\}$$

2. *the manifold*

$$\mathbf{Q}_q = \{x^s : x = \mathrm{col}\{q^s, q^{s\cdot}\}\} \tag{5.21}$$

is called a phase space of the rigid body E_s in the simple free motion w.r.t. the root frame $\mathbf{E}_0$;

3. *the* 6*–dimensional submanifold* $\mathbf{K}_q$ *of* $\mathbf{Q}_q$ *is called a configuration space of the rigid body* E_s *in the simple free motion w.r.t. the root frame* $\mathbf{E}_0$*.*

Comments

1. The canonical generalized coordinates of the rigid body E_s are not unique (ones) but they are convenient for further constructions. All obtained below results can be calculated in any other generalized coordinates of the rigid body if the relation between them and the canonical coordinates are known.
2. The dimension of each of columns (5.21) equals always to 12 in the simple free motion of a rigid body, and $o^s_p \neq 0, \theta^s_{p+3} \neq 0$, for $p = \overline{1,3}$.

Definition 5.8 *Let the rigid body motion w.r.t. the root frame* $\mathbf{E}_0$ *be a simple free motion of shift (5.3). Then:*

1. *the derivative*

$$v^{00}_s = o^{00\bullet}_s \equiv o^{s\bullet} \tag{5.22}$$

is called a vector of velocities of E_s *shift in the simple free motion w.r.t. the root frame* $\mathbf{E}_0$*;*

2. *the vector*

$$v^{0s}_s = c^{0,T}_s v^{00}_s \tag{5.23}$$

is called a vector of quasi–velocities of the shift of E_s *in the simple free motion w.r.t. the root frame* $\mathbf{E}_0$*.*

Comments

1. The vector of the quasi–velocities of shift is a result of calculating the vector of shift velocities in the basis $[\mathbf{e}^s]$ of the frame $\mathbf{E}_s$ attached to the rigid body E_s. The pointed vector depends on the rigid body motion velocities and on its orientation in the root frame. There do not exist functions–coordinates whose derivatives are equal to quasi–velocities vector, *i.e.*, the quasi–velocities are not the velocities of any motion (this explains the term 'quasi').
2. The coordinates of the vector of shift quasi–velocities are linear combinations of the generalized velocities of shift (see (5.21))

$$v^{0s}_s = c^{0,T}_s v^{00}_s \tag{5.24}$$

whose coefficients are the elements of the transpose $c^{0,T}_s$ of the rotation matrix.

3. The 6–dimensional submanifold

$$\{x^s : x = \mathrm{col}\{o^s, o^{s\bullet}\}\} \tag{5.25}$$

of the rigid body shift motions is the vector space $\mathbf{R}_6$.

Proposition 5.4 *Let: 1. the rigid body motion w.r.t. the root frame* $\mathbf{E}_0$ *be a simple free motion;*

2. *$c_s^{s-1} \equiv c^s$ be the rigid body rotation matrix w.r.t. the basis $[\mathbf{e}^0]$ of the frame $\mathbf{E}_0$ (see (5.7))*

$$c_s^0 \equiv c^s = c_1(\theta_4^s)c_2(\theta_5^s)c_3(\theta_6^s) \tag{5.26}$$

3. *$c_s^{0\bullet}$ be the derivative of the rigid body rotation matrix.*

Then: 1. the vector w_s^0 whose coordinate column w_s^{00} in the basis $[\mathbf{e}^0]$ of the frame $\mathbf{E}_0$ generates the skew–symmetric matrix (Konoplev 1985).

$$< \omega_s^0 >^0 = c^{s\bullet} c^{s,T} \tag{5.27}$$

is called a vector of the instantaneous angular velocity of the rotation of E_s

$$\omega_s^0 = [\mathbf{e}^0]\omega_s^{00} \tag{5.28}$$

2. *the vector straight–line $D(\omega)$ with the unit vector $e_\omega^0 = \omega_s^0 \|\omega_s^0\|^{-1}$ is called an instantaneous axis of the rigid body rotation.*

Proof See relation (5.7).

Definition 5.9 *The coordinate column e_ω^{0s} of the rigid body angular velocity vector e_ω^0 in the basis $[\mathbf{e}^s]$ of the frame $\mathbf{E}_s$ attached to the rigid body is called a rigid body angular quasi–velocities vector (Konoplev 1989a and 1989b, Konoplev et al. 2001).*

By analogy with relation (5.23), we have

$$\omega_s^{0s} = c_s^{0,T}\omega_s^{00} \tag{5.29}$$

Proposition 5.5 *The vectors of angular velocities and of quasi–velocities are the coordinate columns of one and the same vector of the instantaneous angular velocity of the rigid body w.r.t. the different bases (the root one $[\mathbf{e}^0]$ and $[\mathbf{e}^s]$ attached to the rigid body)*

$$\omega_s^0 = [\mathbf{e}^0]\omega_s^{00} = [\mathbf{e}^s]\omega_s^{0s} \tag{5.30}$$

Proposition 5.6 *Let:*

1. *$< \omega_s^0 >^0$ and $< \omega_s^0 >^s$ be skew–symmetric matrices generated by the vectors of instantaneous angular velocities (5.28) and of the quasi–velocities (5.29) of the rigid body, respectively;*
2. *$c_s^0 \equiv c^s \in \mathcal{SO}_{yt}^q(\mathcal{R}, 3)$ be the matrix of the rigid body rotation.*

Then the matrix c^s is a solution of one of the two matrix differential equations of the kinematics of the rigid body simple free rotation (Konoplev et al. 2001)

$$c^{s\bullet} = < \omega_s^0 >^0 c^s \tag{5.31}$$

$$c^{s\bullet} = c^s < \omega_s^0 >^s \tag{5.32}$$

Proof The first equation is obtained from (5.27), and the second one by representation of equality (5.27) in the basis $[\mathbf{e}^s]$ by the operation of conjugation $c^{s,T}(\cdot)c^s$.

Comments

1. Each of the matrix differential equations (5.31) and (5.32) is equivalent to three scalar differential equations (because of the skew–symmetry of the matrix coefficients $< \omega_s^0 >^0$ and $< \omega_s^0 >^s$).
2. The vectors of instantaneous angular velocities and of the quasi–velocities are not mathematical models of any independent rotation of the rigid body ('left–hand side' or 'right–hand side' because of left–hand and right–hand side decompositions of the skew–symmetric matrices in equations (5.31) and (5.32)). These velocities are coordinate columns of one and the same vector of the rigid body instantaneous angular velocity w.r.t. the root frame basis in different bases – the root one and attached to the rigid body one – see (5.30).

Proposition 5.7 *Let: 1.* $\mathbf{E}_0 \to \mathbf{E}_1 \to \mathbf{E}_2$ *be the simple composite motion of the rigid body w.r.t. the root frame* $\mathbf{E}_0$*;*

2. c_2^{00}*,* c_1^{00}*,* c_2^{11} *be the matrices of the resulting rotation* $\mathbf{E}_0 \to \mathbf{E}_2$ *and of the intermediate (transport and relative) rotations* $\mathbf{E}_0 \to \mathbf{E}_1$*,* $\mathbf{E}_1 \to \mathbf{E}_2$*, respectively, where the upper inside index indicates the basis in which the matrix is calculated;*

3. ω_2^{00}, ω_1^{00}, ω_2^{11}, ω_2^{02}, ω_1^{02}, ω_2^{12} *be the vectors of the corresponding angular velocities and quasi–velocities;*

4. ω_1^{00}*,* ω_2^{10} *be the vectors of transport rotation and of relative rotation of the rigid body that are calculated in the basis* $[\mathbf{e}^0]$ *of the root frame* $\mathbf{E}_0$*.*

Then the following relations are fulfilled (they are traditionally called theorems of adding angular velocities)

$$\omega_2^{00} = \omega_1^{00} + \omega_2^{10}, \ \omega_2^{02} = \omega_1^{02} + \omega_2^{12} \tag{5.33}$$

$$\omega_2^{00} = \omega_1^{00} + c_1^{00}\omega_2^{11} \tag{5.34}$$

Proof Since c_2^{00}, c_1^{00}, $c_2^{11} \in \mathcal{SO}_{yt}^q(\mathcal{R}, 3)$, we have

$$c_2^{00} = c_1^{00}c_2^{11} \to c_2^{00\cdot} = c_1^{00\cdot}c_2^{11} + c_1^{00}c_2^{11\cdot} \tag{5.35}$$

Using kinematics matrix equations (5.31) we obtain

$$< \omega_2^0 >^0 = < \omega_1^0 >^0 c_2^{00} + c_1^{00} < \omega_2^1 >^1 c_2^{11} \tag{5.36}$$

Multiplying on the right–hand side the above equation by the matrix $c_2^{00,T}$ and taking into account that $c_1^{00} < \omega_2^1 >^1 c_2^{11} c_2^{00,T} = c_1^{00} < \omega_2^1 >^1 c_1^{00,T} = < \omega_2^1 >^0$, we arrive at relation (5.33). If transferring (5.33) w.r.t. the basis $[\mathbf{e}^2]$, the second result is obtained. The third equation (5.34) is obtained from the first equation in (5.33) when the second term in it is written in the basis $[\mathbf{e}^1]$, *i.e.*, $\omega_2^{10} = c_1^{00}\omega_2^{11}$.

Comments

1. Since equalities (5.33) coincide in different bases, the upper inside indexes can be omitted, and we obtain the proposition not for the coordinate columns but for the angular velocities vectors, *i.e.*,

$$\omega_2^0 = \omega_1^0 + \omega_2^1 \tag{5.37}$$

2. Equalities (5.33) and (5.34) can be naturally generalized in the case of an arbitrary composite rotation of a rigid body $\mathbf{E}_0 \to \mathbf{E}_1 \to \mathbf{E}_2 \cdots \to \mathbf{E}_n$

$$\omega_n^{00} = \omega_1^{00} + \omega_2^{10} + \cdots + \omega_n^{n-1,0} \tag{5.38}$$
$$\omega_n^{00} = \omega_1^{00} + c_1^{00}\omega_2^{11} + \cdots + c_{n-1}^{00}\omega_n^{n-1,0} \tag{5.39}$$

3. In all above equalities, the rigid body rotation matrices are not simplest – see (5.6).

Proposition 5.8 *Let: 1.* $c_p(\theta_{p+3}^s)$ *be the simplest rotation matrix (5.6) of s–body,* $p = \overline{1,3}$*;*

2. ω_s^{00} *be the vector of angular velocities in the pointed rotation;*

3. θ_{p+3}^s, $\theta_{p+3}^{s\cdot}$ *be the generalized coordinate and the generalized velocity in the pointed rotation.*

Then the following relations are valid (Konoplev et al. 2001):

$$< \omega_s^0 >^0 \; = \; < e_p^0 > \theta_{p+3}^{s\cdot} = < e_p^s > \theta_{p+3}^{s\cdot} \tag{5.40}$$
$$\omega_s^{00} = e_p^0\,\theta_{p+3}^{s\cdot} = e_p^s\,\theta_{p+3}^{s\cdot} \tag{5.41}$$

Proof According to equations (5.27) and (5.31), we have $< \omega_s^0 >^0 = c^{s\cdot}c^{s,T} = c_p^{\cdot}(\theta_{p+3}^{s\cdot})c_p^T(\theta_{p+3}^{s\cdot}) = < e_p^0 > \theta_{p+3}^{s\cdot} = < e_p^s > \theta_{p+3}^{s\cdot}$ since $e_p^0 = e_p^s$ in the case of the simplest rotation. Equality (5.41) is obtained from (5.40) by transferring to vectors what generate skew–symmetric matrices.

Proposition 5.9 *Let: 1.* $\mathbf{E}_{s-1} \to \mathbf{E}_s$ *be the simple free rotation of s–body w.r.t. the root frame* $\mathbf{E}_{s-1}$ *in the kinematic couple* $(s-1; s)$*;*

2. ε_s^{s-1} *be the* 3×3*–matrix of the form*

$$\varepsilon_s^{s-1} = \left[\; c_3^T(\theta_6^s)c_2^T(\theta_5^s)e_1^s \quad c_3^T(\theta_6^s)e_2^s \quad e_3^s \;\right] \tag{5.42}$$

with 3×1*–columns* $c_3^T(\theta_6^s)c_2^T(\theta_5^s)e_1^s$, $c_3^T(\theta_6^s)e_2^s$ *and* e_3^s.

Then the kinematics differential vector equations of the rigid body simple free rotation are of the form of linear transformation of the column $\theta^{s\cdot}$ *with a matrix* ε_s^{s-1} *(Konoplev et al. 2001)*

$$\omega_s^{0s} = \varepsilon_s^{s-1}\theta^{s\cdot} \tag{5.43}$$

Proof To simplify the writing let us rename $\omega_s^{0s} \equiv \omega_3^{0s}$. Let in equality (5.39) all matrices of intermediate rotations be simplest $c_s^{s-1;s-1} = c_p^{\cdot}(\theta_{p+3}^s)$, $s = \overline{1,3}$, $p = \overline{1,3}$. Then

$$\omega_s^{0s} \equiv \omega_3^{0s} = \omega_1^{00} + c_1^{00}\omega_2^{11} + c_2^{00}\omega_3^{22} \tag{5.44}$$

Multiplying equality (5.44) on the left–hand side by the matrix $c_2^{00,T}$ and taking into account (5.29), we obtain

$$\omega_s^{0s} = c_2^{00,T}\omega_1^{00} + c_3^{11,T}\omega_2^{11} + \omega_3^{22} \tag{5.45}$$

Due to $c_2^{00,T} = c_3^T(\theta_6^s)c_2^T(\theta_5^s)c_1^T(\theta_4^s)$, $c_1^T(\theta_4^s)\omega_1^{00} = \varepsilon_1^s\theta_4^s$, $c_3^{11,T}\omega_2^{11} = c_3^T(\theta_6^s)c_2^T(\theta_5^s)\omega_2^{11} = c_3^T(\theta_6^s)\omega_2^{12} = c_3^T(\theta_6^s)\varepsilon_2^s\theta_5^s$, $\omega_3^{22} = \varepsilon_3^s\theta_6^s$, we will obtain the result in the matrix form.

Definition 5.10 *Let:*

1. $\mathbf{E}_{s-1} \to \mathbf{E}_s$ *be the simple free motion of s–body w.r.t. the root frame* $\mathbf{E}_{s-1}$ *in the kinematic couple* $(s-1; s)$*;*
2. $q^{s\cdot} = \mathrm{col}\{o^{s\cdot}, \theta^{s\cdot}\}$ *be the generalized velocities of the rigid body simple free motion;*
3. M_s^{s-1} *be* 6×6*–matrix of the form*

$$M_s^{s-1} = \mathrm{diag}\{c_s^{s-1,T}, \varepsilon_s^{s-1}\} \tag{5.46}$$

Then: 1. the 6*–dimensional vector*

$$V_s^{s-1;s} = \mathrm{col}\{v_s^{s-1;s}, \omega_s^{s-1;s}\} \tag{5.47}$$

is called quasi–velocity vector of the simple free motion of the s–body w.r.t. the root frame $\mathbf{E}_{s-1}$*;*

2. *the relation that is the union of relations (5.24) and (5.43)*

$$V_s^{s-1;s} = M_s^{s-1} q^{s\cdot} \tag{5.48}$$

is called an equation of the kinematics of the simple free motion of s–body w.r.t. the root frame $\mathbf{E}_{s-1}$ *(Konoplev et al. 2001).*

Comments

1. In particular, when s–body moves w.r.t. the root frame $\mathbf{E}_0$, we have

$$V_s^{0s} = M_s^0 q^{s\cdot} \tag{5.49}$$

2. The equality is easily written in the form

$$q^{s\cdot} = (M_s^{s-1})^{-1} V_s^{s-1;s} \tag{5.50}$$

if the inverse exists.

3. From (5.48) it follows that the vector space of the quasi–velocities $V_s^{s-1;s}$ in the simple free motion of s–th rigid body decomposes into two invariant subspaces of the matrix M_s^{s-1}, namely the subspace of the quasi–velocities of shift and of rotation. Thus, the mentioned simple motions of the rigid body are kinematically independent.
4. From p. 3 it does not follow that these motions are dynamically independent (in view to motion equations (see section 5.4)).

5.2.3. *Kinematics of simple free motion in presence of constructive shifts and rotations*

Definition 5.11 *Let:*

1. *unlike requirement (5.10), in the instant of the start of the motion* $\mathbf{E}_{s-1} \to \mathbf{E}_s$

$$\mathbf{E}_{s-1} \neq \mathbf{E}_s \tag{5.51}$$

2. *in the kinematical couple* $(s-1;s)$, *besides the variables of shift (see (5.3)) and of rotation (see (5.15)) there exist a constant shift and a constant rotation;*
3. $\mathbf{E}_{sc} = (o_{sc}, [\mathbf{e}^{sc}])$ *be an additional conditional frame that considers the presence of a shift and of a rotation in the kinematical couple* $(s-1;s)$;
4. p_{sc}^{s-1} *be the vector of the constant shift of* $\mathbf{E}_{sc}$ *w.r.t.* $\mathbf{E}_{s-1}$

$$p_{sc}^{s-1;s-1} = p^{sc} = \mathrm{col}\{p_1^{sc}, p_2^{sc}, p_3^{sc}\} \tag{5.52}$$

It is possible some of the vector coordinates (5.52) to be equal to 0;

5. φ_{sc}^{s-1} *be a triple of constant angles of rotation of* $[\mathbf{e}^{sc}]$ *w.r.t.* $[\mathbf{e}^{s-1}]$

$$\varphi_{sc}^{s-1} = \varphi^{sc} = \mathrm{col}\{\varphi_1^{sc}, \varphi_2^{sc}, \varphi_3^{sc}\} \tag{5.53}$$

It is possible some of the column elements (5.53) to be equal to 0.

Then: 1. the frame $\mathbf{E}_{sc}$ *is called constructive (Konoplev et al. 2001);*

2. *the vector* p^{sc} *is called a constructive shift vector in the kinematical couple* $(s-1;s)$;
3. *the column* φ^{sc} *is called a column of the constructive angles of rotation in the kinematical couple* $(s-1;s)$;
4. *the vector* o^s *is called a vector of the functional shift in the kinematical couple* $(s-1;s)$ *under condition that the constructive frame* $\mathbf{E}_{sc}$ *is considered as a root frame;*
5. *the column* θ^s *(see (5.15)) is called a column of the functional angles of the rotation in the kinematical couple* $(s-1;s)$ *under condition that the constructive frame* $\mathbf{E}_{sc}$ *is considered as a root frame.*

Comments

1. According to the assumed research scheme, when the simple free motion $\mathbf{E}_{s-1} \to \mathbf{E}_s$ starts not under condition $\mathbf{E}_{s-1} = \mathbf{E}_s$, this motion is formed as $\mathbf{E}_{s-1} \to \mathbf{E}_{sc} \to \mathbf{E}_s$, *i.e.*, as a simple composite motion (see D 5.3) where the first motion $\mathbf{E}_{s-1} \to \mathbf{E}_{sc}$ (the constructive one) is realized by a constant vector of shift and by constant angles of rotation.
2. The functional vector $o_s^{sc} = o^s$ of shift and rotation angles $\theta_s^{sc} = \theta^s$ coincide with the defined above generalized coordinates (5.21) under condition that the constructive frame $\mathbf{E}_{sc}$ is considered as a root frame.

Proposition 5.10 *In the case of constructive shift and rotation* $\mathbf{E}_{s-1} \to \mathbf{E}_{sc} \to \mathbf{E}_s$ *the kinematics equation of the rigid body simple free motion coincides with the kinematics equation of the rigid body simple free motion if no shift and rotation exist (see (5.49)) under condition that the matrix* M_s^{s-1} *(see (5.46)) is substituting by the matrix* M_s^{sc}

$$V_s^{s-1;s} = M_s^{sc} q^{s\cdot},\ q^s = \mathrm{col}\{o_s^{sc;sc}, \theta_s^{sc}\} \tag{5.54}$$

Proof It is necessary to prove that the quasi–velocities vector $V_s^{s-1;s}$ coincides with the vector $V_s^{sc;s}$ of quasi–velocities of the rigid body E_s w.r.t. the constructive frame $\mathbf{E}_{sc}$. For the resulting vector $o_s^{s-1;s-1}$ of the shift we have

$$o_s^{s-1;s-1} = p_{sc}^{s-1;s-1} + c_{sc}^{s-1;s-1} o_s^{sc;sc} \tag{5.55}$$

Differentiating equality (5.55) and taking into account that the matrix $c_{sc}^{s-1;s-1}$ of the constructive rotations is constant, we obtain

$$v_s^{sc;sc} = 0 + c_{sc}^{s-1;s-1} v_s^{sc;sc} = v_s^{sc;s-1} \tag{5.56}$$

For the resulting rotation matrix $c_s^{s-1;s-1}$ we have

$$c_s^{s-1;s-1} = c_{sc}^{s-1;s-1} c_s^{sc;sc} \tag{5.57}$$

Differentiating relation (5.57), we obtain

$$c_s^{s-1;s-1\bullet} = c_{sc}^{s-1;s-1} c_s^{sc;sc\bullet} \tag{5.58}$$

Using the kinematics equation (5.32) we arrive at the relation

$$c_s^{s-1;s-1} < \omega_s^{s-1} >^s = c_{sc}^{s-1;s-1} c_s^{sc;sc} < \omega_s^{sc} >^s$$

Whence $< \omega_s^{s-1} >^s = < \omega_s^{sc} >^s$ if (5.57) is taken into account, and therefore

$$\omega_s^{s-1;s} = \omega_s^{sc;s} \tag{5.59}$$

Thus, repeating the proof of equality (5.48), we obtain (see (5.54))

$$V_s^{s-1;s} = V_s^{sc;s} = M_s^{sc} q^{s\bullet}$$

5.2.4. *Kinematics of simple constrained motion*

Comment When a rigid body moves, it can mechanically contact with other rigid bodies. In result, the dynamical screw $F_0^0(\urcorner B,\ B)$ concludes the dynamical screws of these rigid bodies deformations, the motion of the rigid body B being free (6 degrees of freedom). In many practical problems the elasticity of external bodies can be neglected if we suppose that the basic body B and the external bodies are rigid bodies. Such a supposition involves principally new concepts that are not mechanical ones according to D 2.6 but that are useful in the applied mechanics, namely: the configuration submanifold of the phase space of the body (see D 5.7) (constraints) that the body phase point is situated on in the motion process, and the dynamical screw $F_0^0(\urcorner B,\ B)$ of the constraint reactions which hold the body phase point on the above mentioned submanifold. Thus, it is necessary to understand that the dynamical screw of constraint reactions (these are abstract new unknowns which can be considered as dynamical screw coordinates, if desired) is not principally a part of the class of inertial, gravitational and deformation screws, *i.e.*, it does not respond to the primary properties of Galilean mechanics Universe (see Chapter 2), in the nature it does not exist, and it is considered in order to simplify practical problems.

Definition 5.12 *Let the rigid body B_s move in such a way that the phase point x^s moves along some submanifold in the phase space for some indexes p defined by the equalities*

$$o_p^s = \text{const}, \;\; \theta_{p+3}^s = \text{const}, \;\; p \in \overline{1,3} \tag{5.60}$$

under condition that for the rest coordinates, equalities of form (5.60) are absent. Then:

1. *equalities (5.60) are called simplest (canonical) holonomic constraints, imposed on the motion of the rigid body B;*
2. *the simple motion of the rigid body B w.r.t. the root frame $\mathbf{E}_{s-1}$ is called a simple constrained motion or a simple motion with the simplest holonomic constraints;*
3. *the column of the independent (not related with equalities (5.60)) varying elements of the columns o^s and θ^s is called generalized coordinates of the rigid body in the simple constrained motion w.r.t. the root frame $\mathbf{E}_{s-1}$*

$$q^s = \text{col}\{\cdots, o_p^s, \cdots, \cdots, \theta_p^s, \cdots\} \tag{5.61}$$

4. *the column of the variables of the non–zero elements of the columns $o^{s\bullet}$ and $\theta^{s\bullet}$, is called generalized velocities of the rigid body in a simple constrained motion w.r.t. the root frame $\mathbf{E}_{s-1}$*

$$q^{s\bullet} = \text{col}\{\cdots, o_p^{s\bullet}, \cdots, \cdots, \theta_p^{s\bullet}, \cdots\} \tag{5.62}$$

5. *the number n of the generalized coordinates of the rigid body B_s in the simple motion is called a number of degrees of freedom (of the rigid body B_s in the simple motion) (in the case of a simple motion $n = 6$, in the case of a simple constrained motion $n \in \overline{1,5}$).*
6. *the number $6 - n$ is called a class of the kinematical couple $(\mu, k-1; 1k)$.*

Proposition 5.11 *Let the elements of the kinematical chain be rigid bodies (with the frames $\mathbf{E}_{s-1}$ attached to them) that take part in several constructive (from* 0 *to* 6*) and a few (from* 1 *to* 6*) simplest functional motions w.r.t. $\mathbf{E}_{sc}$ (defined by the generalized coordinates q_i^s, $i \in \overline{1,6}$ where $q_i^s = o_i^s$, $i = 1,2,3$, or $q_i^s = \theta_i^s$, $i = 4,5,6$). Then the relative position of the elements of the kinematical couple $(s-1; s)$ is defined by the following 12×1 and 6×1–columns–configurations (Konoplev et al. 2001)*

$$\mathcal{R}_s^{s-1} = \text{col}\{R_{sc}^{s-1}, R_s^{sc}\}, \; R_{sc}^{s-1} = \text{col}\{p^{sc}, \varphi^{sc}\}, \; R_s^{sc} = \text{col}\{o^s, \theta^s\} \tag{5.63}$$

Besides, in the constructive configuration R_{sc}^{s-1} up to 6 constants can differ from 0*, but in the functional one R_s^{sc} from 1 to 5 variables can be different from* 0 *(Konoplev et al. 2001) (in the alternate case it means either a non–existence of a motion, or a free motion).*

Definition 5.13 *Let:*

1. *on the motion* $\mathbf{E}_{sc} \to \mathbf{E}_s$ *of the body* E_s *w.r.t. the constructive frame* $\mathbf{E}_{sc}$ *in the kinematical couple* $(s-1; s)$ *there be imposed the canonical holonomic constraints (5.60)*

$$o_p^s = 0,\ p \in \overline{1,3},\ \theta_p^s = 0,\ p \in \overline{4,6} \tag{5.64}$$

i.e., the motion $\mathbf{E}_{sc} \to \mathbf{E}_s$ *of the body* E_s *concerning these coordinates is 'forbidden';*

2. f_i^s, $i \in \overline{1,6}$, $i \neq p$, *be a* 6*–dimensional unit vector with* 1 *situated on* i*–th place and* 0 *on the other places*

$$f_i^s = \mathrm{col}\{\underset{1}{0}, \underset{2}{0}, \cdots, \underset{i}{1}, \cdots, \underset{6}{0}\} \tag{5.65}$$

Then the vectors are called 6*–dimensional unit vectors of the movability axes in the kinematical couple* $(s-1; s)$ *(i.e., the unit vectors of the axes where the motion of the body with constraints (5.64) is 'allowed' (Konoplev 1992 and 1993, Konoplev et al. 2001).*

Proposition 5.12 *Let:*

1. $R_s^{sc} = \mathrm{col}\{o^s, \theta^s\}$ *be a functional configuration of* s*–body in the simple constrained motion w.r.t. the root frame* $\mathbf{E}_{s-1}$ *in the kinematical couple* $(s-1; s)$*;*

2. $\|f_i^s\|$ *be the matrix of the* 6*–dimensional unit vectors of the movability axes of the kinematical couple* $(s-1; s)$ *(see (5.65));*
3. q^s *be the column of the generalized coordinates of the kinematical couple* $(s-1; s)$ *(i.e., non–zero elements of the functional configuration* R_s^{sc}*).*

Then the following relation is fulfilled (Konoplev et al. 2001)

$$R_s^{sc\bullet} = \|f^s\| q^{s\bullet} \tag{5.66}$$

Proposition 5.13 *Let:*

1. *the configuration of the kinematical couple* $(s-1; s)$ *be given in the form* $\mathcal{R}_s^{s-1} = \mathrm{col}\{R_{sc}^{s-1}, R_s^{sc}\}$ *(see (5.63));*
2. M_f^{sc} *be* 6×6*–matrix of form (5.64) where the index* f *means that this matrix realizes a transfer from the movability axis in the mentioned couple to the constructive frame.*

Then: 1. the kinematic equation of s*–body in the simple constrained motion w.r.t. the root frame* $\mathbf{E}_{s-1}$ *in the kinematical couple* $(s-1; s)$ *is of the form (Konoplev 1984, 1986c, 1989a and 1990, Konoplev et al. 2001)*

$$V_s^{s-1;s} = M_f^{sc} \|f^s\| q^{s\bullet} \tag{5.67}$$

2. *the matrix* M_f^{sc} *is the transfer matrix from the generalized velocities* $q^{s\cdot}$ *of* s*–body to the quasi–velocities* $V_s^{s-1;s}$ *of this body (more exactly, from non–orthogonal basis of the movability axes to the orthogonal attached basis* $[\mathbf{e}^{sc}]$*).*

Proof Let's write the kinematical equation (5.54) in the following equivalent form $V_s^{s-1;s} = V_s^{sc;s} = M_f^{sc} R_s^{sc\cdot}$. It rests to use relation (5.66).

5.2.5. *Kinematics of composite motion*

Let us remind that the composite motion of s–body w.r.t. the root frame $\mathbf{E}_0$ is called the motion of this body as an element of the kinematical chain $(s)_-$, *i.e.*, as an element of the set of contra–accessibility of s–body if the number of the kinematical couples in it is not less than 2.

Proposition 5.14 *Let:*

1. *in some kinematical chain there be presented a composite motion of the element* E_s *w.r.t. an element* E_p *and a composite motion of the element* E_p *w.r.t. an element* E_t*, i.e.,* $E_t \to E_p \to E_s$*,* $t < p < s$*,* $s, p,\ t \in \mathbf{N}$*;*
2. $W_{ss}^{ts} \equiv W_s^{ts}$*,* $W_{pp}^{tp} \equiv W_p^{tp}$*,* $W_{ss}^{ps} \equiv W_s^{ps}$ *be the kinematical screws of the composite motions* $E_t \to E_s$*,* $E_t \to E_p$ *and* $E_p \to E_s$*;*
3. V_s^{ts}*,* V_p^{tp}*,* V_s^{ps} *be the quasi–velocities vectors of the same composite motions – see (5.48);*
4. L_j^i *be* 6×6*–matrix of motions in the screws vector space, the matrix being induced by the motion* $E_i \to E_j$ *(see (7.7)).*

Then the following kinematical equalities are fulfilled (Konoplev 1989a and 1990, Konoplev et al. 2001)

$$W_s^{ts} = L_p^s W_p^{tp} + W_s^{ps} \tag{5.68}$$

$$V_s^{ts} = L_p^{s,T} V_p^{tp} + V_s^{ps} \tag{5.69}$$

Proof Transferring all kinematical screws w.r.t. the 'last' frame $\mathbf{E}_s$, we obtain $W_{ss}^{ts} = W_{ps}^{ts} + W_{ss}^{ps}$. It rests to use the obvious equality $W_{ps}^{ts} = L_p^s W_{pp}^{tp}$. Let's introduce a block–diagonal 6×6–matrix ε such that the blocks situated on the main diagonal are zero and the blocks on the second diagonal are the identity 3×3–matrix. So we make sure that the equalities $V_s^{ts} = \varepsilon W_{ps}^{ts}$, $V_s^{ps} = \varepsilon W_{ps}^{ps}$, $\varepsilon\varepsilon = E$, $\varepsilon L_p^s \varepsilon = L_p^{s,T}$ are true. Using the proved equality (5.68) we obtain $\varepsilon W_{ps}^{ts} = \varepsilon L_p^s \varepsilon W_p^{tp} + \varepsilon W_s^{ps}$and whence relation (5.69).

Comment The kinematical equalities (5.68) and (5.69) permit us to calculate the kinematical screw (the quasi–velocities vector) of the absolute motion $E_t \to E_s$ using the analogous characteristics of an arbitrary motion of 'shift' and of the corresponding 'relative' motion $E_t \to E_p \to E_s$.

Proposition 5.15 *Let: 1.* $\mathbf{E}_0 \to \mathbf{E}_1 \to \mathbf{E}_2 \to \cdots \to \mathbf{E}_s$ *be a composite motion of the element* $\mathbf{E}_s$ *in the kinematical chain* $(s)_-$*;*

2. $\mathbf{E}_{i-1} \to \mathbf{E}_i$ *be simple constrained motions in kinematical couples* $(i-1; i)$*;*

3. *$\mathcal{R}_i^{i-1} = \mathrm{col}\{R_{ic}^{i-1}, R_i^{ic}\}$ be the configurations (constructive and functional) (5.63) of kinematical couples $(i-1; i)$;*
4. *$L_i^{i-1} = L_{ic}^{i-1} L_i^{ic}$ be 6×6–matrices of the motions in kinematical couples $(i-1; i)$ that correspond to the configurations of p. 3 and such that relation (see (7.7))*

$$L_s^p = L_{p+1}^p L_{p+2}^{p+1} \cdots L_s^{s-1} \tag{5.70}$$

5. *$V_i^{i-1,i} = \mathrm{col}\{v_i^{i-1,i}, \omega_i^{i-1,i}\}$ be the quasi–velocities vectors of simple constrained motions in kinematical couples $(i-1; i)$ (see (5.67));*
6. *$q^{i\cdot} = \mathrm{col}\{\cdots, q_\alpha^{i\cdot}, \cdots\}, \alpha \in \overline{1,6}$ be the generalized velocities in kinematical couples $(i-1; i)$ (see (5.62));*
7. *$V_s^{0,s}$ be the quasi–velocities vector of the composite motion of s–body w.r.t. the root frame $\mathbf{E}_0$.*

Then the kinematics equation of the composite motion of s–body w.r.t. the root frame $\mathbf{E}_0$ is of the form (Konoplev et al. 2001)

$$V_s^{0,s} = \sum_{i=1}^{s} L_s^{i,T} M_f^{ic} \|f^i\| q^{i\cdot} \tag{5.71}$$

Proof First, we shall show that the quasi–velocities vector of the composite motion of s–body w.r.t. the root frame $\mathbf{E}_0$ is a linear combination of the quasi–velocities of all kinematical couples (kinematical couples $(i-1; i)$) with matrix coefficients $L_s^{i,T}$, i.e.,

$$V_s^{0,s} = \sum_{i=1}^{s} L_s^{i,T} V_i^{i-1;i} \tag{5.72}$$

Indeed, according to relation (5.69) we obtain $V_s^{0,s} = L_s^{1,T} V_1^{0,1} + V_s^{1;s} = L_{s1}^{1,T} V_1^{0,1} + L_s^{2,T} V_i^{1,2} + V_s^{2;s} = L_s^{1,T} V_1^{0,1} + L_s^{2,T} V_i^{1,2} + L_s^{3,T} V_i^{3,4} + \cdots + V_s^{s-1;s}$. Now, in relation (5.72) we must put the values $V_i^{i-1,i} = \mathrm{col}\{v_i^{i-1,i}, \omega_i^{i-1,i}\} = M_f^{ic}\|f^i\| q^{i\cdot}$ of the quasi–velocities of kinematical couples.

Comments

1. The physical and geometrical sense of the statement is simple: the quasi–velocities vector of s–body composite motion is a sum of the quasi–velocities of the simple constrained motions in the kinematical couples of the kinematical chain $(s)_-$, that are recalculated from the attached frames $\mathbf{E}_i$ to the attached to s–body one $\mathbf{E}_0$ with the help of the matrices $L_s^{i,T}$.
2. In particular, for the simple composite constrained motion of a body (two kinematical couples with simplest holonomic constraints) we obtain

$$V_2^{0,2} = L_2^{1,T} V_1^{0,1} + V_s^{1;s} = L_2^{1,T} M_f^{1c} \|f^1\| q^{1\cdot} + M_f^{2c} \|f^2\| q^{2\cdot} \tag{5.73}$$

Proposition 5.16 *Let:*

1. *s–body be an element of a tree–like graph and participate in a composite motion w.r.t. the root frame $\mathbf{E}_0$;*

2. s_s^t *be* 1×6*–row of the form*

$$s_s^t = \|f^t\|^T M_f^{tc,T} L_s^t \tag{5.74}$$

Then the kinematics equation of the above composite motion of s–body is of the form

$$V_s^{0,s} = \sum_{i \in (s)_-} s_s^{t,T} q^{s\cdot} \tag{5.75}$$

Proof is reached by changing the notations in relation (5.72).
Comments

1. Relations (5.72) and (5.75) permit us (using an algebraic mathematical software, for example MatLab) to calculate the quasi–velocities vector of the composite motion of the rigid body that participate in intermediate simple constrained motions (as many as wanted) without using traditionally awkward scalar equations. Thus, it is sufficient to calculate the coefficients $s_s^{t,T}$ by means of the simple standard relations (5.74). A lot of similar problems are considered in (Konoplev *et al.* 2001).
2. The same relation resolved w.r.t. the generalized velocities of a fixed kinematical couple, are differential equations for determining the program motions in a given kinematical couple that guarantee the s–body motion to be realized with a given quasi–velocities vector $V_s^{0,s}$ (Konoplev *et al.* 2001).
3. If we introduce 1×6–rows of the form

$$s_s^{t-\alpha} = f_\alpha^{t,T} M_f^{tc,T} L_s^t \tag{5.76}$$

where $\alpha \in \overline{1,6}$ is the numbers of the movability axes in the kinematical couple $(t-1;t)$, then the matrix s_s^t in (5.74) can be represented in the form of a column consisting of these rows

$$s_s^t = \mathrm{col}\{\cdots, s_s^{t-\alpha}, \cdots\} \tag{5.77}$$

In particular, if the kinematical couple refers to the 5–th class (one degree of freedom), then

$$s_s^t \equiv s_s^{t-\alpha} \tag{5.78}$$

4. For kinematical couples of 5–th class and of 4–th one (that are cylindrical couples (Konoplev *et al.* 2001) but not screw ones!) we have

$$\|f^t\|^T M_f^{tc,T} L_s^t \equiv \|f^t\|^T \tag{5.79}$$

5.3. Fundamentals for representations of group of rigid body rotations

5.3.1. *Exact representation of group of two–dimensional rotations in Rodrigues–Hamilton rotation group*

Let us consider the ‘plane’ rotations of a rigid body.

Proposition 5.17 *Let:*

1. *the projection of the simple free motion of a rigid body on the submanifold of rotation with a matrix* $c^0 \in \mathcal{SO}^q_{yt}(\mathcal{R}, 3)$ *(a simple free rotation) be considered;*

2. $\mathbf{Q}_4$ *be a 4–dimensional manifold whose elements are* $x^0 = \mathrm{col}\{x_0, \xi^0\}$ *where* x_0 *is a real number,* $\xi^0 = \mathrm{col}\{\xi^0_1, \xi^0_2, \xi^0_3\}$ *is a real vector (the coordinate column of the vector* $\xi \in \mathbf{V}_3$ *in the basis* $[\mathbf{e}^0]$, $\xi = [\mathbf{e}^0]\xi^0$*)*

$$\mathbf{Q}_4 = \{x^0 : x^0 = \mathrm{col}\{x_0, \xi^0\},\ \xi^0 = \mathrm{col}\{\xi^0_1, \xi^0_2, \xi^0_3\} \in \mathbf{R}_3\} \tag{5.80}$$

3. $H_{x0} = x_0 E$ *be a homothety with a coefficient* x_0 *where* E *is a* 4×4*–identity matrix;*
4. $<\xi>^0_4$ *be a skew-symmetric* 4×4*–matrix generated by the vector* ξ^0 *of the form*

$$< \xi >^0_4 = \begin{bmatrix} 0 & -\xi^0_1 & -\xi^0_2 & -\xi^0_3 \\ \xi^0_1 & 0 & -\xi^0_3 & \xi^0_2 \\ \xi^0_2 & \xi^0_3 & 0 & -\xi^0_1 \\ \xi^0_3 & -\xi^0_2 & \xi^0_1 & 0 \end{bmatrix} \tag{5.81}$$

Then the set of 4×4*–matrices*

$$\mathcal{G}(\mathcal{R}, 4) = \{u^0_x : u^0_x = H_{x0} + < \xi >^0_4,\ x \neq 0\} \tag{5.82}$$

is a subgroup of the group $\mathcal{GO}(\mathcal{R}, 4) = \{u^0_x : u^{0,T}_x u^0_x = \|x^0\|^2 E\}$ *of similitudes where the upper index* 0 *means that the matrix is calculated in the basis* $[\mathbf{e}^0]$ *(Konoplev 1996b).*

Proof is accomplished by a simple verification of equalities

$$u^{0,T}_x u^0_x = \|x^0\|^2 E,\ (u^0_x)^{-1} = \|x^0\|^{-2} u^{0,T}_x \tag{5.83}$$

that determine the similitude.

Comments

1. The skew–symmetric matrix $< \xi >^0_4$ in (5.81) is a similitude, too, as $< \xi >^{0,T}_4 < \xi >^0_4 = \|\xi^0\|^2 E$ but the set of these matrices is not a subgroup of the similitude group (since the product of these skew–symmetric matrices is not a skew–symmetric matrix).
2. The matrix $< \xi >^0_4$ can be represented in the following blocked form using known concepts

$$< \xi >^0_4 = \begin{bmatrix} 0 & -\xi^{0,T} \\ \xi^0 & < \xi >^0 \end{bmatrix} \tag{5.84}$$

3. The skew–symmetric matrix $< \xi >^0_4$ is non–singular if $\xi^0 \neq 0$ (unlike the skew–symmetric 3×3–matrix $< \xi >^0$) as

$$\det < \xi >^0_4 = \|\xi^0\|^4 \tag{5.85}$$

4. In analogous way, for the matrix u_x^0 we have (Konoplev 1996b)

$$\det u_x^0 = \|x^0\|^4 \tag{5.86}$$

Proposition 5.18 *Let: 1. $\mathcal{G}(\mathcal{R},4)$ be group (5.82) of matrices with determinants (5.85);*

2. $\mathcal{G}^+(\mathcal{R},4) \subset \mathcal{G}(\mathcal{R},4)$ be the subgroup of the group $\mathcal{G}(\mathcal{R},4)$ consisting of the matrices with positive determinant

$$\mathcal{G}^+(\mathcal{R},4) = \{u_x^0 : u_x^0 = H_{x0} + <\xi>_4^0, \det u_x^0 = \|x^0\|^4\} \tag{5.87}$$

Then: 1. the group $\mathcal{G}^+(\mathcal{R},4)$ is a subgroup of the group consisting of proper similitudes

$$\mathcal{G}^+(\mathcal{R},4) \subset \mathcal{GO}^+(\mathcal{R},4) \tag{5.88}$$

2. the group $\mathcal{G}^+(\mathcal{R},4)$ is called a quartgroup generated by elements of the manifold $\mathbf{Q}_4$ (see (5.80)).

Comments

1. If we add the 0–matrix u_0^0 to the quartgroup and introduce the matrix addition and the multiplication of a matrix with a number on the obtained set, then the vector space

$$\mathcal{G}_{16}^+(\mathcal{R},4) = \mathcal{G}^+(\mathcal{R},4) \cup u_0^0 \tag{5.89}$$

is the isomorphism of $\mathbf{R}_{16}$.

2. $u_s^0 \in \mathcal{T}(\mathcal{R},4)$ are similitudes of quartbody (5.89). The 4–dimensional unit vectors e_s^4 having 1 on s–th place form the basis of $\mathcal{G}_{16}^+(\mathcal{R},4)$. Thus, for any similitude u_s^0 in the above basis we have the following decomposition

$$u_x^0 = \sum_s x_s u_s^0 \tag{5.90}$$

3. The sets $\mathcal{G}^+(\mathcal{R},4)$, $\mathcal{G}_{16}^+(\mathcal{R},4)$ and $\mathcal{T}(\mathcal{R},4)$ consist of ones and the same elements (except 0–matrix u_0^0), but they are different structures in principle on which different 'rules of game' are determined in principle: the elements of the first set can be multiplied and inverses can be found, the elements of the second one can be added and multiplied by a number, the elements of the third one can be added, multiplied and inverses can be found.

Proposition 5.19 *Let $\mathcal{Y}(\mathcal{R},4)$ be the subgroup of the group $\mathcal{G}^+(\mathcal{R},4)$ (see (5.88)) generated by the normed elements $\lambda^0 = x^0\|x^0\|^{-1}$ of the manifold $\mathbf{Q}_4$ (see (5.80)), i.e., by the elements of the submanifold in $\mathbf{Q}_4$, determined by the relation $\|\lambda^0\|_4 = 1$,*

$$\mathcal{Y}(\mathcal{R},4) = \{u_\lambda^0 : u_\lambda^{0,T} u_\lambda^0 = E, \|\lambda^0\|_4 = 1\} \tag{5.91}$$

Then: 1. the group $\mathcal{Y}(\mathcal{R},4)$ is a subgroup of the rotation group $\mathcal{SO}(\mathcal{R},4)$;

2. the group $\mathcal{Y}(\mathcal{R},4)$ is called a Rodrigues–Hamilton group;

3. *the normed element* $\lambda^0 = \mathrm{col}\{\lambda_0^0, \lambda_1^0, \lambda_2^0, \lambda_3^0\}$ *of manifold (5.80) is called a Rodrigues–Hamilton parameter (parameters);*
4. *the submanifold*

$$\mathbf{Y}_\lambda = \{\lambda^0 : \lambda^0 \in \mathbf{Q}_4,\ \|\lambda^0\|_4 = 1\} \tag{5.92}$$

of the normed elements of the manifold $\mathbf{Q}_4$ *is called a Rodrigues–Hamilton sphere (Konoplev 1996b).*

Proof follows from relation (5.83).

Comment One more subgroup of the group of 4–dimensional rotations was found in Chapter 4 (see (4.51)).

Proposition 5.20 *Let: 1.* $\lambda^0 = \mathrm{col}\{\lambda_0^0, \lambda_1^0, \lambda_2^0, \lambda_3^0\} = \mathrm{col}\{\lambda_0^0, \Lambda^0\}$ *be a Rodrigues–Hamilton parameter (5.92),* $\|\Lambda^0\|_3$ *be Euclidean norm of* Λ^0 *in* $\mathbf{R}_3$*;*

2. *for a time* Δt*, the rotation of k–body performs with a constant unit vector* $d_\omega^0 = \mathrm{col}\{d_{w1}^0, d_{w2}^0, d_{w3}^0\}$ *of the rotation axis* D_ω *of the rigid body in* $\mathbf{V}_3$ *(see P 5.4), and with an angular velocity* ω_k^{00}*;*
3. α *be the angle of rotation of k–body with the axis* D_ω *for the time* Δt

$$\alpha = \int \chi_{\Delta t} \omega_k^{00} \mu_1(dt) \tag{5.93}$$

Then among other representations there exists the Rodrigues–Hamilton rotation one of the form

$$u_\lambda^0 = H_0^{\cos\alpha} + < d_\omega >_4^0 \sin\alpha \tag{5.94}$$

Proof $\lambda^0 = \mathrm{col}\{\lambda_0^0, \lambda_1^0, \lambda_2^0, \lambda_3^0\} = \mathrm{col}\{\lambda_0^0, \|\lambda^0\|_3 \,\mathrm{col}\{\overline{\lambda}_1^0, \overline{\lambda}_2^0, \overline{\lambda}_3^0\}\}$, $\overline{\lambda}_i^0 = \|\lambda^0\|_3^{-1} \lambda_i^0$, $i = \overline{1,3}$. But $\|\lambda^0\|_4 = 1$ (see (5.92)) and hence there exists such an angle φ that $\lambda_0 = \cos\varphi$, $\|\Lambda^0\|_3 = \sin\varphi$. The essence of the proposition consists of the statement $\overline{\lambda}_i^0 = d_{wi}^0$, $\varphi = \alpha$.

Comments

1. P 5.20 is basic for a construction of an algebraic theory of 'finite rotations' of a rigid body (Konoplev 1996b).
2. In the evident representation of the Rodrigues–Hamilton parameter $\lambda_0 = \mathrm{col}\{\cos\varphi, \sin\varphi \,\mathrm{col}\{\overline{\lambda}_1^0, \overline{\lambda}_2^0, \overline{\lambda}_3^0\}\}$, the angle φ coincides with the angle α of the body rotation for a time Δt with a constant unit vector d_ω^0 of the rotation axis, and the unit vector $\mathrm{col}\{\overline{\lambda}_1^0, \overline{\lambda}_2^0, \overline{\lambda}_3^0\}$ coincides with the unit vector (itself) d_ω^0 for the same time. This proposition permits us to obtain useful theoretical and applied results.

Proposition 5.21 *Let:*

1. $\mathbf{P}(\omega) \subset \mathbf{V}_3$ *be a vector plane in* $\mathbf{V}_3$ *(*$o_0 \in \mathbf{P}(\omega)$ *is the origin of the frame* $\mathbf{E}_0$*), being perpendicular to the rotation axis* D_ω *of the rigid body,* $D_\omega \perp \mathbf{P}(\omega)$, $\mathbf{P}(\omega) \cap D_\omega = o_0$*;*
2. $u_\lambda^0 \mid_{\mathbf{P}(\omega)}$ *be the restriction of the Rodrigues–Hamilton operator* $u_\lambda^0 \in \mathcal{Y}(\mathcal{R}, 4)$ *on the plane* $\mathbf{P}(\omega)$*;*

3. representation (5.94) be fulfilled.

Then the plane $\mathbf{P}(\omega)$ *is an invariant space of the operator* u^0_λ

$$u^0_\lambda \mid_{\mathbf{P}(\omega)} \mathbf{P}(\omega) = \mathbf{P}(\omega) \tag{5.95}$$

Proof Let for any $x^0 \in \mathbf{P}(\omega)$ we have $y^0 = u^0_\lambda x^0$, $x^0 = \mathrm{col}\{0, \xi^0\}$, $y^0 = \mathrm{col}\{0, \zeta^0\}$. Let us consider the inner product $\zeta^0 \cdot d^0_\omega =< d_\omega >^0 \xi^0 \cdot d^0_\omega = 0$ regarding the property of three–linear forms, and hence $\zeta^0 \perp d^0_\omega \to y^0 \in \mathbf{P}(\omega)$ for any $x^0 \in \mathbf{P}(\omega)$.

Proposition 5.22 *Let: 1.* $\mathcal{SO}(\mathcal{R}, 2)$ *be the group of* 2*–dimensional (plane) rotations of a rigid body (rotations of the plane* $\mathbf{P}(\omega)$ *(see P 5.21),* $c^0 \in \mathcal{SO}(\mathcal{R}, 2)$*;*

2. representation (5.94) be true.

Then the exact representation of the group $\mathcal{SO}(\mathcal{R}, 2)$ *in the Rodrigues–Hamilton group* $\mathcal{Y}(\mathcal{R}, 4)$ *is*

$$u^0_\lambda \mid_{\mathbf{P}(\omega)} = c^0 \tag{5.96}$$

where c^0 *is the matrix (in the basis* $[\mathbf{e}^0]$*) of the rigid body rotation with an angle* α *w.r.t. a constant rotation axis* D_ω *for a time* Δt*.*

Proof Let us construct the corresponding parameters $\lambda_0 = \mathrm{col}\{\cos\varphi,\ d^0_\omega \sin\varphi\}$ and $u^0_\lambda = H_0^{\cos\alpha} + < d_\omega >^0_4 \sin\alpha$ of a Rodrigues–Hamilton rotation. Let $x^0 \in \mathbf{P}(\omega)$ and $y^0 = u^0_\lambda x^0$, $x^0 = \mathrm{col}\{0, \xi^0\}$, $y^0 = \mathrm{col}\{0, \zeta^0\}$ where ξ^0, ζ^0 are the coordinate columns of the radius–vectors of one and the same point belonging to $\mathbf{P}(\omega)$ before and after the action of the operator $u^0_\lambda \mid_{\mathbf{P}(\omega)}$

$$\zeta^0 = u^0_\lambda \mid_{\mathbf{P}(\omega)} \xi^0 = (H_0^{\cos\alpha} + < d_\omega >^0_4 \sin\alpha) \mid_{\mathbf{P}(\omega)} \xi^0 \tag{5.97}$$

on the point.

From (5.97) follows that $\zeta^0 = \xi^0 \cos\alpha + < d_\omega >^0_4 \xi^0 \sin\alpha = \xi^0 \cos\alpha + \eta^0 \sin\alpha$ where $\eta^0 =< d_\omega >^0_4$, $\eta^0 \perp d^0_\omega$, $\eta^0 \perp \xi^0$, and $\|\eta^0\| = \|d^0_\omega\| \|\xi^0\| \sin\pi/2 = 1\|\xi^0\| 1 = \|\xi^0\|$ since $d^0_\omega \perp \xi^0$. Hence $\left(d^0_\omega, \xi^0, \eta^0\right)$ is an orthogonal basis of $\mathbf{V}_3$ under conditions $\|\eta^0\| = \|\xi^0\|$ and

$$\zeta^0 = \xi^0 \cos\alpha + \eta^0 \sin\alpha \tag{5.98}$$

Thus the result $\zeta^0 = c^0 \xi^0$ follows.

5.3.2. *Covering of group of 3–dimensional rotations by Rodrigues–Hamilton group*

In the previous paragraph we considered only the 'plane' rotations of a rigid body. For the application, 3–dimensional rotations are of interest. In this case it is possible to compare the 3–dimensional rotations $c^0 \in \mathcal{SO}(\mathcal{R}, 4)$ and the Rodrigues–Hamilton rotation group of a definite type, but not bijectively. Here we associate two corresponding Rodrigues–Hamilton rotations to each 3–dimensional rotation. We will prove a few auxiliary statements that are of independent interest (Konoplev 1996b).

Proposition 5.23 *Let: 1.* $x^0 = \mathrm{col}\{0,\ d^0_\omega\}$ *be* 4*–dimensional representation of the* 3*–dimensional unit vector* d^0_ω *of the rotation axis* $D(\omega)$ *of the rigid body in the manifold* $\mathbf{Q}_4$ *(more exactly, its coordinate column in the basis* $[\mathbf{e}^0]$*);*

2. $y^0 = \mathrm{col}\{0, \xi^0\}$ *be* 4*–dimensional representation of the* 3*–dimensional vector* $\xi^0 \in \mathbf{R}_3$ *in the manifold* $\mathbf{Q}_4$*;*
3. $\zeta \in \mathbf{P}(\omega)$ *be the vector* ζ *with a coordinate column* ξ^0 *in the basis* $[\mathbf{e}^0]$ *and belonging to the plane* $\mathbf{P}(\omega) \perp d^0_\omega$*;*
4. $u^0_x \equiv < d_\omega >^0_4$, $u^0_y \equiv < \xi >^0_4 \in \mathcal{SO}(\mathcal{R}, 4)$ *be* 4×4*–similitudes (5.82) generated by the elements* x^0 *and* y^0 *of the manifold* $\mathbf{Q}_4$*;*
5. $e^4_1 = \mathrm{col}\{1, 0, 0, 0\}$.

Then

$$u^0_x u^0_y e^4_1 = -u^0_y u^0_x e^4_1 \tag{5.99}$$

Proof Let us consider the images of the operators $u^0_x u^0_y$ and $u^0_y u^0_x$ on the element $e^4_1 \in \mathbf{Q}_4$

$$u^0_x u^0_y e^4_1 = u^0_x y^0 = -d^0_\omega \cdot \xi^0 + < d_\omega >^0 \xi^0 \tag{5.100}$$

$$u^0_y u^0_x e^4_1 = u^0_y x^0 = -\xi^0 \cdot d^0_\omega + < \xi >^0 d^0_\omega \tag{5.101}$$

Since $d^0_\omega \cdot \xi^0 = \xi^0 \cdot d^0_\omega = 0$ (as $\xi^0 \perp d^0_\omega$) and $< d_\omega >^0 \xi^0 = - < \xi >^0 d^0_\omega$, the proof is reached.

Comments

1. The condition $\zeta \in \mathbf{P}(\omega)$ is a determining one, in the opposite case $d^0_\omega \cdot \xi^0 = \xi^0 \cdot d^0_\omega \neq 0$ and (5.99) is wrong.
2. From (5.99) it does not follow that $u^0_x u^0_y = -u^0_y u^0_x$ since $< d_\omega >^0 \xi^0 \neq -\xi^0 < d_\omega >^0$.

Proposition 5.24 *Let: 1.* $\mathbf{P}(\omega) \subset \mathbf{V}_3$ *be the vector plane being perpendicular to the rotation axis* $D(\omega)$ *of the rigid body;*

2. $u^0_\lambda \in \mathcal{Y}(\mathcal{R}, 4)$ *be the Rodrigues–Hamilton rotation by an angle* $\alpha/2$

$$u^0_\lambda = H_0^{\cos\alpha/2} + < d_\omega >^0_4 \sin\alpha/2 \tag{5.102}$$

3. $\mathcal{SO}(\mathcal{R}, 2)$ *be the group of* 2*–dimensional rotations (i.e., the rotations of* $\mathbf{P}(\omega)$*);*
4. $\Phi^0_\lambda(\cdot)$ *be the operator acting on* $\mathbf{V}_3$ *by the rule*

$$\Phi^0_\lambda(\cdot) = u^0_\lambda < \cdot >^0_4 u^{0,T}_\lambda e^4_1 \tag{5.103}$$

and $\Phi^0_\lambda(\cdot) \mid_{\mathbf{P}(\omega)}$ *be its restriction on the plane* $\mathbf{P}(\omega)$*.*

Then (similarly to (5.96)) the following representation of $\mathcal{SO}(\mathcal{R}, 2)$ *in the group* $\mathcal{Y}(\mathcal{R}, 4)$ *exists*

$$\Phi^0_\lambda(\cdot) \mid_{\mathbf{P}(\omega)} = c^0 \tag{5.104}$$

Proof Let $x^0 = \mathrm{col}\{0, \xi^0\}$, $y^0 = \mathrm{col}\{0, \zeta^0\}$, $\zeta, \xi \in \mathbf{P}(\omega)$ and $y^0 = \Phi^0_\lambda(\cdot) \mid_{\mathbf{P}(\omega)} x^0 = (u^0_\lambda)^{-1} < x >^0_4 u^0_\lambda e^4_1$. It should be proved that $\zeta^0 = c^0 \xi^0$ in this case. Let us consider $y^0 = \Phi^0_\lambda(\cdot) \mid_{\mathbf{P}(\omega)} x^0 = (u^0_\lambda)^{-1} < x >^0_4 u^0_\lambda e^4_1 = (E \cos\alpha/2 + < d_\omega >^0_4 \sin\alpha/2) < \xi >^0_4 (E \cos\alpha/2 + < d_\omega >^{0,T}_4 \sin\alpha/2) e^4_1 = (< \xi >^0_4 \cos^2\alpha/2 + < d_\omega >^0_4 < \xi >^0_4 \cos\alpha/2 \ \sin\alpha/2) + < \xi >^0_4 < d_\omega >^{0,T}_4 \cos^2\alpha/2 \ \sin\alpha/2 +$

$< d_\omega >_4^0 < \xi >_4^0 < d_\omega >_4^{0,T} \ \sin^2 \alpha/2)\, e_1^4 \ = < \xi >_4^0 e_1^4 \ \cos^2 \alpha/2 + < d_\omega >_4^0 < \xi >_4^0 e_1^4 \sin\alpha + < d_\omega >_4^0 < \xi >_4^0 < d_\omega >_4^{0,T} e_1^4 \ \sin^2\alpha/2 = < \xi >_4^0 e_1^4 \cos^2\alpha/2 + < d_\omega >_4^0 < \xi >_4^0 \ e_1^4 \sin\alpha - < d_\omega >_4^{0,T} < d_\omega >_4^0 < \xi >_4^0 e_1^4 \ \sin^2\alpha/2 = < \xi >_4^0 e_1^4 \ \cos^2\alpha/2 + < d_\omega >_4^0 < \xi >_4^0 e_1^4 \sin\alpha - < \xi >_4^0 e_1^4 \sin^2\alpha/2 \ = < \xi >_4^0 e_1^4 \ \cos\alpha + < d_\omega >_4^0 < \xi >_4^0 e_1^4 \sin\alpha = \mathrm{col}\{0, \xi^0\}\cos\alpha + < d_\omega >_4^0 \mathrm{col}\{0, \xi^0\} \sin\alpha = \mathrm{col}\{0, \xi^0\}\cos\alpha + \mathrm{col}\{0, \eta^0\} \sin\alpha \to \zeta^0 = \xi^0 \cos\alpha + \eta^0 \sin\alpha \to \zeta^0 = c^0\xi^0.$

In order to make the above transformations, we have taken into account that $< \xi >_4^0 < d_\omega >_4^0 e_1^4 = - < d_\omega >_4^0 < \xi >_4^0 e_1^4$ – see (5.99), $< d_\omega >_4^0 < d_\omega >_4^{0,T} = E$, $< d_\omega >_4^0 = \eta^0$ and $< \xi >_4^0 e_1^4 = \mathrm{col}\{0, \xi^0\}$.

Comments

1. Note that

$$u_\lambda^{0,T} = (u_\lambda^0)^{-1},\ \|\lambda^0\|_4 = 1 \tag{5.105}$$

2. The operation $u_\lambda^0 < \cdot >_4 u_\lambda^{0,T}$ is a conjugation on the set of 4–dimensional skew–symmetric matrices $< \cdot >_4$. The subset of these matrices $u_\lambda^0 < \cdot >_4 u_\lambda^{0,T}$ is called conjugate to the set of all matrices of the type $< \cdot >_4$. Now it can be said that all skew–symmetric matrices belonging to the conjugated set are generated by 3–dimensional vectors which are results of the rotation of those 3–dimensional vectors that generate the set of all skew–symmetric matrices of the form $< \cdot >_4$.

Proposition 5.25 *Map of the group $\mathcal{Y}(\mathcal{R}, 4)$ into the group $\mathcal{SO}(\mathcal{R}, 2)$ is surjective – see (5.104), moreover to each image $c^0 \in \mathcal{SO}(\mathcal{R}, 2)$ correspond two proimages u_λ^0 and $u_{-\lambda}^0$.*

Proof $\Phi_\lambda^0(\cdot) = u_{-\lambda}^0 < \cdot >_4 u_{-\lambda}^{0,T} e_1^4 = (-u_\lambda^0) < \cdot >_4 (-u_\lambda^{0,T}) e_1^4 = u_\lambda^0 < \cdot >_4 u_\lambda^{0,T} e_1^4.$

Definition 5.14 *The surjective map $\Phi_\lambda^0 : \mathcal{Y}(\mathcal{R}, 4) \to \mathcal{SO}(\mathcal{R}, 2)$ is called a covering of the group $\mathcal{SO}(\mathcal{R}, 2)$ of the 'plane' rotations of the rigid body by the Rodrigues–Hamilton group $\mathcal{Y}(\mathcal{R}, 4)$ and is marked as follows*

$$\mathcal{Y}(\mathcal{R}, 4) = \mathrm{Spin}\,\mathcal{SO}(\mathcal{R}, 2) \tag{5.106}$$

Proposition 5.26 *Let: 1. $u_\lambda^0 \in \mathcal{Y}(\mathcal{R}, 4)$ be the Rodrigues–Hamilton rotation by the angle $\alpha/2$*

$$u_\lambda^0 = H_0^{\cos\alpha/2} + < d_\omega >_4^0 \sin\alpha/2 \tag{5.107}$$

2. $\mathcal{SO}(\mathcal{R}, 3)$ be the group of 3–dimensional rotations of a rigid body (i.e., the rotations of $\mathbf{V}_3$), $c^0 \in \mathcal{SO}(\mathcal{R}, 3)$;

3. $\Phi_\lambda^0(\cdot)$ be the operator acting on $\mathbf{V}_3$ (including $\mathbf{P}(\omega)$) by the rule

$$\Phi_\lambda^0(\cdot) = u_\lambda^0 < \cdot >_4^0 u_\lambda^{0,T} e_1^4 \tag{5.108}$$

Then the representation of the group $\mathcal{SO}(\mathcal{R}, 3)$ in the group $\mathcal{Y}(\mathcal{R}, 4)$ is

$$\Phi_\lambda^0(\cdot) = c^0 \tag{5.109}$$

Proof The idea of proof is simple.

1. Let us decompose each 3–dimensional vector into two components, one of them belonging to the plane $\mathbf{P}(\omega)$, and the another one being perpendicular to it and belonging to the rotation axis $D(\omega)$;

2. Let us act on the sum of these vectors by operator (5.108). It rests the components of $D(\omega)$ fixed while the component belonging to $\mathbf{P}(\omega)$ will be rotated by an angle α.

If the obtained results are added for each vector, we have proved the proposition.

Definition 5.15 *The surjection $\Phi^0_\lambda(\cdot) : \mathcal{Y}(\mathcal{R}, 4) \to \mathcal{SO}(\mathcal{R}, 3)$ is called a covering of the group of 3–dimensional rotations of a rigid body by the group of Rodrigues–Hamilton:*

$$\mathcal{Y}(\mathcal{R}, 4) = \mathrm{Spin}\, \mathcal{SO}(\mathcal{R}, 3) \tag{5.110}$$

5.3.3. *Kinematics equation on Rodrigues–Hamilton sphere*

Definition 5.16 *Let:*

1. *$\mathcal{Y}_\lambda$ be the Rodrigues–Hamilton sphere – see (5.91), $\lambda^0 \in \mathcal{Y}_\lambda$ be parameter of Rodrigues–Hamilton – see (5.92), $\lambda^{0\cdot}$ be its derivative w.r.t. the time, $\lambda^k = \mathrm{col}\{\lambda^k_0, \lambda^k_1, \lambda^k_2, \lambda^k_3\}$;*
2. *$\omega^{00}_k, \omega^{0k}_k$ be the vectors of instantaneous angular velocities and quasi–velocities of k–body – see (5.28), (5.29).*

Then the relation of the form

$$\lambda^{0\cdot} = f(\lambda^0, \omega^{00}_k), \quad \lambda^{k\cdot} = f(\lambda^k, \omega^{0k}_k) \tag{5.111}$$

is called kinematics equation on the Rodrigues–Hamilton sphere.

Proposition 5.27 *Let: 1. ξ^0 be the coordinate column (in the basis $[\mathbf{e}^0]$) of the radius–vector ξ in $\mathbf{E}_0$ of an arbitrary point of the k–body;*

2. *ω^{00}_k be the vector of the angular velocities of k–body (see (5.28));*
3. *the kinematics equation of the simple free rotation of k–body (see (5.31)) be*

$$c^{0\cdot} = <\omega^0_k>^0 c^0 \tag{5.112}$$

4. *v^{00}_ξ be the velocity of a point of k–body*

$$v^{00}_\xi = <\omega^0_k>^0 \xi^0$$

Then the following identities are fulfilled

$$\begin{aligned}
<v^0_\xi>^0 &= <\omega^0_k>^0 <\xi>^0 - <\xi>^0 <\omega^0_k>^0 && (5.113)\\
<v^0_\xi>^0 &= <\omega^0_k>^0 <\xi>^0 + (\omega^{00}_k \cdot \xi^0)E - \omega^{00}_k \xi^{0,T} && (5.114)\\
<v^0_\xi>^0_4 &= <\omega^0_k>^0_4 <\xi>^0_4 + (\omega^{00}_k \cdot \xi^0)E && (5.115)\\
<\xi>^0_4 <\omega^0_k>^0_4 &= -<\omega^0_k>^0_4 <\xi>^0_4 - 2(\omega^{00}_k \cdot \xi^0)E && (5.116)
\end{aligned}$$

where E are the identity matrices of the corresponding dimensions.

Proof is realized by comparing elements of the matrices in both parts of the equalities after all operations having been made.

Proposition 5.28 *Let: 1.* S_ω^k *be* 4×4*–matrix of the form*

$$S_\omega^k = \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & 2 < \omega_k^0 >^k \end{bmatrix} \tag{5.117}$$

2. K^k *be* 4×4*–skew–symmetric matrix of the form*

$$K^k = < \omega_k^0 >_4^k - S_\omega^k \tag{5.118}$$

Then the following differential equations on the Rodrigues–Hamilton sphere are fulfilled (in the bases $[\mathbf{e}^0]$ *and* $[\mathbf{e}^k]$*, respectively) (Konoplev 1996b)*

$$\lambda^{0\bullet} = \frac{1}{2} < \omega_k^0 >_4^k \lambda^0,\ \lambda^{0\bullet} = \frac{1}{2} K^k \lambda^k \tag{5.119}$$

Proof Let t_0 and t be the initial instant and an arbitrary one, respectively, $\xi^0(t) = c^0 \xi^0(t_0)$. Then, relative to relation (5.104)

$$\xi^0(t) = u_\lambda^0(t) < \xi(t_0) >_4^k u_\lambda^{0,T}(t) \tag{5.120}$$

After a time Δt passing, the vector $\xi(t)$ takes a position $\xi(t+\Delta t)$ where by definition $\Delta\xi(t) = \xi(t+\Delta t) - \xi(t) \to 0$ when $\Delta t \to 0$. For an instant $t + \Delta t$, similarly to (5.104) we have

$$< \xi(t+\Delta t) >_4^k = u_\lambda^0(t+\Delta t) < \xi(t_0) >_4^k u_\lambda^{0,T}(t+\Delta t) \tag{5.121}$$

From relations (5.120) and (5.121) for Rodrigues–Hamilton rotations we get

$$\begin{aligned} u_\lambda^0(t) &= < \xi(t) >_4^k u_\lambda^0(t)(< \xi(t_0) >_4^k)^{-1} \\ u_\lambda^0(t+\Delta t) &= < \xi(t+\Delta t) >_4^k u_\lambda^0(t+\Delta t)(< \xi(t_0) >_4^k)^{-1} \end{aligned}$$

We form the difference between the above two equalities

$$\begin{aligned} u_\lambda^0(t+\Delta t) - u_\lambda^0(t) &= < \xi(t+\Delta t) >_4^k u_\lambda^0(t+\Delta t)(< \xi(t_0) >_4^k)^{-1} - \\ &\quad < \xi(t) >_4^k u_\lambda^0(t)(< \xi(t_0) >_4^k)^{-1} \end{aligned}$$

But $< \xi(t_0) >_4^k = u_\lambda^{0,T}(t) < \xi(t) >_4^k u_\lambda^0(t)$ and $u_\lambda^0(t+\Delta t) - u_\lambda^0(t) = u_{\Delta\lambda}^0(t)$, $u_\lambda^0(t+\Delta t) = u_\lambda^0(\Delta t) u_\lambda^0(t)$ where $\Delta\lambda = \lambda(t+\Delta t) - \lambda(t)$. That is why $u_{\Delta\lambda}^0(t)$ $= [< \xi(t+\Delta t) >_4^k u_\lambda^0(\Delta t)(< \xi(t) >_4^k)^{-1} - E] u_\lambda^0(t)$.
Later, if we use the equality $< \xi(t+\Delta t) >_4^k = < \xi(t) >_4^k + < \xi(\Delta t) >_4^k$ we get

$$\begin{aligned} u_{\Delta\lambda}^0(t) &= [< \xi(t) >_4^k u_\lambda^0(\Delta t) - E)(< \xi(t) >_4^k)^{-1} + \\ &\quad < \xi(\Delta t) >_4^k u_\lambda^0(\Delta t)(< \xi(t) >_4^k)^{-1}] u_\lambda^0(t) \end{aligned}$$

We multiply the obtained equality by the unit vector e_0 from the right, then we divide the equality by Δt and pass to the limit using the equalities $\lim \Delta\lambda^0(t)/\Delta t = \lambda^{0\bullet}(t)$, $\lim(u_\lambda^0(\Delta t) - E)/\Delta t = \lim < d_\omega(t) >_4^0 (\sin\Delta\alpha/2)/\Delta t + E \lim(\cos\Delta\alpha/2 - 1)/\Delta t = \frac{1}{2} < d_\omega(t) >_4^0 \|\omega_k^{00}\| + 0 = \frac{1}{2} < \omega_k^0 >_4^0 \lim < \xi(\Delta t) >_4^0 /\Delta t = < v_\xi^0 >_4^0$,

$\lim u^0_\lambda(\Delta t) = E$, $\lambda^{0\cdot}(t) = [-\frac{1}{2} < \xi(t) >^0_4 < \omega^0_k >^0_4 (< \xi(t) >^0_4)^{-1} + < v^0_\xi >^0_4 (< \xi(t) >^0_4)^{-1}]\lambda^0(t)$.

It rests to use the equalities from p.p. 3 and 4 of the previous proposition for the matrices $< \xi(t) >^0_4 < \omega^0_k >^0_4$ and $< v^0_\xi >^0_4$

$$\begin{aligned} \lambda^{0\cdot}(t) &= [-\frac{1}{2} < \omega^0_k >^0_4 < \xi(t) >^0_4 (< \xi(t) >^0_4)^{-1} - (\omega^{00}_k \cdot \xi^0)(< \xi(t) >^0_4)^{-1} + \\ &\quad < \omega^0_k >^0_4 < \xi(t) >^0_4 (< \xi(t) >^0_4)^{-1} + (\omega^{00}_k \cdot \xi^0)(< \xi(t) >^0_4)^{-1}]\lambda^0(t) \\ &= \frac{1}{2} < \omega^0_k >^0_4 \lambda^0(t) \end{aligned}$$

Introduce 4×4–matrix of the form

$$C^0_k = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & c^0_k \end{bmatrix} \tag{5.122}$$

where c^0_k is 3×3–matrix that transfers the basis $[\mathbf{e}^0]$ of the root frame $\mathbf{E}_0$ to the basis $[\mathbf{e}^k]$ of the frame $\mathbf{E}_k$ attached to k–body, $[\mathbf{e}^k] = [\mathbf{e}^0]c^0_k$. Then $\lambda^0 = C^0_k\lambda^k$ and from the first equation in (5.119) we get $\lambda^{k\cdot}(t) = \frac{1}{2}C^0_k C^{0\cdot}_k \lambda^k = \frac{1}{2}[< \omega^0_k >^0_4 - 2S^k_\omega]\lambda^k = \frac{1}{2}K^k\lambda^k$.

5.3.4. *Cayley–Klein group and its representation in Rodrigues–Hamilton group*

Proposition 5.29 *Let: 1.* $\mathbf{C}_2 = \mathbf{C} \times \mathbf{C}$ *be Cartesian product of the complex number field;*

2. $\mathcal{GO}^+(\mathcal{C}_2, 2)$ *be the group of the proper similitudes of* $\mathbf{C}_2$ *with an unit* $\mathbf{1}$

$$\mathcal{GO}^+(\mathcal{C}_2, 2) = \{v_{ab} : v^*_{ab}v_{ab} = \alpha^2\mathbf{1}, \alpha \in \mathbf{R}_1, \det v_{ab} > 0\} \tag{5.123}$$

3. $a = a_0i_0 + a_1i_1 \in \mathbf{C}$, $b = b_0i_0 + b_1i_1 \in \mathbf{C}$ *be complex numbers,* a_0, a_1, b_0 *and* $b_1 \in \mathbf{R}_1$,

$i_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $i_0^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = i_0$,

$i_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $i_1^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -i_0$.

Then each similitude v_{ab} *has one of the following* 4 *possible forms*

$$\begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}, \begin{bmatrix} a & -\overline{b} \\ b & \overline{a} \end{bmatrix}, \begin{bmatrix} \overline{a} & -\overline{b} \\ b & a \end{bmatrix}, \begin{bmatrix} \overline{a} & b \\ -\overline{b} & a \end{bmatrix} \tag{5.124}$$

Proof Let c_a, c_b, c_d and $c_e \in \mathbf{C}$ be complex numbers. Consider the matrix $S = \begin{bmatrix} c_a & c_b \\ c_d & c_e \end{bmatrix}$ and determine its elements so that it will be a matrix of similitude, *i.e.*, $S^*S = \alpha^2E, \det S = \alpha^2$.

Hence we obtain 5 equations

$$\begin{aligned} c_a\overline{c}_a + \overline{c}_bc_b &= \alpha^2,\ c_d\overline{c}_d + \overline{c}_ec_e = \alpha^2,\ c_ac_e + c_bc_d = \alpha^2 \\ \overline{c}_ac_d + \overline{c}_bc_e &= 0,\ c_a\overline{c}_d + c_b\overline{c}_e = 0 \end{aligned}$$

The solution of this system of equations gives the result that should be proved.

Definition 5.17 *The subgroup $\mathcal{P}(\mathcal{C}_2, 2) \subset \mathcal{GO}^+(\mathcal{C}_2, 2)$ that consists of the matrices v_{ab} of the first type is called a Pauli group:*

$$\mathcal{P}(\mathcal{C}_2, 2) = \{v_{ab} : v_{ab} = \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}\} \tag{5.125}$$

Definition 5.18 *Let: 1. $\mathcal{P}(\mathcal{C}_2, 2)$ be the Pauli group (5.125);*
2. P_0, P_1, P_2 and $P_3 \in \mathcal{P}(\mathcal{C}_2, 2)$ be 2×2–matrices of the form

$$\begin{aligned} P_0 &= i_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, P_1 = i_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ P_2 &= i_0 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, P_3 = i_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned} \tag{5.126}$$

Then they are called Pauli matrices.

Proposition 5.30 *The following relations are true: 1. $P_0P_0 = P_0$, $P_0P_1 = P_1$, $P_0P_2 = P_2$, $P_0P_3 = P_3$;*
2. $P_1P_1 = -P_0$, $P_2P_2 = -P_0$, $P_3P_3 = -P_0$;
3. $P_1P_2 = -P_2P_1 = P_3$, $P_2P_3 = -P_3P_2 = P_1$, $P_3P_1 = -P_1P_3 = P_2$;
4. $\mathrm{trace}(P_0^T P_1) = \mathrm{trace}(P_1^T P_2) = \mathrm{trace}(P_2^T P_3) = 0$.

Proof is realized by a simple verification.

Proposition 5.31 *Any similitude $v_{ab} \in \mathcal{P}(\mathcal{C}_2, 2)$ can be represented as a linear combination of the Pauli matrices*

$$v_{ab} = a_0P_0 + a_1P_1 + b_0P_2 + b_1P_3 \tag{5.127}$$

Proof is realized by a simple verification.

Comments

1. If we add 0–matrix to the multiplicative Pauli group $\mathcal{P}(\mathcal{C}_2, 2)$, and if on the set $\mathcal{P}(\mathcal{C}_2, 2) \cup 0$ we introduce the evident matrix operations of addition and of multiplication by a number, then the set

$$\mathcal{P}_2(\mathcal{C}_2, 2) = \mathcal{P}(\mathcal{C}_2, 2) \cup 0 \tag{5.128}$$

   transforms into an unitary 2–dimensional vector space where the Pauli matrices are its basis: $[\mathbf{P}] = (P_0, P_1, P_2, P_3)$.
2. The basis is orthogonal (it follows from in P 5.30-4).
3. The matrix P_0 is the group unit (it follows from P 5.30 -1).
4. If we designate $a_2 = b_0$ and $a_3 = b_1$, then (5.127) takes the form

$$v_a = a_0P_0 + a_1P_1 + a_2P_2 + a_3P_3 = \begin{bmatrix} a_0i_0 + a_1i_1 & a_2i_0 + a_3i_1 \\ -a_2i_0 + a_3i_1 & a_0i_0 - a_1i_1 \end{bmatrix}$$

   where

$$z_a = (a_0i_0 + a_1i_1)\gamma_0 + (a_2i_0 + a_3i_1)\gamma_1 \tag{5.129}$$

   $[\gamma] = \mathrm{col}\{\gamma_0, \gamma_1\}$, $\gamma_0 = \mathrm{col}\{i_0, 0\}$ and $\gamma_0 = \mathrm{col}\{0, i_1\}$. We will say that in this case the similitude v_a is generated by the vector $z_a \in \mathbf{C}_2$ or by the element of the manifold $\mathbf{Q}_4$. In some cases it is convenient to use new indexes:

$$a = \mathrm{col}\{a_1, a_2, a_3, a_4\} \tag{5.130}$$

$$z_a = (a_1i_1 + a_2i_2)\gamma_1 + (a_3i_1 + a_4i_2)\gamma_2 \tag{5.131}$$

Proposition 5.32 *Let: 1.* $v_x^0 \in \mathcal{P}(\mathcal{C}_2, 2)$ *be a Pauli similitude that is generated by the vector* $z_x^0 \in \mathbf{C}_2$ *– see (5.129),* $x^0 = \mathrm{col}\{x_0, \xi^0\}$ *– see (5.80);*

2. $< \xi >_2^0$ *be a skew-symmetric matrix (generated by the coordinate column* $\xi^0 = \mathrm{col}\{\xi_1^0, \xi_2^0, \xi_3^0\}$ *of a vector* $\xi \in \mathbf{V}_3$ *in the basis* $[\mathbf{e}^0]$*) of the form (that is an analog of (5.81))*

$$< \xi >_2^0 = \begin{bmatrix} \xi_1^0 i_1 & \xi_2^0 i_0 + \xi_3^0 i_1 \\ \xi_2^0 i_0 + \xi_3^0 i_1 & -\xi_1^0 i_1 \end{bmatrix} \tag{5.132}$$

3. $H_{x0} = i_0 E x_0$ *be a homothety on* $\mathbf{C}_2$ *with a coefficient* x_0*.*

Then the similitude v_x^0 *can be represented as follows*

$$v_x^0 = H_{x0} + < \xi >_2^0 \tag{5.133}$$

Comment The Pauli group $\mathcal{P}(\mathcal{C}_2, 2)$ is an analog of the quart-group, and representation (5.133) is an analog of (5.87).

Proposition 5.33 *The subgroup* $\mathcal{Y}(\mathcal{C}_2, 2)$ *of the Pauli group* $\mathcal{P}(\mathcal{C}_2, 2)$ *generated by Rodrigues–Hamilton parameter* $\lambda^0 = \mathrm{col}\{\lambda_0^0, \lambda_1^0, \lambda_2^0, \lambda_3^0\}$ *is a subgroup* $\mathcal{SO}(\mathcal{C}_2, 2)$ *of the rotations group of the vector space* $\mathbf{C}_2$

$$\mathcal{Y}(\mathcal{C}_2, 2) = \{v_\lambda^0 : v_\lambda^0 = \begin{bmatrix} \lambda_0 i_0 + \lambda_1 i_1 & \lambda_2 i_0 + \lambda_3 i_1 \\ -\lambda_2 i_0 + \lambda_3 i_1 & \lambda_0 i_0 - \lambda_1 i_1 \end{bmatrix}\} \tag{5.134}$$

Comment The group $\mathcal{Y}(\mathcal{C}_2, 2)$ is an analog of the Rodrigues–Hamilton group $\mathcal{Y}(\mathcal{R}, 4)$ – see (5.91).

Definition 5.19 *1. The couple of complex numbers that is a coordinate column of the vector* z_λ *in the basis* $[\gamma]$ *(see (5.129))*

$$k = \mathrm{col}\{k_1, k_2\},\ k_1 = \lambda_0 i_0 + \lambda_1 i_1,\ k_2 = \lambda_2 i_0 + \lambda_3 i_1 \tag{5.135}$$

is called a Cayley–Klein parameter;

2. the group $\mathcal{Y}(\mathbf{C}_2, 2)$ *is called the Cayley–Klein group of rotations.*

Comment The next problem is to find out an isomorphism between the quart-group and the Cayley–Klein group that is an exact mutual representation of each of these groups in the other. The pointed fact will be later used when we will present the theory of the 'finite' rotations of a rigid body by using the Cayley–Klein group.

Proposition 5.34 *The bijection* $v : \mathcal{G}^+(\mathcal{R}, 4) \to \mathcal{P}(\mathcal{C}_2, 2)$ *is the isomorphism* $\mathcal{G}^+(\mathcal{R}, 4) \cong \mathcal{P}(\mathcal{C}_2, 2)$ *which realizes the exact representation*

$$\mathcal{G}^+(\mathcal{R}, 4) \to \mathcal{P}(\mathcal{C}_2, 2) \tag{5.136}$$

with the representing space $\mathbf{C}_2$ *and with the character* $\chi_a(v) = \mathrm{trace}\, v_x^0 = 2x_0 i_0$*.*

Proof is realized by a simple verification of the equalities $v(u_x^0 u_y^0) = v(u_{f(x,y)}^0) = v(u_x^0)v(u_y^0) = v_x^0 v_y^0 = v_{f(x,y)}^0$, $v(u_x^0)^{-1} = (v_x^0)^{-1}$, $x, y, f(x,y) \in \mathbf{Q}_4$.

Comment The obtained result is transferred automatically on the Rodrigues–Hamilton subgroup of rotations (see (5.91)) and on the Cayley–Klein subgroup of rotations

$$v : \mathcal{Y}(\mathcal{R}, 4) \to \mathcal{Y}(\mathcal{C}_2, 2) \tag{5.137}$$

5.3.5. *Representation of rotation group in Cayley–Klein rotation group*

Proposition 5.35 *The representation of the Cayley–Klein rotation of the form (Konoplev 1996b)*

$$u_\lambda^0 = H_0^{\cos\alpha} + < d_\omega >_4^0 \sin\alpha \tag{5.138}$$

takes place where all rotations are the same as the introduced ones in subsection 4.3.1.

Proof is achieved with the help of representation (5.137).

Proposition 5.36 *Let* $u_\lambda^0 \in \mathcal{Y}(\mathcal{C}_2, 2)$ *be Cayley–Klein rotation – see (5.134). Then the restriction of the operator* u_λ^0 *on the plane* $P(\omega)$ *(the plane being perpendicular to the axis* $\mathbf{D}_\omega$ *of the* 2*–dimensional rotation of the rigid body) is a* 2*–dimensional rotation of this rigid body by an angle* α *during the time* Δt *– see (5.93):*

$$u_\lambda^0 \mid_{P(\omega)} = c^0 \tag{5.139}$$

Proof is realized by translation of result (5.95) in the terms of the Cayley–Klein group of rotations with the help of representation (5.137).

Proposition 5.37 *Let: 1. similarly to relation (5.102)*

$$v_\lambda^0 = H_0^{\cos\alpha/2} + < d_\omega >_4^0 \sin\alpha/2 \tag{5.140}$$

2. $\Phi_\lambda^0(\cdot)$ *be the operator acting on* $\mathbf{V}_3$ *by the rule*

$$\Phi_\lambda^0(\cdot) = v_\lambda^0 < \cdot >_2^0 v_\lambda^{0,*}\gamma \tag{5.141}$$

Then the restriction of $\Phi_\lambda^0(\cdot)$ *on the plane* $P(\omega)$ *(the plane being perpendicular to the axis* $\mathbf{D}_\omega$ *of the* 2*–dimensional rotation of the rigid body) is a* 2*–dimensional rotation of this rigid body by the angle* α *during the time* Δt *(see (5.93))*

$$\Phi_\lambda^0(\cdot) \mid_{P(\omega)} = c^0 \tag{5.142}$$

Proof is realized by translation of result (5.104) in the terms of the Cayley–Klein group of rotations with the help of representation (5.137).

Proposition 5.38 *Let: 1. similarly to relation (5.102)*

$$v_\lambda^0 = H_0^{\cos\alpha/2} + < d_\omega >_4^0 \sin\alpha/2 \tag{5.143}$$

2. $\Phi^0_\lambda(\cdot)$ *be the operator acting on* $\mathbf{V}_3$ *by the law*

$$\Phi^0_\lambda(\cdot) = v^0_\lambda < \cdot >^0_2 v^{0,*}_\lambda \gamma \tag{5.144}$$

Then the restriction of $\Phi^0_\lambda(\cdot)$ *on the vector space* $\mathbf{V}_3$ *is a rotation of this rigid body by the angle* α *during the time* Δt *(see (5.93))*

$$\Phi^0_\lambda(\cdot) = c^0 \tag{5.145}$$

Proof is realized by translation of result (5.108) in the terms of the Cayley–Klein group of rotations with the help of representation (5.137).

Proposition 5.39 *Representation (5.145) of the Rodrigues–Hamilton group of rotations (5.106) in the rigid body rotations group (see P 5.17) is a covering of the first group by the second one.*

Proof $\Phi^0_\lambda(\cdot) = \Phi^0_{-\lambda}(\cdot) = c^0$.

Comment If isomorphism (5.137) is in presence, then an analog of kinematics equations (5.119) on the Cayley–Klein group of rotations is absent.

Proposition 5.40 *Let: 1.* $k^0 = \mathrm{col}\{k^0_1, k^0_2\}$ *be a Cayley–Klein parameter (5.135),* $z^0_\lambda = [\gamma] z^{0\gamma}_\lambda \equiv [\gamma] k^0$ *where the index '0' shows that in the construction of the parameter* k^0*, the Rodrigues–Hamilton parameter* k^0 *is used (in the basis* $[\mathbf{e}^0]$*),* $\lambda^0 = \mathrm{col}\{\lambda_0, \Lambda^0\}$ *(see (5.92));*

2. $\widehat{k}^0 = \mathrm{col}\{\widehat{k}^0_0, -\widehat{k}^0_1\}$ *be the coordinate column of some vector* $\widehat{z}^0_\lambda \in \mathbf{C}_2$ *in the basis* $[\gamma]$, $\widehat{z}^0_\lambda = [\gamma]\, \widehat{z}^{0\gamma}_\lambda \equiv [\gamma]\, \widehat{k}^0$*;*

3. $< \omega^0_k >^0_k$ *be a skew-symmetric* 2×2*–matrix of form (5.132) generated by the vector of the angular velocity of* k*–body.*

Then there is the kinematics equation of the form

$$\widehat{k}^{0\cdot} = \frac{1}{2} < \omega^0_k >^0_k \widehat{k}^0 \tag{5.146}$$

5.3.6. *Representation of rotation group in multiplicative group of quart–body*

Proposition 5.41 *Let: 1.* $\mathcal{G}(\mathcal{R}, 4)$ *be quart-group (5.82);*

2. $\mathcal{T}(\mathcal{R}, 4)$ *be the union of the quart-group and of* 0*–matrix* u^0_0

$$\mathcal{T}(\mathcal{R}, 4) = \mathcal{G}(\mathcal{R}, 4) \cup u^0_0 \tag{5.147}$$

3. the set $\mathcal{T}(\mathcal{R}, 4)$ *be closed w.r.t. the natural matrix operations of addition and of multiplication and in the presence of the corresponding properties (i.e., an additive commutative group w.r.t. the first operation, non–commutative group w.r.t. the second operation (without* u^0_0*), distributivity).*

Then: 1. the set $\mathcal{T}(\mathcal{R}, 4)$ *is a body.*

2. the body $\mathcal{T}(\mathcal{R}, 4)$ *is called a quart–body.*

Proof concludes in a direct verification of corresponding properties of the pointed operations: an additive commutative group w.r.t. the first operation, non–commutative group w.r.t. the second operation (without u_0^0), *i.e.*, distributivity.

Definition 5.20 *Let: 1.* $\mathcal{T}(\mathcal{R},4)$ *be quart–body (5.147);*

2. $e_1^4 = \mathrm{col}\{1,0,0,0\}$ *be the unit vector from* $\mathbf{R}_4$*;*

3. $\mathbf{K}$ *be a body whose elements are images of elements* $u_x^0 \in \mathcal{T}(\mathcal{R},4)$ *on the unit vector* e_1^4

$$\mathbf{K} = \{z_x : z_x = u_x^0 e_1^4\} \tag{5.148}$$

such that

$$z_x z_y = u_x^0 u_y^0 e_1^4 \tag{5.149}$$

$$z_x^{-1} = (u_x^0)^{-1} e_1^4 \tag{5.150}$$

$$z_x + z_y = (u_x^0 + u_y^0) e_1^4 \tag{5.151}$$

$$-z_x = -u_x^0 e_1^4 \tag{5.152}$$

Then: 1. the body $\mathbf{K}$ *is called a body of quaternions;*

2. the elements of $\mathbf{K}$ *are called quaternions.*

Comments

1. If we concentrate our interest only in the Abel group of the quart–body $\mathcal{T}(\mathcal{R},4)$ and if on $\mathcal{T}(\mathcal{R},4)$ we introduce the natural operation of a multiplication of a matrix by a number, *i.e.*, μu_x^0, $\mu \in \mathbf{R}_1$, then with account of

$$\mu z_x^0 = \mu u_x^0 e_1^4 \tag{5.153}$$

equality (5.148) realizes an isomorphism of the 4–dimensional vector space $\mathbf{R}_4$ and of the mentioned above Abel group of the quart–body $\mathcal{T}(\mathcal{R},4)$, *i.e.*, for each $x^0 \in \mathbf{R}_4$

$$x^0 \Leftrightarrow u_x^0 \Leftrightarrow z_x^0 \tag{5.154}$$

$$\mathbf{R}_4 \Leftrightarrow \mathcal{T}(\mathcal{R},4) \Leftrightarrow \mathbf{K} \tag{5.155}$$

2. In this case (when operations (5.149) and (5.153) are valid only) we will say that the quaternion $z_x^0 \in \mathbf{K}$ is generated by the vector $x^0 \in \mathbf{R}_4$ and the set $\mathbf{K}$ can be called the vector space of quaternions. We will mark this set by $\mathbf{K}_4$ (considered as a vector space).
3. If we consider the multiplicative group $\mathcal{G}(\mathcal{R},4)$ of the quart–body $\mathcal{T}(\mathcal{R},4)$ (with the internal operation (5.149) of multiplication), then the fours x^0 of numbers are elements of the manifold $\mathbf{Q}_4$ and we may speak about the isomorphism

$$\mathbf{Q}_4 \Leftrightarrow \mathcal{T}(\mathcal{R},4) \Leftrightarrow \mathbf{K} \tag{5.156}$$

In this case the set $\mathbf{K}$ (without u_0^0) is a multiplicative non–commutative group also and it will be marked by

$$\mathcal{K}^o = \mathbf{K}_4 \backslash \{u_0^0\} \tag{5.157}$$

Proposition 5.42 *Let: 1.* $u_s^0 \in \mathcal{T}(\mathcal{R},4)$, $s = 1,2,3,4$, *be similitudes of quart–body (5.147) generated by* 4*–dimensional unit vector* e_s^4 *with* 1 *situated on* s*–th place (relevant to (5.90));*

2. $[\mathbf{i}] = (i_1, i_2, i_3, i_4) \in \mathbf{K_4}$ *be the set of vectors such that*

$$i_s = u_s^0 e_1^4,\ i_s \equiv e_s^4 \tag{5.158}$$

Then $[\mathbf{i}]$ *is a basis of the vector space* $\mathbf{K}_4$ *of quaternions.*

Proof For each $s = 1,2,3,4$ we have $i_s = e_s^4$ and for each quaternion z_x, $x^0 = \mathrm{col}\{x_1, x_2, x_3, x_4\}$ as a vector. Thus, taking into account (5.148) we obtain the decomposition $z_x = u_x^0 e_1^4 = \sum_s x_s u_s^0 e_1^4 = \sum_s x_s i_s$.

Comment The basis of the vector space $\mathbf{K}_4$ of quaternions is generated by the basis of the vector space $\mathcal{G}_{16}^+(\mathcal{R},4)$ of similitudes (5.90) due to (5.158).

Proposition 5.43 *Let: 1.* $\mathcal{K}^o$ *be multiplicative group of quaternions (5.157);*

2. $u_s^0 \in \mathcal{T}(\mathcal{R},4)$, $s = 1,2,3,4$, *be similitudes from quart–body (5.147) that are generated by* 4*–dimensional unit vectors* e_s^4 *with* 1 *situated on* s*–th place;*

3. $[\mathbf{i}] = (i_1, i_2, i_3, i_4) \in \mathbf{K}_4$ *be the set of vectors such that*

$$i_s = u_s e_1^4,\ i_s \equiv e_s^4 \tag{5.159}$$

Then: 1. the set $[\mathbf{i}]$ *in* $\mathbf{K}^o$ *(but not in* $\mathbf{K}_4$*!) has the following properties:*

1.1. $i_1 i_1 = i_1^2 = u_s^2 e_1^4 = u_s e_1^4 = i_1$; $i_1 i_k = i_k i_1 = u_1 u_k e_1^4 = u_s e_1^4 = i_k$, $k = 2,3,4$;

1.2. $i_k i_k = i_k^2 = u_k^2 e_1^4 = u_k e_1^4 = -i_1$, $k = 2,3,4$;

1.3. $i_1 i_2 = -i_2 i_1 = i_3$; $i_2 i_3 = -i_3 i_2 = i_1$; $i_3 i_1 = -i_1 i_3 = i_2$;

2. the element i_1 *is the unit of the group* $\mathcal{K}^o$*, and thus for each* $z_x \in \mathcal{K}^o$ *there is* $i_1 z_x = z_x$.

Comments

1. According to p. 1.2 the elements i_k of the group $\mathcal{K}^o$ are called 'imaginary' units. Every quaternion has three 'imaginary' units. That is why we may use the term 'hypercomplex numbers' speaking about the quaternions.
2. But without any base, the vectors $i_k \in \mathbf{K}_4$ are also called imaginary units. It is not possible as in the vector space the objects i_k are 4–dimensional vectors with 1 situated on k–th place and the operation i_k^2 is not defined at all.
3. The same takes place in the definition of the complex numbers: the set of complex numbers, as a vector space, concludes the unit vector $i_2 = \mathrm{col}\{0,1\}$ and the operation i_2^2 has no sense; the set of complex numbers as a body concludes the same element i_2 but the operation i_2^2 is determined and $i_2^2 = -i_1$ where $i_1 = \mathrm{col}\{1,0\}$ is the unit of the field and is not '1' of the arithmetic as it is assumed.

Definition 5.21 *The quaternion*

$$z_x^* = u_x^{0,T} e_1^4 \tag{5.160}$$

is called conjugated to the quaternion z_x.

Proposition 5.44 *The map $\varphi : z_x \to z_x^*$ is involute, i.e., ($\varphi^2 = E$).*

Proof Indeed, $(z_x^*)^* = (u_x^{0,T})^T e_1^4 = u_x^0 e_1^4 = z_x$.

Proposition 5.45 *Let: 1. $\mathbf{K}_4 = \mathbf{K}_0 \times \mathbf{K}_\xi$ where $\mathbf{K}_0$ is the vector straight-line with the unit vector i_0, $\mathbf{K}_\xi$ is 3–dimensional vector space generated by the vector $\xi \in \mathbf{R}_3$;*

2. z_x and $z_x^* \in \mathbf{K}_4$.

Then

$$z_x + z_x^* \in \mathbf{K}_0,\ z_x z_x^* \in \mathbf{K}_0 \tag{5.161}$$

Proof $z_x + z_x^* = (u_x^0 + u_x^{0,T})e_1^4 = 2x_0 E e_1^4 = 2x_0 i_0 \in \mathbf{K}_0$, $z_x z_x^* = u_x^0 u_x^{0,T} e_1^4 = \|x_0\|_4^2 i_0 \in \mathbf{K}_0$.

Definition 5.22 *1. The number $2x_0$ is called a reduced trace of the quaternion.*

2. The number $\|x_0\|_4^2 = \|z_\lambda^0\|_4^2$ is called a reduced norm of the quaternion.

Definition 5.23 *The image of the Rodrigues–Hamilton sphere $\mathcal{Y}_\lambda$ (see (5.82)) in $\mathbf{K}_4$ (or in $\mathcal{K}^o$) is called a Rodrigues–Hamilton sphere $\mathbf{K}_4$ (or in $\mathcal{K}^o$) and is marked by $\mathbf{K}_\lambda$.*

Comments

1. Elements of the Rodrigues–Hamilton sphere $\mathbf{K}_\lambda$ are quaternions with an unit reduced norm $\|z_\lambda^0\|_4$.
2. The coordinates of the quaternion $z_\lambda \in \mathbf{K}_\lambda$ in basis (5.158) are Rodrigues–Hamilton parameters λ^0 (see (5.92)).

Proposition 5.46 *Let: 1. $\mathcal{K}^o$ be the multiplicative group of quaternions (see (5.157));*

2. $z_\lambda \in \mathbf{K}_\lambda$ be the quaternion generated by the Rodrigues–Hamilton parameter (see (5.92), (5.102))

$$z_\lambda^0 = \mathrm{col}\{\cos\alpha/2,\ d_{w1}^0 \sin\alpha/2,\ d_{w2}^0 \sin\alpha/2,\ d_{w3}^0 \sin\alpha/2\} \tag{5.162}$$

Then the restriction of the operator $z_\lambda^0 z_{(\cdot)}^0 z_\lambda^{0,*}$ on $\mathbf{V}_3$ is a rigid body rotation. This operator realizes a covering of the group of 3–dimensional rotations of a rigid body by the group $\mathcal{K}^o$ of quaternions in accordance with

$$z_\lambda^0 z_{(\cdot)}^0 z_\lambda^{0,*}|_{\mathbf{V}_3} = c^0 \tag{5.163}$$

Proof The operator $z_\lambda^0 z_{(\cdot)}^0 z_\lambda^{0,*}|_{\mathbf{V}_3}$ coincides with $\Phi_\lambda^0(\cdot) = u_\lambda^0 < \cdot >_4^0 u_\lambda^{0,T} e_1^4$ on $\mathbf{V}_3$ (see (5.109)).

5.4. Equations of rigid body motion

5.4.1. *Equations of simple free motion*

Let us consider the case of simple free motion.

Proposition 5.47 *Let: 1. B be a rigid body, $\mathbf{E}_s$ be a frame attached to it;*

2. *$\mathbf{E}_0$ be a frame in $\mathbf{A}_3^\mu$;*
3. *$V_{ss}^{0s} = V_s^{0s} = \operatorname{col}\{v_s^{0s}, \omega_s^{0s}\}$ be the vector of quasi velocities of $\mathbf{E}_s$ w.r.t. $\mathbf{E}_0$ in the basis $[\mathbf{e}^s]$ (see (5.23), (5.47));*
4. *$l_s^{v(x),s}$ be a sliding vector generated by the vector $v(x) \equiv v_x^{0s}$ of quasi velocities of shift of a point $x \in B$ w.r.t. $\mathbf{E}_0$;*
5. *x^s be the radius vector of a point $x \in B$, $x^{ss} = \operatorname{col}\{x_1^{ss}, x_2^{ss}, x_2^{ss}\}$ be its coordinate column in the basis $[\mathbf{e}^s]$;*
6. *Θ_x^s be 6×6–dimensional matrix of the kind*

$$\Theta_x^s = \begin{bmatrix} E & <x^s>^{s,T} \\ <x^s>^s & -<x^s>^s<x^s>^s \end{bmatrix} \tag{5.164}$$

Then the sliding vector $l_s^{v(x),s}$ is the following linear transformation of the linear (coordinate) space of quasi–velocities of the rigid body

$$l_s^{v(x),s} = \Theta_x^s V_s^{0s} \tag{5.165}$$

Proof From $<\omega_s^0>^s x^{ss} = -<x^s>^s \omega_s^{0s} = -<x^s>^{s,T} \omega_s^{0s}$ follows that

$$\begin{aligned} l_s^{v(x),s} \quad = \quad & \begin{bmatrix} E & O \\ O & <x^s>^s \end{bmatrix} \begin{pmatrix} v_x^{0s} \\ v_x^{0s} \end{pmatrix} = \\ & \begin{bmatrix} E & O \\ O & <x^s>^s \end{bmatrix} \begin{pmatrix} v_s^{0s}+<\omega_s^0>^s x^{ss} \\ v_s^{0s}+<\omega_s^0>^s x^{ss} \end{pmatrix} = \\ & \begin{bmatrix} E & <x^s>^{s,T} \\ <x^s>^s & -<x^s>^s<x^s>^s \end{bmatrix} \begin{pmatrix} v_x^{0s} \\ \omega_x^{0s} \end{pmatrix} = \Theta_x^s V_s^{0s} \end{aligned}$$

Definition 5.24 *Let B be a rigid body, $\mathbf{E}_s$ be a frame attached to it.*

Then 1. the vector r_c^s and its coordinate column r_c^{ss} defined in the basis $[\mathbf{e}^s]$ by the following relations

$$r_c^s = m^{-1}(B) \int \chi_B x^s m(dx), \;\; r_c^{ss} = m^{-1}(B) \int \chi_B x^{ss} m(dx) \tag{5.166}$$

are called the radius vector of the centroid of the rigid body B in $\mathbf{E}_s$ and its coordinate column in the basis $[\mathbf{e}^s]$;

2. *the point $c \in B$, settled in $\mathbf{E}_s$ by the radius vector r_c^s, is the center of mass of the rigid body B;*
3. *3×3–dimensional matrix*

$$\theta_s^s = -\int \chi_B <x^s>^s<x^s>^s m(dx) \tag{5.167}$$

is the inertia matrix of the body B with respect to the point o_s (see lower index) in the basis $[\mathbf{e}^s]$ (see upper index).

Definition 5.25 *Let: 1. B be a rigid body;*

2. *$\mathbf{E}_0$ be an inertial frame;*

3. $l_s^{v(x),s}$ *be a sliding vector generated by the vector* $v(x) \equiv v_x^{0s}$ *of quasi velocity of shift of a point* $x \in B$ *w.r.t.* $\mathbf{E}_0$.

Then the Radon vector measure (Milnor 1965, Wallace 1968) on σ*–algebra* σ_3^μ *of mechanical systems of the kind*

$$Q_{Bs}^{0s} = \int \chi_B l_0^{v(x)s} m_i(dx) \tag{5.168}$$

is called as a kinetic screw of the body B *moving w.r.t. the inertial frame* $\mathbf{E}_0$ *(the inner indexes) in the attached one* $\mathbf{E}_s$ *(the outer indexes) (Konoplev 1985 and 1987a, Konoplev et al. 2001).*

Comments

1. For the first time, the necessity in the notion 'kinetic screw' has been arisen for constructing the rigid body motion equations. The Radon vector measure (5.168) is not necessary when one studies a local linearly changing medium.
2. Density (2.31) of the inertial dynamical measure connects with the density of (5.168) due to

$$\rho_0^{v^{\cdot}(x),\,0} = d/dt\ \rho_0^{v(x),\,0} \tag{5.169}$$

3. The kinetic screw of the body B in the inertial frame $\mathbf{E}_0$ is defined as in (5.168)

$$Q_{B0}^{00} = \int \chi_B l_0^{v(x),0} m_i(dx) \tag{5.170}$$

Notation Henceforth 6×6–dimensional inertia matrices Θ_0^0 and Θ_s^s (Mises ones) of the rigid body B defined at the points o_0 and o_s (the lower index) in the bases $[\mathbf{e}^0]$ and $[\mathbf{e}^s]$ (the upper index) are defined in the form

$$\Theta_k^k = \begin{bmatrix} m_i(B)E & -< r_c^k >^{k,T} m_i(B) \\ < r_c^k >^k m_i(B) & \theta_k^k \end{bmatrix} \tag{5.171}$$

where $k = 0$ or $k = s$.

Proposition 5.48 *Let* $L_s^0 \in \mathcal{L}(\mathcal{R}, 6)$ *be the transformation of the coordinates of sliding vectors that is induced by the motion* $\mathbf{E}_s \to \mathbf{E}_0$*. Then:*

– *the kinetic screw* Q_{Bs}^{0s} *of* B *being in motion in* $\mathbf{E}_s$ *taken with respect to* $\mathbf{E}_0$ *is a linear transformation of the space of quasi–velocities of* B *(Konoplev 1985, 1987a and 1990, Konoplev et al. 2001)*

$$Q_{Bs}^{0s} = \Theta_s^s V_{ss}^{0s} = \Theta_s^s V_s^{0s} \tag{5.172}$$

– *the kinetic screws* Q_{B0}^{00} *and* Q_{Bs}^{0s} *are connected by the following relation*

$$Q_{B0}^{00} = L_s^0 Q_{Bs}^{0s} \tag{5.173}$$

– *the Mises matrices* Θ_0^0 *and* Θ_s^s *are connected by*

$$\Theta_0^0 = L_s^0 \Theta_s^s L_s^{0,T} \tag{5.174}$$

Proof 1. With the help of (5.168), (5.165) we obtain

$$Q^{0s}_{Bs} = \int \chi_B l^{vs}_s m(dx) = \int \chi_B \Theta^s_x V^{0s}_{ss} m(dx) = \int \chi_B \Theta^s_x m(dx) V^{0s}_{ss} = \Theta^s_s V^{0s}_s$$

2. Equation (5.173) is fulfilled due to the matrix L^0_s definition.

3. The proof of the equality $Q^{00}_{Bs} = \Theta^0_0 V^{00}_{s0}$ is similar to the one in the proof beginning. Because of relation (7.7), we get $V^{00}_{s0} = L^{s,T}_0 V^{0s}_{ss}$ and therefore $Q^{00}_{Bs} = \Theta^0_0 L^{s,T}_0 V^{0s}_{ss}$. But from relations (5.173) and (5.172) follows that $Q^{00}_{B0} = L^0_s Q^{0s}_{Bs} = L^0_s \Theta^s_s V^{0s}_s$. Therefore $\Theta^0_0 L^{s,T}_0 = L^0_s \Theta^s_s$, and finally we obtain relation (5.174).

Comment From relation (5.171) follows that the Mises matrix is the one of some quadratic form.

Proposition 5.49 *The Mises matrix has two three–dimensional invariant subspaces of quasi–velocities (shift and rotation ones). In the appropriative frame it has the form*

$$\Theta^s_{sc} = \mathrm{diag}\{m_i(B)E, \theta^k_{kc}\},\ \mathbf{E}_{sc} = (o_{sc}, [\mathbf{e}^s]) \tag{5.175}$$

Proof We arrive at relation (5.175) from (5.171) if we take the frame origin at the mass center of B, *i.e.*, $r^{ss}_c = 0$.

Definition 5.26 *A frame with origin at the mass center of B is called central frame of inertia.*

Proposition 5.50 *The Mises matrix has 6 one–dimensional invariant subspaces of quasi–velocities. In the appropriative basis it has the form*

$$\Theta^s_{sc} = \mathrm{diag}\{m_i(B), m_i(B), m_i(B), I^{sc}_{44}, I^{sc}_{55}, I^{sc}_{66}\} \tag{5.176}$$

where I^{sc}_{44}, I^{sc}_{55} and I^{sc}_{66} are the main inertial torques.

Proof follows from the fact that the Mises matrix is symmetric.

Definition 5.27 *The frame of inertia where the Mises matrix is diagonal is called the main central one.*

Definition 5.28 *Let: 1. the Mises matrix Θ^s_s be defined w.r.t. $\mathbf{E}_s$;*
2. the its origin do not coincide with the mass center of B, i.e.,

$$o_s = c \tag{5.177}$$

Then the body B is called dynamically unbalanced in $\mathbf{E}_s$.

Comment The last notion has two meaning: in the physical meaning the body is dynamically unbalanced if its mass center is not on the rotation axis (defined by corresponding hinges); in the mathematical sense it is dynamically unbalanced in any frame attached to it if $r^{ss}_s \neq 0$ (*e.g.*, in the frame with the origin on the physical rotation axis that does not pass through the mass center).

Definition 5.29 *Let: 1. the Mises matrix Θ_c^c be defined in the main central frame of inertia – see (5.176);*

2. the main inertial torques such that

$$I_{44}^{sc} \neq I_{55}^{sc} \neq I_{66}^{sc} \tag{5.178}$$

Then the body B is called dynamically asymmetric in $\mathbf{E}_c$.

Proposition 5.51 *Let: 1. B be a rigid body;*

2. $I_0^0(\urcorner B, B)$ be the inertial dynamical screw of the body (see (2.29)) in the inertial frame $\mathbf{E}_0$;

3. $Q_{B0}^{00} \equiv Q_{s0}^{00}$ be the kinetic screw (5.170) of the body in the inertial frame $\mathbf{E}_0$.

Then the inertial dynamical screw of B is defined by the following relation

$$I_0^0(\urcorner B, B) = -Q_{B0}^{00\cdot} \tag{5.179}$$

Proof follows from relations (2.29), (5.170) and (5.169), (2.47).

Comments

1. Relation (5.179) is the continual analog 'theorems about changing momentum and moment of momentum', proved for the case of a finite set of points with concentrated masses. Here it is proved only for the rigid body B (which has the defined kinetic screw Q_{B0}^{00} (see (5.170)).
2. Relation (5.179) is not a law of the nature as it is usually considered. In the correct meaning it is not a property of Galilean mechanics Universe as there are neither points with concentrated masses, nor rigid bodies. Moreover, it does not arisen in mechanics of locally changed media at all. In other words, relation (5.179) is an element of the applied mechanics having sense if a real mechanical system can be modeled as a finite set of points with concentrated masses or as a rigid body.
3. All terms of the right–hand side of relation (5.179) have the inertial nature, *i.e.*, are components of the dynamical screw of the inertia. They can have different form in different frames. In practice, the frame attached to the body has the most interest.

Proposition 5.52 *Let: 1. B be a rigid body;*

2. $\mathbf{E}_s$ be a frame attached to it; $\mathbf{E}_0$ be an inertial frame;

3. $Q_{B0}^{00} \equiv Q_{s0}^{00}$ be the kinetic screw (5.170) of the body in the inertial frame $\mathbf{E}_0$;

4. $F_0^0(\urcorner B, B)$ be the dynamical screw of action of the outer medium $\urcorner B$ with non–inertial origination (the supplement of B in σ-algebra σ_3^μ (5.28) defined in $\mathbf{E}_0$

$$F_0^0(\urcorner B, B) = G_0^0(\urcorner B, B) - D_0^0(B, \urcorner B) \tag{5.180}$$

where $G_0^0(\urcorner B, B)$ and $D_0^0(B, \urcorner B)$ are the dynamical screws (2.36) and (2.42) of gravitation (of B) and deformation of the outer medium (by the body B).

Then the free motion of B (w.r.t. $\mathbf{E}_0$ in its basis) is described by the following equation

$$Q_{B0}^{00\cdot} = F_0^0(\urcorner B, B) \tag{5.181}$$

Proof Due to Axiom D 4 (2.48) and the additivity of the dynamical measure on σ_3^μ we have

$$-\rho_0^{v^\cdot(x),\ 0} + \rho_0^{g(\urcorner x,\ x),\ 0} + \rho_0^{\Delta(\urcorner x,\ x),\ 0}\rho_y^{-1} = 0 \tag{5.182}$$

i.e., in the dynamical balanced Galilean mechanics Universe the sum of the densities (w.r.t. $m_i(d(x))$ of dynamical screws (of inertia, gravitation and deformation) is equal 0 at every point the absolutely continual medium.

After integrating (5.182) by the scalar inertial measure $m_i(dx)$ we have $-\int \chi_B \rho_0^{v^\cdot(x),\ 0} m_i(dx) + \int \chi_B \rho_0^{g(\urcorner x,\ x),\ 0} m_i(dx) + \int\ \chi_B \rho_0^{\Delta(\urcorner x,\ x),\ 0}\rho_y^{-1} m_i(dx) = 0.$

But $\int\ \chi_B \rho_0^{g(\urcorner x,\ x),\ 0} m_i(dx) = \int\ \chi_B(\rho_0^{g(\urcorner B\setminus\{x\},\ x),\ 0} + rho_0^{g(\urcorner B,\ x),\ 0}) m_i(dx) = 0 +$ $\int\ \chi_B \rho_0^{g(\urcorner B,\ x),\ 0} m_i(dx) = G_0^0(\urcorner B, B)$ and, similarly, $\int\ \chi_B \rho_0^{\Delta(\urcorner x,\ x),\ 0}\rho_y^{-1} m_i(dx) =$ $\int\ \chi_B(\rho_0^{\Delta(B\setminus\{x\},\ x),\ 0}\rho_y^{-1} + \rho_0^{\Delta(\urcorner B,\ x),\ 0}\rho_y^{-1}) m_i(dx) = 0 + \int\ \chi_B \rho_0^{\Delta(\urcorner B,\ x),\ 0}\rho_y^{-1} m_i(dx)$ $= D_0^0(B, \urcorner B)$. Thus $\int\ \chi_B \rho_0^{v(\urcorner x),\ 0\cdot} m_i(dx) = F_0^0(\urcorner B, B)$. Hence

$$d/dt \int \chi_B \rho_0^{v^\cdot(x),\ 0} m_i(dx) = F_0^0(\urcorner B, B) \text{ or } d/dt\ Q_{B0}^{00} = F_0^0(\urcorner B, B)$$

Proposition 5.53 *Let: 1. B be a rigid body;*

2. Q_{Bs}^{0s} be the kinetic screw of B w.r.t. $\mathbf{E}_0$ defined in $\mathbf{E}_s$ (see (5.168));

3. $V_s^{0s} = \mathrm{col}\,\{v_s^{0s}, \omega_s^{0s}\}$ be the vector of quasi–velocities of B w.r.t. $\mathbf{E}_0$;

4. Φ_s^{0s} be 6×6 matrix generated by V_s^{0s} in the following form (Konoplev 1987a)

$$\Phi_s^{0s} = \begin{bmatrix} <\omega_s^0>^s & 0 \\ <v_s^0>^s & <\omega_s^0>^s \end{bmatrix} \tag{5.183}$$

5. the asterisk * be the differentiation symbol in the frame $\mathbf{E}_s$ attached to B.

Then the free motion of B (w.r.t. $\mathbf{E}_0$ in the basis of $\mathbf{E}_s$) is described by the following equation

$$Q_{Bs}^{0s*} + \Phi_s^{0s} Q_{Bs}^{0s} = F_s^s(\urcorner B, B) \tag{5.184}$$

Proof Inserting relation (5.173) in equation (5.181) we arrive at $(L_s^0 Q_{Bs}^{0s})^\cdot =$ $F_0^0(\urcorner B, B)$ or $L_s^{0\cdot} Q_{Bs}^{0s} + L_s^0 Q_{Bs}^{0s\cdot} = F_0^0(\urcorner B, B)$. Hence multiplying the relation obtained from left–hand side on the matrix $(L_s^0)^{-1} = L_0^s$ and using (5.183) and (7.7) we have the result to be proved.

Comment When we pass to the basis of $\mathbf{E}_s$ attached to B in its motion equation there arise the second term of the inertial origination (the second component $\Phi_s^{0s} Q_{Bs}^{0s}$ of the inertial dynamical screw). This term depends on the elements of Φ_s^{0s} and Q_{Bs}^{0s}, but does not depend on velocities of their change.

Proposition 5.54 *Let: 1. B be a rigid body;*

2. $V_s^{0s} = \mathrm{col}\,\{v_s^{0s}, \omega_s^{0s}\}$ be the vector of quasi–velocities of B w.r.t. $\mathbf{E}_0$;

3. Θ_s^s be the Mises matrix (the constant matrix (5.171) defined w.r.t. $\mathbf{E}_s$).

Then the free motion of B (w.r.t. $\mathbf{E}_0$ in the basis of $\mathbf{E}_s$) is described by the following equation

$$\Theta_s^s V_s^{0s*} + \Phi_s^{0s}\Theta_s^s V_s^{0s} = F_s^s(\urcorner B, B) \tag{5.185}$$

in quasi–velocities (Konoplev 1985, Konoplev et al. 2001).

Proof The last equation issues directly from relations (5.184) and (5.172).

Comments

1. The term $\Phi_s^{0s}\Theta_s^s V_s^{0s}$ in (5.185) depends on second power of the quasi–velocity coordinates. It is traditionally separated in three groups: Coriolis, gyroscopic and centrifugal.
2. Equation (5.185) given in the basis of $\mathbf{E}_s$ attached to the body is convenient for using as, firstly, the Mises matrix Θ_s^s is constant and, secondly, the vectors of quasi-accelerations V_s^{0s*} and quasi–velocities V_s^{0s} can be easily measured by devices attached to the body (airplane, ship or satellite).
3. In the basis of $\mathbf{E}_0$ the Mises matrix Θ_s^s is not constant. What is why from equation (5.181) we have $Q_{B0}^{00\cdot} = F_0^0(\urcorner B, B) \to (\Theta_0^0 V_s^{00})^{\cdot} = F_0^0(\urcorner B, B)$ or

$$\Theta_0^0 V_s^{00\cdot} + \Theta_0^{0\cdot} V_s^{00} = F_0^0(\urcorner B, B) \tag{5.186}$$

Proposition 5.55 *Let: 1. B be a rigid body;*

2. the kinematics equation of simple free motion have the following form

$$V_s^{0s} = M_s^0 q^{s\cdot} \tag{5.187}$$

Then the free motion of B in the generalized coordinates q^s (see (5.21)) is described by the following equation (Konoplev 1985, Konoplev et al. 2001)

$$A_s^s(q^s)q^{s\cdot\cdot} + B_s^s(q^s, q^{s\cdot})q^{s\cdot} = F_s^s(\urcorner B, B) \tag{5.188}$$

$$A_s^s(q^s) = \Theta_s^s M_s^s,\ B_s^s(q^s, q^{s\cdot}) = \Theta_s^s M_s^{s\cdot} + \Phi_s^{0s}\Theta_s^s M_s^s \tag{5.189}$$

Comments

1. In equation (5.188) the right–hand side is given by the dynamical screw in $\mathbf{E}_s$ (*i.e.*, it is defined in coordinates of the frame $\mathbf{E}_s$ but not in the generalized coordinates q^s). What is why $A_s^s(q^s)$ is not a matrix of a quadratic form.
2. By multiplying the matrix $M_s^{0,T}$ and $F_s^s(\urcorner B, B)$ we may express the screw in the generalized coordinates q^s. Then equation (5.188) has the following form

$$A_s^s(q^s)q^{s\cdot\cdot} + B_s^s(q^s, q^{s\cdot})q^{s\cdot} = Q_s^s(\urcorner B, B) \tag{5.190}$$

$$A_s^s(q^s) = M_s^{0,T}\Theta_s^s M_s^s,\ Q_s^s(\urcorner B, B) = M_s^{0,T} F_s^s(\urcorner B, B) \tag{5.191}$$

$$B_s^s(q^s, q^{s\cdot}) = M_s^{0,T}(\Theta_s^s M_s^{s\cdot} + \Phi_s^{0s}\Theta_s^s M_s^s) \tag{5.192}$$

where the matrix $A_s^s(q^s)$ is symmetrical and therefore it may be the matrix of a quadratic form.

3. Relations (5.185), (5.188) and (5.190) are various forms of six differential equations of second order describing the free motion of the body B. Everyone of them contains the full set of coordinates of quasi–velocities and quasi-accelerations (generalized velocities and accelerations) for the body shift and rotation. That is why in general it is impossible to separate them on equations of partial motions: shift and rotation. In other words, from the free motion equations written in an arbitrary frame $\mathbf{E}_s$ attached to B we may define its motion of shift and rotation, but they are not inertially independent (due to inertial terms in the motion equations, without taking in account possible dependence of these motion generated by dynamical screws).
4. The inertial independence of shift and rotation motions of the body is not their kinematic independence. The last notion is absolute: if a phase point moves along some submanifold in $\mathbf{Q}_n$ (see (5.60)) in the phase space which depends on the coordinates of the both motions then these motions are kinematically dependent. If no such submanifolds are then these motions are kinematically independent. The first notion is conditional: the corresponding condition is the choice of a frame attached to the body, and in particular the choice of the frame origin. It is clear that from the kinematical dependence follows the dynamical one (through dynamical screws of constraints) (if some particular cases are omitted), but the return result is not true.

Definition 5.30 *The quadratic form*

$$T = \frac{1}{2}V_s^{0s,T}\Theta_s^s V_s^{0s} = \frac{1}{2}V_s^{00,T}\Theta_0^0 V_s^{00} = \frac{1}{2}q^{s,T\cdot} A_s^s q^{s\cdot} \tag{5.193}$$

is called the kinematical energy of the body in quasi–velocities, velocities and generalized velocities, respectively.

Comment From (5.193) follows that the Mises matrix Θ_s^s in relation (5.171) is the matrix of the kinematical energy of the body in quasi–velocities.
The question is: do frames exit where motions of shift and rotation w.r.t. the inertial frame are inertially independent? The answer is positive and it can be seen in the following propositions.

Proposition 5.56 *Let: 1. B be a rigid body;*

2. the equation of the free motion of B w.r.t. the inertial frame in the central inertial frame $\mathbf{E}_{sc}$ (see (5.175)) have the following form (Konoplev 1985)

$$\Theta_c^s V_c^{0s*} + \Phi_c^{0s}\Theta_c^s V_c^{0s} = F_c^s(\urcorner B, B) \tag{5.194}$$

3. $F^s(\urcorner B, B)$ and $F_c(\urcorner B, B)$ are main vector and momentum *of the dynamical screw $F_c^s(\urcorner B, B)$ (see (5.188)) such that*

$$F_c^s(\urcorner B, B) = \mathrm{col}\{F^s(\urcorner B, B),\ F_c(\urcorner B, B)\} \tag{5.195}$$

Then 6×6–dimensional matrix equations (5.174) in quasi–velocities separate into two 3×3–dimensional matrix equations, the first one

$$(V_c^{0s*} + <\omega_s^0>^s v_c^{0s})m_i(B) = F^s(\urcorner B, B) \tag{5.196}$$

depending on quasi–velocities of shift and rotation, and the second one being the following independent equation of the free rotation of B in quasi–velocities

$$\theta^s_c \omega^{0s*}_s + < \omega^0_s >^s \theta^s_c \omega^{0s}_s = F_c(\urcorner B, B) \tag{5.197}$$

Proof The simple check confirms that the proposition is true.

Comment From (5.196) does not follow that the shift of the mass center does not depend inertially on the angle velocity. This is in spite of that the angle velocity ω^{0s}_s is in the left–hand side of (5.196) (but we cannot say the same about the velocity ω^{00}_s). Indeed, differentiating (5.196) in the inertial frame we have $v^{0s}_c = c^{0,T}_s v^{00}_c$, $< \omega^0_s >^0 = c^0_s < \omega^0_s >^s c^{0,T}_s$, $F^s(\urcorner B, B) = c^{0,T}_s F^0(\urcorner B, B)$ and

$$\begin{aligned} m_i(B) v^{00\cdot}_c &= F^0(\urcorner B, B) && (5.198)\\ \theta^s_c \omega^{0s*}_s &+ < \omega^0_s >^s \theta^s_c \omega^{0s}_s = F_c(\urcorner B, B) && (5.199)\end{aligned}$$

Comments

1. In fact, the proposition above is the answer on the question about existence of frames where shift and rotation motions of B are inertially independent: in order that shift and rotation motions are inertially independent it is necessary and sufficient that the motion (w.r.t. $\mathbf{E}_s$) equations are written in the central frame.
2. In fact, relation (5.196) is the equation of mass center motion, the body B being considered as a continual continuous mechanical system where $m_i(B) = \int \chi_B \rho^i_y \mu_3(dx)$. In contrast with the equations for a finite set of points with concentrated masses (which is called as a rigid body) that is the standard object of 'analytical mechanics' we have always $m_i(B) = \sum_p m_p \neq \int \chi_B \rho^i_y \mu_3(dx)$.
3. Equation (5.198) is not the second law of Newtown. It is precisely formulated and proved without using the notion of points with concentrated masses that do not exist in the nature.
4. When solving the applied problems one must remember that equations (5.198) and (5.199) are written in the different bases.

Proposition 5.57 *Let $\mathbf{E}_c$ be the main central frame (see D 5.27).*

Then: 1. there is the known form of Newtown-Euler equation

$$\begin{aligned} m_i(B) v^{00\cdot}_c &= F^0(\urcorner B, B) && (5.200)\\ \theta^c_s \omega^{0c*}_s &+ < \omega^0_c >^c \theta^c_c \omega^{0c}_c = F_c(\urcorner B, B) && (5.201)\end{aligned}$$

2. relation (5.201) is the Euler dynamical equation for the body considered as a continual continuous mechanical system where $\theta^c_c = -\int \chi_B < r^c_x >^{c2} \rho_i \mu_3(dx)$; it is clear in contrast with the above for a finite set of points with concentrated masses (which is called as a rigid body) that is a standard object of 'analytical mechanics' we have always

$$\theta^c_c = -\sum_p < r^c_p >^{c2} m_p \neq -\int \chi_B < r^c_x >^{c2} \rho_i \mu_3(dx) \tag{5.202}$$

3. *the shift and rotation motions of B are inertially independent when their equations are written in the main central frame.*

Proof Only the first point of the proposition is need to be proved. It is clear that equations (5.200) and (5.201) are the result of multiplying the matrices in the left–hand side of relation (5.194) with the help of (5.176).

5.4.2. *Equations of simple constrained motion of rigid body*

Proposition 5.58 *Let: 1. the body motion be such that the phase point x^s (see (5.21)) moves along some submanifold in the configuration part of the phase space $\mathbf{Q}_q$. This submanifold is defined by the following relations*

$$o_p^s = \text{const},\ \text{for some } p \in \overline{1,3},\ \theta_p^s = \text{const},\ \text{for some } p \in \overline{4,6} \tag{5.203}$$

under the condition that for other coordinates no relations of the kind (5.203) are (the simplest holonomic constraints);

2. *$\|f^s\|$ be the matrix of 6–dimensional unit orthogonal vectors (with 1 at i–th for some $i \in \overline{1,6}$) of the body movability axes ($(o;s)$–kinematical couple) (see (5.65))).*
3. *q^s be the column of the body generalized coordinates – see D (5.7;*
4. *the kinematical equations of simple constrained motion of the body have the following form (see (5.67)) (Konoplev 1985, Konoplev et al. 2001)*

$$V_s^{0s} = M_f^0 \|f^s\| q^s \tag{5.204}$$

5. *M_f^0 and $M_f^{0\cdot}$ be the matrices from (5.67) and (6.22).*

Then the kinematical equations of simple constrained motion of the body in generalized coordinates have the following form

$$\begin{aligned} A_s^s(q^s)q^{s\cdot\cdot} \quad &+ \quad B_s^s(q^s, q^{s\cdot})q^{s\cdot} = Q_s^s(\urcorner B, B) && (5.205)\\ A_s^s(q^s) \quad &= \quad \|f^s\|^T M_f^{0,T} \Theta_s^s M_f^0 \|f^s\| && (5.206)\\ B_s^s(q^s, q^{s\cdot}) \quad &= \quad \|f^s\|^T M_f^{0,T} (\Theta_s^s M_f^{0\cdot} + \Phi_s^{0s} \Theta_s^s M_f^0) \|f^s\| && (5.207)\\ Q_s^s(\urcorner B, B) \quad &= \quad \|f^s\|^T M_f^{0,T} (G_s^s(\urcorner B, B) - D_s^s(B, \urcorner B)) && (5.208) \end{aligned}$$

Proof is obtained by replacing the matrix $M_f^{0,T}$ with $\|f^s\|^T M_f^{0,T}$ in equations (5.190).

Comments

1. After multiplying equation (5.188) with the matrix $M_f^{0,T}$ from left–hand side, we obtain the new equation in the non–orthogonal basis of the body movability axes. Moreover the components of the dynamical screw of the constraint reactions are zero (but no other components are zero, *i.e.*, by these axes where the body motion is forbidden by constraints (5.203)). After multiplying equation with the matrix $\|f^s\|^T M_f^{0,T}$ from left–hand side, we obtain equations w.r.t. axes along which the motion is permitted by constraints (5.203), *i.e.*,

the equations of simple constrained motion of the body. In result we have $\|f^s\|^T M_f^{0,T} R_s^s(\urcorner B, B) = 0$. Due to the last relation the dynamical screw coordinates do not enter in the equations obtained. These equations have k scalar equations with k generalized coordinates of the body (where $k < 6$).

2. If the body has one power of freedom or two cylinder (but not screw), then the motion equations have the more simple form as in this case

$$M_f^0 \|f^s\| = \|f^s\| \tag{5.209}$$

$$A_s^s(q^s) = \|f^s\|^T \Theta_s^s \|f^s\|, \tag{5.210}$$

$$B_s^s(q^s, q^{s\cdot}) = \|f^s\|^T \Phi_s^{0s} \Theta_s^s \|f^s\|$$

$$Q_s^s(\urcorner B, B) = \|f^s\|^T F_s^s(\urcorner B, B) \tag{5.211}$$

Definition 5.31 *The quadratic form*

$$T = \frac{1}{2} V_s^{0s,T} \Theta_s^s V_s^{0s} = \frac{1}{2} q^{s,T\cdot} \|f^s\|^T M_f^{0,T} \Theta_s^s M_f^0 \|f^s\| q^{s\cdot} \tag{5.212}$$

is called the kinematical energy of the body in simple constrained motion with the simplest holonomic constraints (5.203).

5.4.3. *Motion equations of body bearing dynamically unbalanced and asymmetric rotation bodies*

Proposition 5.59 *Let: 1. B be a rigid body;*

2. *B_i be i-rotating body attached to B_s, $\mathbf{E}_i$ be a frame attached to B_i, Θ_i^s be its Mises of the kind (5.171) under condition (5.178) (B_i is a dynamically unbalanced and asymmetric rotation body in general);*
3. *$R_{ic}^s = \mathrm{col}\{p_1, p_2, p_3, \varphi_4, \varphi_5, 0\}$ be the constructive configuration of the kinematical couple (s, i) where $p_i^{si} = \mathrm{col}\{p_1, p_2, p_3\}$ is the constructive vector of B_i position in E_i computed in the basis $[\mathbf{e}^s]$; φ_4 and φ_5 are the constructive attitude angles of the basis $[\mathbf{e}^{ik}]$ computed in the basis $[\mathbf{e}^s]$ (see (5.63));*
4. *$R_i^{ic} = \mathrm{col}\{\mathbf{0}, 0, 0, \theta_6^i\}$ be the function configuration of the kinematical couple (s, i) where θ_6^i is the rotation angle of $[\mathbf{e}^i]$ (i.e., the body B_i) w.r.t. the basis $[\mathbf{e}^{ik}]$ with the unit orthogonal vector $e_6^{ik} = e_6^i$, $f_6^i = \mathrm{col}\{\mathbf{0}, e_3^i\}$ (see (5.63));*
5. *$L_i^s = L_{ic}^s L_i^{ic}$ be the transform matrices of the (coordinate) screws generated by motions $\mathbf{E}_s \to \mathbf{E}_{ic} \to \mathbf{E}_i$ in the correspondence with the configurations R_{ic}^s and R_i^{ic};*
6. *$n(s)$ be the number of rotating bodies attached to the body B_s;*
7. *the velocities of changing the angles θ_6^i, $i = 1, 2, \ldots, n(s)$, do not depend on the B motion (the feedback of the bearing body to the rotating bodies is absent);*
8. *for the writing simplicity $Q_s^s + \sum_{i=1}^{n(s)} L_i^s Q_i^i \Longleftrightarrow Q_s^s$.*

Then the motion equation of the body B_s with the attached bodies B_i has the following form (Konoplev 1986a, 1986c and 1987b, Konoplev et al. 2001)

$$A_s^s V_s^{0s*} + B_s^s V_s^{0s} = F_s^s(\urcorner B, B) + T_s^s(\cup B_i, B) \tag{5.213}$$

where

$$A_s^s = \Theta_s^s + \sum_{i=1}^{n(s)} L_i^s \Theta_i^i L_i^{s,T}, B_s^s = \Phi_s^{0s} A_s^s + A_s^{s\cdot}$$

$$A_s^{s\cdot} = \sum_{i=1}^{n(s)} L_i^s([< e_3^i >]\Theta_i^i - \Theta_i^i[< e_3^i >]) L_i^{s,T} \theta_6^i$$

$$T_s^s(\cup B_i, B) = \sum_{i=1}^{n(s)} L_i^s G_i^i - \sum_{i=1}^{n(s)} (I_i^s \theta_6^{i\cdot\cdot} + J_i^i \theta_6^{i\cdot}), I_i^s = L_i^s \Theta_i^i f_6^i$$

$$J_i^i = (\Phi_s^{0s} L_i^s + L_i^s[< e_3^i >]\theta_6^{i\cdot})\Theta_i^i f_6^i$$

Proof Any kinematical screw is a vector measure on σ-algebra σ_3^μ of mechanical systems of the mechanics Universe and therefore for the system $B = B_s \cup B_1 \cup B_2 \cup \cdots \cup B_{n(s)}$ we have $Q_{B0}^{00} = Q_{s0}^{00} + Q_{10}^{00} + Q_{20}^{00} + \cdots + Q_{n(s)0}^{00}$ where we use the notation $Q_{B_m 0}^{00} = Q_{m0}^{00}$, $m = s, 1, 2, \ldots, n(s)$. Thus equation (5.181) takes the following form

$$(Q_{s0}^{00} + Q_{10}^{00} + Q_{20}^{00} + \cdots + Q_{n(s)0}^{00})^{\cdot} = F_0^0(\urcorner B, B) \tag{5.214}$$

Further on we have the obvious transformation with using relations (5.172), (5.173) and $V_i^{0i} = L_s^{0,T} V_s^{0s} + V_i^{si}$ (see (5.69)), namely,

$$L_s^0 \Theta_s^s V_s^{0s} + \sum_{i=1}^{n(s)} L_i^s \Theta_i^i V_i^{oi\cdot} = L_s^0 Q_s^s,\ L_s^0(\Theta_s^s V_s^{0s} + \sum_{i=1}^{n(s)} L_i^s \Theta_i^i V_i^{oi})^{\cdot} = L_s^0 Q_s^s$$

$$L_s^{0\cdot}(\Theta_s^s V_s^{0s} + \sum_{i=1}^{n(s)} L_i^s \Theta_i^i V_i^{oi}) + L_s^0(\Theta_i^i A_s^s V_s^{0s} + \sum_{i=1}^{n(s)} L_i^s \Theta_i^i V_i^{oi\cdot}) = L_s^0 Q_s^s$$

After multiplying the relation obtained on $(L_s^0)^{-1} = L_0^s$ from the left–hand side we have to–be–proved proposition due to the relations $\Phi_i^{si} = [< \omega_i^s >^s] = [< e_3^i >]\theta_6^{i\cdot}$, $V_i^{si} = f_6^i \theta_6^{i\cdot}$, $V_i^{si*} = f_6^i \theta_6^{i\cdot\cdot}$.

Comments

1. Equation (5.213) is not the motion equation for the system '$B = B_s \cup B_1 \cup B_2 \cup \cdots \cup B_{n(s)}$'. This is the motion equation for the free body B_s (with the attached bodies – their rotation does not depend on the B motion).
2. The term $\sum_{i=1}^{n(s)} (I_i^s \theta_6^{i\cdot\cdot} + J_i^i \theta_6^{i\cdot})$ in the right–hand side of equation (5.213) defines only one part of the inertial dynamical screws of the rotating bodies, the other part is in the term $A_s^s V_s^{0s*} + B_s^s V_s^{0s}$.
3. Equation (5.213) permits us to study the influence of the bodies B_i rotation on the B motion when the feedback is eliminated by control systems or it is considered as neglected small.

5.4.4. *Free rigid body motion in inertial external medium*

Proposition 5.60 *Let: 1. B_s be a rigid body with attached dynamically unbalanced and asymmetric rotation bodies B_i, $i = 1, 2, \ldots, n(s)$, V_s^{0s} be the quasi–velocity of B_s w.r.t. the inertial frame $\mathbf{E}_0$;*

2. *in Galilean mechanics Universe there exit an open mechanical system D such that:*

   2.1. *D is a locally linear changing continuous medium (see D 2.4);*

   2.2 *the deforming of the system D in result of the B_s motion is potential (non–circular) (see (3.94));*

   2.3 *the kinematical screw Q^{0s}_{D0} of the system D in $\mathbf{E}_s$ for motion w.r.t. $\mathbf{E}_0$ is*

$$Q^{0s}_{D0} = \Lambda^s_s V^{0s}_s \tag{5.215}$$

where Λ^s_s is a linear map.

Then the motion equation (for the body B_s w.r.t. $\mathbf{E}_0$) in the basis $[\mathbf{e}^s]$ of the frame $\mathbf{E}_s$ has the following form (Konoplev 1986a and 1987b, Konoplev et al. 2001)

$$A^s_s V^{0s*}_s + B^s_s V^{0s}_s = Z^s_s + H^s_s + N^s_s \tag{5.216}$$

where $A^s_s = \Sigma^s_s + \sum\limits_{i=1}^{n(s)} L^s_i \Theta^i_i L^{s,T}_i$, $B^s_s = \Phi^{0s}_s A^s_s + A^{s\cdot}_s$, $A^{s\cdot}_s = \sum\limits_{i=1}^{n(s)} L^s_i([< e^i_3 >]\Theta^i_i - \Theta^i_i[< e^i_3 >])L^{s,T}_i \theta^i_6$, $T^s_s = \sum\limits_{i=1}^{n(s)} L^s_i G^i_i - \sum\limits_{i=1}^{n(s)} (I^s_i \theta^{i\cdot\cdot}_6 + J^i_i \theta^{i\cdot}_6)$, $I^s_i = L^s_i \Theta^i_i f^i_6$, $J^i_i = (\Phi^{0s}_s L^s_i + L^s_i[< e^i_3 >]\theta^{i\cdot}_6)\Theta^i_i f^i_6$, $\Sigma^s_s = \Lambda^s_s + \Theta^s_s$.

Proof of equation (5.216) is similar to proving (5.213) with taking into account the relation $Q^{00}_{B0} = Q^{00}_{s0} + Q^{00}_{D0} + Q^{00}_{10} + Q^{00}_{20} + \cdots + Q^{00}_{n(s)0}$ where $B = B_s \cup B_D \cup B_1 \cup B_2 \cup \cdots \cup B_{n(s)}$.

Definition 5.32 *The constant symmetrical matrix Λ^s_s is traditionally called the attached mass matrix of the B_s body.*

5.4.5. *Simple motion equations for free and constrained rigid body in Lagrange form of II kind*

Proposition 5.61 *Let: 1. M^s_f be 6×6–dimensional transform matrix from the basis $[\mathbf{e}^s] \times [\mathbf{e}^s]$ to the non–orthogonal movability basis $[\mathbf{f}^s]$ such that (see (5.67))*

$$V^{0s}_s = M^s_f q^{s\cdot} \tag{5.217}$$

2. $[c^0_s] = \mathrm{diag}\{c^0_s, c^0_s\}$ *where c^0_s is the rigid body rotation matrix;*
3. *M^0_f be 6×6–dimensional transform matrix from the basis $[\mathbf{e}^0] \times [\mathbf{e}^0]$ to the non–orthogonal movability basis $[\mathbf{f}^s]$ such that (see (5.67))*

$$V^{00}_s = M^0_f q^{s\cdot}\ , M^0_f = [c^0_s] M^s_f \tag{5.218}$$

4. *q^s and $q^{s\cdot}$ be generalized coordinates and velocities of the B free motion* ($\dim q^s = 6$)*;*

5. π–coordinates of B in ($\pi_s^{00\cdot} = V_s^{00}$) be continuous functions of the time.

Then

$$dV_s^{00}/dq^{s\cdot} = M_f^0 \tag{5.219}$$

$$dV_s^{00}/dq^s = M_f^{0\cdot} \tag{5.220}$$

Proof 1. $dV_s^{00}/dq^{s\cdot} = dM_f^0 q^{s\cdot}/dq^{s\cdot} = M_f^0$; 2. $dV_s^{00}/dq^s = d\pi_s^{00\cdot}/dq^s = (d\pi_s^{00}/dq^s)^\cdot = M_f^{0\cdot}$. We arrive at (5.220) with using $\pi_s^{00} = \int V_s^{00}\mu_1(dt) = \int M_f^0 q^{s\cdot}\mu_1(dt) \to d\pi_s^{00}/dq^s = M_f^0$.

Comment Constructing the matrix $d\pi_s^{00}/dq^s$ is connected with computing 36 partial derivatives relatively to the generalized coordinate. In the case under consideration we exclude this computation as relations (5.46) and (5.42) give the matrix M_f^0 in the form convenient for calculating with the help of computers.

Proposition 5.62 *Let: 1. the kinetic energy of the free motion of B be*

$$T = T(V_s^{00}) = \frac{1}{2} V_s^{00,T}\Theta_0^0 V_s^{00} \tag{5.221}$$

where $V_s^{00} = V_s^{00}(q^s, q^{s\cdot}) = M_f^0 q^{s\cdot}$;

2. $\mathrm{grad}_q T$, $\mathrm{grad}_{q^\cdot} T$, $\mathrm{grad}_{V0} T$ *be gradients of the kinetic energy of the body in the subspace of generalized velocities, the configuration subspace of the phase space Q_q of the body in the vector space of velocities $V_s^{00} = [c_s^0]V_s^{0s}$, respectively.*

Then there are

$$\mathrm{grad}_q T = M_f^{0\cdot,T}\,\mathrm{grad}_{V0} T \tag{5.222}$$

$$\mathrm{grad}_{q^\cdot} T = M_f^{0,T}\,\mathrm{grad}_{V0} T \tag{5.223}$$

$$\mathrm{grad}_{q^\cdot} T = M_f^{0,T}(M_f^{0\cdot})^{-T}\,\mathrm{grad}_q T \tag{5.224}$$

where $(M_f^{0\cdot})^{-T} = (M_f^{0\cdot,T})^{-1}$.

Proof The first two relations are obtained with the help of relations (5.219) and (5.220) with using common rules for computing derivatives of composite functions. The last one is the result of excluding the vector $\mathrm{grad}_{V0}T$ from them.

Proposition 5.63 *Let: 1. B be a rigid body and its free motion equation w.r.t. $\mathbf{E}_0$ in its basis be (see (5.181))*

$$Q_{B0}^{00\cdot} = F_0^0(\urcorner B, B) \tag{5.225}$$

$$F_0^0(\urcorner B, B) = G_0^0(\urcorner B, B) - D_0^0(B, \urcorner B) \tag{5.226}$$

2. $Q(\urcorner B, B) = M_f^{0,T} F_0^0(\urcorner B, B)$ be the generalized forces computed in generalized coordinates of the body (the dynamical screw $F_0^0(\urcorner B, B)$ computed in non–orthogonal movability basis $[\mathbf{f}^s]$).

Then the free motion of B (w.r.t. $\mathbf{E}_0$ in its basis) is described by the following Lagrange equation of the second kind

$$(\mathrm{grad}_{q^\cdot} T)^\cdot - \mathrm{grad}_q T = Q(\urcorner B, B) \tag{5.227}$$

Proof With the help of relations (5.222), (5.223)

$$(\mathrm{grad}_{q^{\cdot}} T)^{\cdot} - \mathrm{grad}_q T = M_f^{0,T} (\mathrm{grad}_{V0} T)^{\cdot} + M_f^{0\cdot,T} \mathrm{grad}_{V0} T -$$
$$(M_f^{0\cdot})^T \mathrm{grad}_{V0} T = M_f^{0,T} (\mathrm{grad}_{V0} T)^{\cdot} = M_f^{0,T} Q_{B0}^{00\cdot} =$$
$$M_f^{0,T} F_0^0(\urcorner B, B) = Q(\urcorner B, B)$$

In this simplest proof it is supposed that the equation form is known. Let us give another proof where this supposition is not made. Due to (5.251) $Q_{B0}^{00\cdot\cdot} = \mathrm{grad}_{V0} T = (M_f^0)^{-T} \mathrm{grad}_{q^{\cdot}} T$ and (5.222) we arrive at $d[(M_f^0)^{-T} \mathrm{grad}_{q^{\cdot}} T]/dt = F_0^0(\urcorner B, B)$. After differentiating in the last relation we have $(M_f^0)^{-T} d(\mathrm{grad}_{q^{\cdot}} T)/d\, t + (M_f^0)^{-T\cdot} \mathrm{grad}_{q^{\cdot}} T = F_0^0(\urcorner B, B)$ or $d(\mathrm{grad}_{q^{\cdot}} T)/dt + M_f^{0,T} (M_f^0)^{-T\cdot} \mathrm{grad}_{q^{\cdot}} T = M_f^{0,T} F_0^0(\urcorner B, B) = Q(\urcorner B, B)$. Hence with the help of (5.222) there is

$$d(\mathrm{grad}_{q^{\cdot}} T)/dt + M_f^{0,T} (M_f^0)^{-T\cdot} M_f^{0,T} (M_f^{0\cdot})^{-T} \mathrm{grad}_q T = Q(\urcorner B, B) \qquad (5.228)$$

where it is clear that $(M_f^0)^{-T\cdot} \neq (M_f^{0\cdot})^{-T}$ as $M_f^0 (M_f^0)^{-1} = E \to [M_f^0 (M_f^0)^{-1}]^{\cdot} = 0 \to -M_f^{0\cdot} = M_f^0 (M_f^0)^{-1\cdot} M_f^0 \to -(M_f^{0\cdot})^T = M_f^{0,T} (M_f^0)^{-T\cdot} M_f^{0,T}$. Inserting the last result in (5.228) and taking in account that $(M_f^{0\cdot})^T (M_f^{0\cdot})^{-T} = E$ we obtain equation (5.227).

Proposition 5.64 *Let: 1. the B body motion be simple with the simplest holonomic constraints of kind (5.203);*

2. $\|f^s\|$ *be the matrix of* 6*–dimensional unit orthogonal vectors (with* 1 *at i–th for some* $i \in \overline{1,6}$*) of the body movability axes ((o; s)–kinematical couple) (see (5.65)));*
3. *the kinematical equations of simple constrained motion of the body have the following form (see (5.67)) (Konoplev 1985, Konoplev et al. 2001)*

$$V_s^{0s} = M_f^s \|f^s\| q^{s\cdot} \qquad (5.229)$$

4. M_f^s *be* 6×6*–dimensional transform matrix from the basis* $[\mathbf{e}^0] \times [\mathbf{e}^0]$ *to the non–orthogonal movability basis* $[\mathbf{f}^s]$ *such that (see (5.218))*

$$V_s^{00} = M_f^0 q^{s\cdot},\ M_f^0 = [c_s^0] M_f^s \qquad (5.230)$$

where the first relation is the kinematic equation in velocities;

5. q^s *and* $q^{s\cdot}$ *be generalized coordinates and velocities of the B free motion (*$\dim q^s < 6$*);*
6. π_s^{00}*–coordinates of B in* $\mathbf{E}_0$ *be continuous functions of the time.*

Then

$$dV_s^{00}/dq^{s\cdot} = M_f^0 \|f^s\| \qquad (5.231)$$
$$dV_s^{00}/dq^s = M_f^{0\cdot} \|f^s\| \qquad (5.232)$$

Proof coincides literally with the proof of P 5.61.

Proposition 5.65 *Let: 1. the kinetic energy of the free motion of B be*

$$T = T(V_s^{00}) = \frac{1}{2} V_s^{00,T} \Theta_0^0 V_s^{00} \tag{5.233}$$

where $V_s^{00} = V_s^{00}(q^s, q^{s\cdot}) = M_f^0 \|f^s\| q^{s\cdot}$;

2. $\mathrm{grad}_q T, \mathrm{grad}_{q\cdot} T, \mathrm{grad}_{V0} T$ *be gradients of the kinetic energy of the body in the subspace of generalized velocities, the configuration subspace of the phase space* Q_q *of the body and the vector space of velocities* V_s^{00}($\dim \mathrm{grad}_{q\cdot} T = \dim \mathrm{grad}_q T = \dim q^s < 6$).

Then there are

$$\mathrm{grad}_q T \;=\; (M_f^{0\cdot} \|f^s\|)^T \, \mathrm{grad}_{V0} T \tag{5.234}$$

$$\mathrm{grad}_{q\cdot} T \;=\; (M_f^{0} \|f^s\|)^T \, \mathrm{grad}_{V0} T \tag{5.235}$$

Proof The two relations are obtained with the help of using common rules for computing derivatives of composite functions.

Proposition 5.66 *Let: 1. B be a rigid body (by supposition) (see (5.181))*

$$Q_{B0}^{00\cdot} \;=\; F_0^0(\urcorner B, B) \tag{5.236}$$

$$F_0^0(\urcorner B, B) \;=\; G_0^0(\urcorner B, B) - D_0^0(B, \urcorner B) + R_0^0(\urcorner B, B) \tag{5.237}$$

be its free motion equation w.r.t. $\mathbf{E}_0$ *in its basis with the simplest holonomic constraints of kind (5.203) where* $R_0^0(\urcorner B, B)$ *is the dynamical screw of external reactions;*

2. $Q(\urcorner B, B) = \|f^s\|^T M_f^{0,T} F_0^0(\urcorner B, B)$ *be the generalized forces computed in generalized coordinates of the body (the dynamical screw* $F_0^0(\urcorner B, B)$ *computed in the non–orthogonal movability basis* $[\mathbf{f}^s]$ $\dim\, Q(\urcorner B, B) = \dim q^s < 6$);

3. $\|f^s\|^T M_f^{0,T} R_0^0(\urcorner B, B) = 0$.

Then the free motion of B (w.r.t. $\mathbf{E}_0$ *in its basis) is described by the following Lagrange equation of the second kind*

$$(\mathrm{grad}_{q\cdot} T)^{\cdot} - \mathrm{grad}_q T = Q(\urcorner B, B) \tag{5.238}$$

Proof With the help of relations (5.234), (5.235)

$$\begin{aligned}
(\mathrm{grad}_{q\cdot} T)^{\cdot} \quad - \quad & \mathrm{grad}_q T = \|f^s\|^T M_f^{0,T} (\mathrm{grad}_{V0} T)^{\cdot} + \\
& \|f^s\|^T M_f^{0\cdot,T} \mathrm{grad}_{V0} T - \|f^s\|^T (M_f^{0\cdot})^T \mathrm{grad}_{V0} T = \\
& \|f^s\|^T M_f^{0,T} (\mathrm{grad}_{V0} T)^{\cdot} = \|f^s\|^T M_f^{0,T} Q_{B0}^{00\cdot} = \\
& \|f^s\|^T M_f^{0,T} F_0^0(\urcorner B, B) = Q(\urcorner B, B)
\end{aligned}$$

Comments

1. Relations (5.227) and (5.238) are Lagrange equation for a rigid body considered as a continual continuity medium that does not be a finite set of points with concentrated masses (they do not exist in the nature).

2. Proving (5.227) and (5.238) we use no 'classical' motion equations such as Newton laws, general equation of mechanics, central Lagrange equation, *etc.* For real rigid bodies considered as continual continuity media these equations do not exist as they are finite sums terms that contain zero masses (as inertial measures of continuum points).
3. Proving relations (5.227) and (5.238) are more short (in two rows!) and more transparent (in mathematical and physical senses) than proving 'classical' Lagrange equations for a finite set of points with concentrated masses under the same conditions.
4. Relation (5.227) is not intended for integrating (excepting cases of obtaining integrals). This is only an algorithm for construct scalar motion equations of rigid bodies. Matrix motion equations (5.205) obtained from relations (5.181) are ready to be used for integrating on computers with the help of the matrix software. The same equations can be use in systems of analytical computation on computers (Gerdt *et al.* 1980, Klimov 1989).

5.4.6. *Motion in potential field*

Definition 5.33 *Let:*

1. *in the configuration space of a rigid body B there be defined a scalar field (may be, the point q^s satisfies constraint (5.203))*

$$P = P(q^s) = \sum_i P_i(q^s) \tag{5.239}$$

2. *the generalized force $Q(\urcorner B, B)$ (action of the external medium on B) contain the additive terms*

$$Q_{pi}(\urcorner B, B) = -\operatorname{grad}_q P_i(q^s) \tag{5.240}$$

Then: 1. the scalar field (5.239) is called potential, the function $P(q^s)$ – the field potential;

2. the dynamical screw $Q_{pi}(\urcorner B, B)$ is called the potential dynamical one.

Proposition 5.67 *Let: 1. $Q_{pi}(\urcorner B, B)$ be potential dynamical screws;*

2. L be a scalar function such that $L : \mathbf{Q}_q \to \mathbf{R}_1$, it is equal the difference between the kinetic energy T of B and the field potential P where the body B is, i.e.,

$$L = T - P \tag{5.241}$$

3. $Q_s^s(\urcorner B, B)$ be the generalized forces of non–potential origination

$$Q_s^s(\urcorner B, B) = Q(\urcorner B, B) - \sum_i Q_{pi}(\urcorner B, B) \tag{5.242}$$

Then: 1. the kinematical equations of simple motion of the body in generalized velocities have the following form

$$A_s^s(q^s) q^{s\bullet\bullet} + B_s^s(q^s, q^{s\bullet}) q^{s\bullet} = Q_s^s(\urcorner B, B) - \operatorname{grad}_q P_i(q^s) \tag{5.243}$$

2. *the free motion of B (w.r.t. $\mathbf{E}_0$ in its basis) is described by the following Lagrange equation of the second kind*

$$d/dt \ \operatorname{grad}_{q^{\cdot}} L - \operatorname{grad}_q L = Q_s^s(\urcorner B, B) \tag{5.244}$$

Example: Dynamical screw of Earth gravitation

Let us suppose that

1. the whole Galilean mechanics Universe constitutes from the Earth planet (Z) and some body B_s, $x \in Z$ and $y \in B_s$;
2. the Earth planet has the spherical form with the radius R_Z;
3. the densities of Z and B_s are constant;
4. the inertial frame is chosen such that the coordinate plane e_1^0, o_0, e_2^0 is the horizon plane, but the unit e_3^0 is directed from the Earth center along its radius.

Then with using relation (2.36) we have

$$P_g = m(B) g r_{c3}^{00} \tag{5.245}$$

where r_{c3}^{00} is the third coordinate of the mass center radius vector of B_s in $\mathbf{E}_0$; $g = \gamma m(Z)(R_Z + r_{c3}^{00})^{-2}$ is the acceleration of the Earth gravitation.
For the main vector τ^0 of the dynamical screw of Earth gravitation (of B_s) in $[\mathbf{e}^0]$ we have

$$\tau^0 = \operatorname{grad}_q P_g = -m(B) g^0, \ g^0 = \operatorname{col}\{0, 0, g\} \tag{5.246}$$

In the attached frame $\mathbf{E}_s$ there are $\tau^s = c_s^{0,T} \tau^0$ and

$$G_s^s(Z, B) = G_{cs}^s [c_s^{0,T}] \operatorname{col}\{\tau^s, \tau^s\} \tag{5.247}$$

Hence the equation of B motion has the following form

$$A_s^s(q^s) q^{s\cdot\cdot} + B_s^s(q^s, q^{s\cdot}) q^{s\cdot} = Q_s^s(Z, B) - G_s^s(Z, B) \tag{5.248}$$

Comment Do not mix up the dynamical screw of Earth gravitation (of B) with the dynamical screw of B_s weight. If the body foundation does not move w.r.t. the Earth) they coincide one with another. If the foundation acceleration is directed along the free downfall acceleration the norm of the dynamical screw of B_s weight is less that the norm of the dynamical screw of Earth gravitation (of B_s), if it is directed in reverse side then there is the reverse relation. If the accelerations coincide (the body foundation downs freely then the dynamical screw of B_s weight is equal zero (the state of weightlessness).

5.5. Secondary properties of Galilean mechanics Universe

Proposition 5.68 *Let:*

1. *T be the kinetic energy of B (see (5.193), (5.221) and (5.233))*

$$T = \frac{1}{2} V_s^{0s,T} \Theta_s^s V_s^{0s} = \frac{1}{2} V_s^{00,T} \Theta_0^0 V_s^{00} = \frac{1}{2} q^{s,T\cdot} A_s^s(q^s) q^{s\cdot} \tag{5.249}$$

2. *Q_{B0}^{00}, Q_{Bs}^{0s} be the kinetic screws of the body in the motion w.r.t. the inertial frame $\mathbf{E}_0$ calculated in $\mathbf{E}_0$ and $\mathbf{E}_s$ (see (5.170), (5.168));*
3. *$Q_B^{0q} \equiv A_0^0(q^s) q^{s\cdot}$ be the generalized kinetic screws of the body in the motion w.r.t. the inertial frame $\mathbf{E}_0$ (in generalized coordinates).*

Then

$$T = \frac{1}{2} V_s^{0s,T} Q_{B0}^{00} = \frac{1}{2} V_s^{0s,T} Q_{Bs}^{0s} = \frac{1}{2} q^{s,T\cdot} Q_B^{0q} \tag{5.250}$$

Proof follows from the definitions of kinetic energy and kinetic screws in the terms of velocities, quasi–velocities and generalized velocities – see (5.172).

Proposition 5.69 *In the previous proposition notations*

$$Q_{B0}^{00} = \mathrm{grad}_{V0}\, T,\ Q_{Bs}^{0s} = \mathrm{grad}_{Vs}\, T,\ Q_B^{0q} = \mathrm{grad}_{q^\cdot}\, T \tag{5.251}$$

Proof is obtained with the help of computing gradients of the right–hand side of (5.249).

Comments

1. Relation (5.251) has been used many times before.
2. From relations (5.250) and (5.251) follows that

$$V_s^{00} = (\Theta_0^0)^{-1}\, \mathrm{grad}_{V0}\, T, V_s^{0s} = (\Theta_s^s)^{-1}\, \mathrm{grad}_{Vs}\, T \tag{5.252}$$

Proposition 5.70 *Let: 1. θ_0^0 and θ_s^s be 3×3–dimensional inertial matrices (bottom right 3×3–dimensional blocks of Mises matrices Θ_0^0 and Θ_s^s in $\mathbf{E}_0$ and $\mathbf{E}_s$);*

2. *$o_s^{00}, r_c^{00}, r_c^{ss}$ be coordinate columns of the vectors defining the positions of the $\mathbf{E}_s$ origin o_s^0 and the mass center of B in $\mathbf{E}_0$ and $\mathbf{E}_s$ and in their bases.*

Then

$$\theta_0^0 = c_s^0 \theta_s^s c_s^{0,T} - (< r_c^0 >^0 < o_c^0 >^0 + < o_s^0 >^0 < r_c^s >^0) m(B) \tag{5.253}$$

Proof is obtained by multiplying the matrices in relation (5.174).

Comment In particular if the attached frame is central (see (5.175)) then $r_c^{ss} = 0$ and therefore

$$\theta_0^0 = c_s^0 \theta_s^s c_s^{0,T} - < r_c^0 >^{02} m(B) \tag{5.254}$$

Proposition 5.71 *Let $D_{V0}(T)$, $D_{Vs}(T)$ and $D_{q^{\cdot}}(T)$ be the level surfaces of the kinetic energy in velocities, quasi–velocities and generalized velocities. Then*

$$Q_{B0}^{00} \perp D_{V0}(T),\ Q_{Bs}^{0s} \perp D_{Vs}(T),\ Q_{B}^{0q} \perp D_{q^{\cdot}}(T) \tag{5.255}$$

Proof follows from (5.251) and the fact that $\mathrm{grad}_0\, T \perp D(T)$ in spaces of any variables.

Proposition 5.72 *Let:*

1. *T be the kinetic energy of B (see (5.250)), B be in the field with potential $P = P(q^s)$;*
2. *$Q_s^s(\urcorner B, B)$ be the generalized forces of non–potential origination – see (5.242).*

Then there is the following relation

$$d/dt\ (q^{s\cdot} \cdot \mathrm{grad}_{q^{\cdot}}\, T - L) = q^{s\cdot} \cdot Q_s^s(\urcorner B, B) \tag{5.256}$$

Proof Relation (5.256) is true as it follows from Lagrange equation (5.242) due to the relations (5.241) and $P = P(q^s)$.

Proposition 5.73 *Let:*

1. *T be the kinetic energy of B (see (5.250)), B be in the field with potential $P = P(q^s)$;*
2. *all non–potential dynamical screws of the external medium action on B be equal zero (without the inertial one)*

$$Q_s^s(\urcorner B, B) \equiv 0$$

Then

$$T + P = \mathrm{const} \tag{5.257}$$

Proof follows from relation (5.256) considered along with (5.241), (5.250) and (5.251). Indeed $d/dt\,(q^{s\cdot} \cdot \mathrm{grad}_{q^{\cdot}}\, T - L) = 0 \to q^{s\cdot} \cdot \mathrm{grad}_{q^{\cdot}}\, T - L = \mathrm{const} \to T + P = \mathrm{const}$.

Comment Under conditions of P 5.73 the relations $T + P = \mathrm{const}$ and $q^{s\cdot} \cdot \mathrm{grad}_{q^{\cdot}}\, T - L = \mathrm{const}$ are integrals of equation (5.256).

Proposition 5.74 *Let:*

1. *$iF_0^0(\urcorner B, B)$ be the impulse of the dynamical screw $F_0^0(\urcorner B, B)$ in $\mathbf{E}_0$ for the time Δt*

$$iF_0^0(\urcorner B, B) = \int \chi_{\Delta t} F_0^0(\urcorner B, B)\mu_1(dt) \tag{5.258}$$

2. *$Q_{B0}^{00}(t), Q_{Bs}^{00}(t + \Delta t)$ be the kinetic screws of the body in the motion w.r.t. the inertial frame $\mathbf{E}_0$ calculated in t and $t + \Delta t$.*

Then

$$dQ^{00}_{B0}(t) = F^0_0(\urcorner B, B)\mu_1(dt) \quad (5.259)$$

$$Q^{00}_{Bs}(t+\Delta t) - Q^{00}_{B0}(t) = iF^0_0(\urcorner B, B) \quad (5.260)$$

Proof of relation (5.260) is obtained with the help of integrating relation (5.259) or (5.181) on the interval Δt.

Comment We may represent relation (5.260) in the following form

$$\Theta^0_0(t+\Delta t)V^{00}_s(t+\Delta t) - \Theta^0_0(t)V^{00}_s(t) = \int \chi_{\Delta t}F^0_0(\urcorner B, B)\mu_1(dt) \quad (5.261)$$

Proposition 5.75 *Let:*

1. $iF^s_s(\urcorner B, B)$ *be the impulse of the dynamical screw* $F^s_s(\urcorner B, B)$ *in* $\mathbf{E}_s$ *for the time* Δt

$$iF^s_s(\urcorner B, B) = \int \chi_{\Delta t}F^s_s(\urcorner B, B)\mu_1(dt) \quad (5.262)$$

2. $I^s_s(\urcorner B, B)$ *be the inertial screw of the body in the motion w.r.t. the inertial frame* $\mathbf{E}_s$ *such that*

$$I^s_s(\urcorner B, B) = I^s_{s1}(\urcorner B, B) + I^s_{s2}(\urcorner B, B) \quad (5.263)$$

where

$$I^s_{s1}(\urcorner B, B) = -\Theta^s_s V^{s*}_s, I^s_{s2}(\urcorner B, B) = -\Phi^{0s}_s\Theta^s_s V^s_s$$

3. *the screw impulses*

$$iI^s_{s1}(\urcorner B, B) = -\int \chi_{\Delta t}I^s_{s1}(\urcorner B, B)\mu_1(dt) \quad (5.264)$$

$$iI^s_{s2}(\urcorner B, B) = -\int \chi_{\Delta t}I^s_{s2}(\urcorner B, B)\mu_1(dt) \quad (5.265)$$

4. $Q^{0s}_{B0}(t), Q^{0s}_{Bs}(t+\Delta t)$ *be the kinetic screws of the body in the motion w.r.t. the frame* $\mathbf{E}_s$ *calculated in* t *and* $t+\Delta t$.

Then

$$Q^{0s}_{Bs}(t+\Delta t) - Q^{0s}_{B0}(t) = iF^s_s(\urcorner B, B) + iI^s_{s2}(\urcorner B, B) \quad (5.266)$$

Proof of relation (5.266) is obtained with the help of integrating relation (5.184) on the interval Δt with taking in account (5.172).

Comment We may represent relation (5.266) in the following form

$$\Theta^s_s(t+\Delta t)V^{0s}_s(t+\Delta t) - \Theta^s_s(t)V^{0s}_s(t) =$$
$$\int \chi_{\Delta t}F^s_s(\urcorner B, B)\mu_1(dt) - \int \chi_{\Delta t}I^s_{s2}(\urcorner B, B)\mu_1(dt)$$

Proposition 5.76 *Under the P 5.75 conditions*

$$V_s^{0s}(t+\Delta t) - V_s^{0s}(t) = (\Theta_s^s)^{-1} \int \chi_{\Delta t}[F_s^s(\urcorner B, B) - \Phi_s^{0s}\Theta_s^s V_s^s]\mu_1(dt) \quad (5.267)$$

Comment Relation (5.267) written in the form

$$V_s^{0s}(t_1) - V_s^{0s}(0) = (\Theta_s^s)^{-1} \int_0^{t_1} [F_s^s(\urcorner B, B) - \Phi_s^{0s}\Theta_s^s V_s^s(t)]\mu_1(dt) \quad (5.268)$$

is the integral motion equation in quasi–velocities with the kernel $F_s^s(\urcorner B, B) - \Phi_s^{0s}\Theta_s^s V_s^{s*}(t)$.

Proposition 5.77 *The power of the quadratic (w.r.t. quasi–velocities) part of the inertial screw $I_{s2}^s(\urcorner B, B)$ is equal zero in the frame $\mathbf{E}_s$*

$$I_{s2}^s(\urcorner B, B) \cdot V_s^{0s} = \Phi_s^{0s}\Theta_s^s V_s^{0s} \cdot V_s^{0s} = 0 \quad (5.269)$$

Proof $\Phi_s^{0s}\Theta_s^s V_s^{0s} \cdot V_s^{0s} = V_s^{0s,T}\Phi_s^{0s}\Theta_s^s V_s^{0s} = m(B)v_s^{0s,T} < \omega_s^0 >^s v_s^{0s} + m(B)$ $v_s^{0s,T} < \omega_s^0 >^s < r_c^0 >^{s,T} \omega_s^{0s} + m(B)\omega_s^{0s,T} < v_s^0 >^s < r_c^0 >^{s,T} \omega_s^{0s} + m(B)\omega_s^{0s,T}$ $< \omega_s^0 >^s < r_c^0 >^{s,T} v_s^{0s} + m(B)\omega_s^{0s,T} < v_s^0 >^s v_s^{0s} + m(B)\omega_s^{0s,T} < \omega_s^0 >^s \Theta_s^s\omega_s^{0s}$ $= -m(B)v_s^{0s,T} < v_s^0 >^s \omega_s^{0s} - m(B)\omega_s^{0s,T} < v_s^0 >^s < r_c^0 >^{s,T} \omega_s^{0s} + m(B)\omega_s^{0s,T}$ $< v_s^0 >^s v_s^{0s} + m(B)\omega_s^{0s,T} < v_s^0 >^s < r_c^0 >^{s,T} \omega_s^{0s} + m(B)\omega_s^{0s,T} < \omega_s^0 >^s < r_c^0 >^{s,T}$ $v_s^{0s} + (B)\omega_s^{0s,T} < \omega_s^0 >^s \Theta_s^s\omega_s^{0s}$ due to the skew-symmetrical matrix property $< x > y = - < y > x$.

Proposition 5.78 *Let: 1. T be the kinetic energy of B;*
2. $I_s^s(\urcorner B, B)$ be the inertial screw of the body in the motion w.r.t. the frame $\mathbf{E}_s$. Then

$$T^{\cdot} = -I_s^s(\urcorner B, B) \cdot V_s^{0s} \quad (5.270)$$

Proof 1. Let us do the inner product between relation (5.184) and the quasi–velocity V_s^{0s}. Then with taking into account (5.263) and (5.269) we arrive at

$$-I_s^s(\urcorner B, B) \cdot V_s^{0s} = V_s^{0s} \cdot Q_{Bs}^{0s*} \quad (5.271)$$

From the derivation $(V_s^{0s} \cdot Q_{Bs}^{0s})^* = V_s^{0s} \cdot Q_{Bs}^{0s*} + V_s^{0s*} \cdot Q_{Bs}^{0s}$ we have $V_s^{0s} \cdot Q_{Bs}^{0s*} = (V_s^{0s} \cdot Q_{Bs}^{0s})^* - V_s^{0s*} \cdot Q_{Bs}^{0s}$. Remind that $(V_s^{0s} \cdot Q_{Bs}^{0s})^* = T^{\cdot}$ (see (5.251)) and $T^{\cdot} = V_s^{0s*} \cdot \mathrm{grad}_{V_s} T = V_s^{0s\cdot} \cdot Q_{Bs}^{0s}$.

Proposition 5.79 *Let:*

1. *T be the kinetic energy of B;*
2. *Q_{B0}^{0s*} be the change velocity of the kinetic screw of the body in the frame $\mathbf{E}_s$.*

Then

$$T^{\bullet} = Q^{0s*}_{B0} \cdot V^{0s}_{s} \tag{5.272}$$

Proof is obtained with substitution of (5.271) in (5.270).

Proposition 5.80 *Let:*

1. *T be the kinetic energy of B;*
2. *$F^0_0(\urcorner B, B) \cdot V^{00}_s$, $F^s_s(\urcorner B, B) \cdot V^{0s}_s$ and $Q^s_s(\urcorner B, B) \cdot q^{s\bullet}$ be powers of the dynamical screws $F^0_0(\urcorner B, B)$ and $F^s_s(\urcorner B, B)$ and the generalized forces $Q^s_s(\urcorner B, B)$.*

Then

$$T^{\bullet} = F^0_0(\urcorner B, B) \cdot V^{00}_s = F^s_s(\urcorner B, B) \cdot V^{0s}_s = Q^s_s(\urcorner B, B) \cdot q^{s\bullet} \tag{5.273}$$

Proof The second equality in (5.273) is the result of multiplying relation (5.184) and the quasi–velocities V^{0s}_s with the help of (5.269) and (5.272). The first equality in (5.273) is the second inner product calculated in the basis of $\mathbf{E}_0$ (powers of the screws $F^0_0(\urcorner B, B)$ and $F^s_s(\urcorner B, B)$ coincide). Let us prove the third equality. Multiplying equation (5.238) with the generalized velocities $q^{s\bullet}$ we have

$$q^{s\bullet} \cdot (\mathrm{grad}_{q^{\cdot}}\ T)^{\bullet} - q^{s\bullet} \cdot \mathrm{grad}_q T) = q^{s\bullet} \cdot Q^s_s(\urcorner B, B) \tag{5.274}$$

Let us calculate $(q^{s\bullet} \cdot \mathrm{grad}_{q^{\cdot}}\, T)^{\bullet} = q^{s\bullet\bullet} \cdot \mathrm{grad}_{q^{\cdot}}\, T + q^{s\bullet} \cdot (\mathrm{grad}_{q^{\cdot}}\, T)^{\bullet}$. Hence $q^{s\bullet} \cdot (\mathrm{grad}_{q^{\cdot}}\, T)^{\bullet} = (q^{s\bullet} \cdot \mathrm{grad}_{q^{\cdot}}\, T)^{\bullet} - q^{s\bullet\bullet} \cdot \mathrm{grad}_{q^{\cdot}}\, T$ and from (5.274) we have $q^{s\bullet} \cdot \mathrm{grad}_{q^{\cdot}}\, T)^{\bullet} - q^{s\bullet\bullet} \cdot \mathrm{grad}_{q^{\cdot}}\, T - q^{s\bullet} \cdot \mathrm{grad}_q\, T = q^{s\bullet} \cdot Q^s_s(\urcorner B, B)$. Due to Euler theorem about homogeneous functions $q^{s\bullet} \cdot \mathrm{grad}_{q^{\cdot}}\, T = 2T$ and the differential law $T^{\bullet} = q^{s\bullet\bullet} \cdot \mathrm{grad}_{q^{\cdot}}\, T + q^{s\bullet} \cdot \mathrm{grad}_q\, T$ we have relation (5.273).

Proposition 5.81 *Let:*

1. *$\omega_{V0}, \omega_{Vs}, \omega_{q^{\cdot}}$ be linear forms generated by the dynamical screws $F^0_0(\urcorner B, B)$ and $F^s_s(\urcorner B, B)$ and the generalized forces $Q^s_s(\urcorner B, B)$ in the inertial frame $\mathbf{E}_0$ (where $\pi^{00\bullet}_s = V^{0s}_s$), in the attached frame $\mathbf{E}_s$ and in generalized coordinates (Cartan 1967)*

$$\begin{aligned} \omega_{V0} &= F^0_0(\urcorner B, B) \cdot d(\pi^{00}_s) = F^0_0(\urcorner B, B) \cdot V^{00}_s\, \mu_1(dt) && (5.275)\\ \omega_{Vs} &= F^s_s(\urcorner B, B) \cdot d(\pi^{0s}_s) = F^s_s(\urcorner B, B) \cdot V^{0s}_s\, \mu_1(dt) && (5.276)\\ \omega_{q^{\cdot}} &= Q^s_s(\urcorner B, B) \cdot d(q^s) = Q^s_s(\urcorner B, B) \cdot q^{s\bullet}\, \mu_1(dt) && (5.277) \end{aligned}$$

2. *the impulses of these forms*

$$\begin{aligned} i\omega_{V0} &= \int \chi_{\Delta t} F^0_0(\urcorner B, B) \cdot V^{00}_s\, \mu_1(dt) && (5.278)\\ i\omega_{Vs} &= \int \chi_{\Delta t} F^s_s(\urcorner B, B) \cdot V^{0s}_s\, \mu_1(dt) && (5.279)\\ i\omega_{q^{\cdot}} &= \int \chi_{\Delta t} Q^s_s(\urcorner B, B) \cdot q^{s\bullet}\, \mu_1(dt) && (5.280) \end{aligned}$$

3. $T(t)$, $T(t+\Delta t)$ be the kinetic energies of the body in t and $t+\Delta t$.

Then the increment of T is

$$T(t+\Delta t) - T(t) = i\omega_{V0} = i\omega_{Vs} = i\omega_q. \tag{5.281}$$

Proof follows from relations (5.273), (5.278), (5.279), and (5.280).

Definition 5.34 *Forms (5.275), (5.275), and (5.277) are called the work of the dynamical screws $F_0^0(\urcorner B, B)$ and $F_s^s(\urcorner B, B)$ and the generalized forces $Q_s^s(\urcorner B, B)$ on the transitions $d(\pi_s^{00})$, $d(\pi_s^{0s})$ and $d(q^s)$, respectively.*

Definition 5.35 *Holonomic constraints (5.203) are called ideal if the work of the dynamical screws $R_0^0(\urcorner B, B)$ and $R_s^s(\urcorner B, B)$ on the transitions $d(\pi_s^{00})$ and $d(\pi_s^{0s})$ is zero*

$$R_0^0(\urcorner B, B) \cdot d(\pi_s^{00}) = R_s^s(\urcorner B, B) \cdot d(\pi_s^{0s}) = 0 \tag{5.282}$$

Comments With multiplying relation (5.273) on $\mu_1(dt)$ and taking into account the first part of (5.275), (5.276), and (5.277) we have the following statements:

1. The differential of the kinetic energy of B coincides with the work of the dynamical screws $F_0^0(\urcorner B, B)$ and $F_s^s(\urcorner B, B)$ and the generalized forces $Q_s^s(\urcorner B, B)$ on the transitions $d(\pi_s^{00})$, $d(\pi_s^{0s})$ and $d(q^s)$:

$$dT = F_0^0(\urcorner B, B) \cdot d(\pi_s^{00}) = F_s^s(\urcorner B, B) \cdot d(\pi_s^{0s}) = Q_s^s(\urcorner B, B) \cdot d(q^s) \tag{5.283}$$

2. The increment of the kinetic energy of B for the time Δt coincides with the work of the dynamical screws $F_0^0(\urcorner B, B)$ and $F_s^s(\urcorner B, B)$ and the generalized forces $Q_s^s(\urcorner B, B)$ on the transitions $d(\pi_s^{00})$, $d(\pi_s^{0s})$ and $d(q^s)$ for the same time.
3. Let us suppose that constraints (5.203) are ideal – see (5.282). Then the dynamical screws $F_0^0(\urcorner B, B)$ and $F_s^s(\urcorner B, B)$ in the relations above do not contain reaction screws due to (5.282).

Chapter 6

Mechanics of multibody systems with tree–like structure

The complete theory under consideration is considered in (Konoplev 1984–2001) with a lot of detailed examples. Here we give only main results. Statement proofs are omitted, one–position indices of s kinematic loop type (see D 5.2) are replaced with two–position indexes of lk system with tree–like structure (Konoplev *et al.* 2001).

6.1. Equations of kinematics of a multibody system

6.1.1. *Graph of tree–like system*

The design kinematic equation problem for multibody systems is reduced to packing up kinematic equations of separate kinematic couples in one block (subsection 5.2.1, 5.2.5) (Mesarovich *et al.* 1978) in correspondence with the system graph. This approach explains using the system analysis terminology (Konoplev *et al.* 2001).

Definition 6.1 *1. E_{lk} is (lk)–element of a graph where l is index of tree trunk, k is the index of the level l, $k \in \mathbf{N}$.*

2. *the graph is oriented by sets of indexes l and k with the priority of l, these set increase from the root E_{10} to the tree tops;*
3. *$(\mathbf{lk})_+$ is the set of accessibility of (lk)–element of the graph, i.e. the set of all graph elements which are accessed from of (lk)–element towards the tree pick (in the direction of index increasing), $E_{lk} \in (\mathbf{lk})_+$. For any tree–like graph the set $(\mathbf{lk})_+$ of accessibility is a subtree of the main tree with (lk)–root.*
4. *$(\mathbf{lk})_-$ is the set of contra–accessibility of (lk)–element of the graph, i.e. the set of all graph elements which are accessed from of (lk)–element towards to the tree root (in the direction of index decreasing), $E_{lk} \in (\mathbf{lk})_-$. For any tree–like graph the set $(\mathbf{lk})_-$ of contra–accessibility is a kinematic chain with the first element E_{10} and the last one E_{lk}.*
5. *$(\mathbf{lk})^+$ is the set of right incidence of lk–th element of the system graph: the set of elements from $(\mathbf{lk})_+$ being accessible from k–th element with one step, $E_{lk} \notin (\mathbf{lk})^+$.*

6. $(\mathbf{lk})^-$ *is the set of left incidence of* lk*–th element of the system graph: the set of elements from* $(\mathbf{lk})_-$ *being accessible from* k*–th element with one step,* $E_{lk} \notin (\mathbf{lk})_-$.
7. *Two graph elements* $E_{\mu,k-1}$ *and* E_{lk} *is* $(\mu, k-1; lk)$*–kinematic couple if* $E_{lk} \in (\mu, k-1)^+$ *and* $\mu \leq 1$.

6.1.2. *Kinematics of elastic* $(\mu, k-1; lk)$*–kinematic couple*

Proposition 6.1 *Let:*

1. $R^{\mu,k-1}_{lkc} = \mathrm{col}\,\{p^{lk}_1, p^{lk}_2, p^{lk}_3, \varphi^{lk}_\alpha, \varphi^{lk}_\beta, \varphi^{lk}_\gamma\}^T$ *be the constructive configuration of* $(\mu, k-1, lk)$*–kinematic couple (5.63);*
2. $R^{lkc}_{lk} = \mathrm{col}\,\{o^{lk}_1, o^{lk}_2, o^{lk}_3, \theta^{lk}_4, \theta^{lk}_5, \theta^{lk}_6\}^T$ *be the functional configuration of* $(\mu, k-1; lk)$*–kinematic couple;*
3. $\mathrm{col}\,\{\delta^{lk}_1, \delta^{lk}_2, \delta^{lk}_3\}^T$ *be the linear deformation vector of* $(\mu, k-1; lk)$*–kinematic couple;*
4. $\mathrm{col}\,\{\delta^{lk}_4, \delta^{lk}_5, \delta^{lk}_6\}^T$ *be the angle deformation vector of* $(\mu, k-1; lk)$*–kinematic couple;*
5. *the next supposition be true:* $\delta^{lk}_i \neq 0$, *if* $o^{lk}_i = 0$ *or* $\theta^{lk}_i = 0$, *and* $\delta^{lk}_i = 0$, *if* $o^{lk}_i \neq 0$ *or* $\theta^{lk}_i \neq 0$, *or* $\delta^{lk}_i = o^{lk}_i = \theta^{lk}_i = 0$, *if no functional motion by* i*–th coordinate of* $(\mu, k-1; lk)$*–kinematic couple is.*

Then: 1. $(\mu, k-1; lk)$*–kinematic couple is called elastic;*

2. *configuration (5.63) of elastic* $(\mu, k-1; lk)$*–kinematic couple has the following form*

$$\mathcal{R}^{\mu,k-1}_{lk} = \{R^{\mu,k-1}_{lkc}, R^{lkc}_{lkc}\}, R^{\mu,k-1}_{lkc} = (p^{lk}, \varphi^{lk}) \tag{6.1}$$

$R^{lkc}_{lk} = \mathrm{col}\,\{o^{lk}_1, o^{lk}_2, o^{lk}_3, \theta^{lk}_4, \theta^{lk}_5, \theta^{lk}_6\}^T + \mathrm{col}\,\{\delta^{lk}_1, \delta^{lk}_2, \delta^{lk}_3, \delta^{lk}_4, \delta^{lk}_5, \delta^{lk}_6\}^T = \mathrm{col}\,\{\cdots, q^{lk}_i, \cdots\}^T$ *where* q^{lk}_i *is equal* o^{lk}_i, θ^{lk}_i, δ^{lk}_i *or* 0*;*

3. *the kinematic equation of elastic* $(\mu, k-1; lk)$*–kinematic couple*

$$V^{\mu,k-1;lk}_{lk} = M^{lkc}_{lk} \| f^{lk} \| q^{lk\boldsymbol{\cdot}}, \;\; M^{lkc}_{lk} = \mathrm{diag}\{c^{lkc,T}_{lk}, \varepsilon^{lkc}_{lk}\} \tag{6.2}$$

$$c^{lkc}_{lk} = c_1(q^{lk}_4) c_2(q^{lk}_5) c_3(q^{lk}_6) \tag{6.3}$$

$$\varepsilon^{lkc}_{lk} = \left[c^T_3(q^{lk}_6) c^T_2(q^{lk}_5) e^{lk}_1 \mid c^T_3(q^{lk}_6) e^{lk}_2 \mid e^{lk}_3 \right] \tag{6.4}$$

Comment Elements of elastic $(\mu, k-1; lk)$–kinematic couple are elastic rigid bodies. By supposition, the shift of k–th body w.r.t. $\mathbf{E}_{lkc}$ is produced by functional motions of the elastic (deformation of $(\mu, k-1)$ –body) and non–elastic nature, the influence of the elastic motions on one-indexes non–elastic motions is neglected.

Let us give kinematical equations in several forms (Konoplev *et al.* 1991 and 2001).

Proposition 6.2 *Let:*

1. $(10)_+$ *be a tree–like multibody system;*
2. $(\mathbf{V}) = \mathrm{col}\{\cdots, V_{nj}^{i,j-1;nj}, \cdots\}$ *be* $6n \times 1$*–dimensional vector of quasi–velocities of kinematic couples of system – see (5.67);*
3. $\mathbf{V} = \mathrm{col}\{\cdots, V_{lk}^{10;lk}, \cdots\}$ *be* $6n \times 1$*–dimensional vector of quasi–velocities of* (lk)*–element (see (5.49));*
4. L *be the block upper–triangular* $6n \times 6n$*–dimensional matrix with* 6×6*–dimensional blocks* L_{lk}^{st} *– see (7.7), if* $(lk) \in (\mathbf{st})_+$*, and* 6×6*–dimensional zero blocks, if* $(lk) \notin (\mathbf{st})_+$*, on the intersection of* (st)*–(matrix) rows and* (lk)*–(matrix) columns.*

Then: 1. the kinematic equation for a tree–like multibody system has the following form

$$\mathbf{V} = L^T(\mathbf{V}) \tag{6.5}$$

2. the matrix L *is called a configuration one.*

Comment The vector $\mathbf{V}$ of quasi–velocities of all system bodies w.r.t. $\mathbf{E}_{10}$ is the linear transformation of the vector of quasi–velocities of these bodies in kinematic couples with the configuration matrix L. From relation (6.5) follows that

$$(\mathbf{V}) = L^{-t}\mathbf{V} \tag{6.6}$$

Proposition 6.3 *Let:*

1. $(10)_+$ *be a tree–like multibody system;*
2. $q^{\cdot} = \mathrm{col}\{q^{11\cdot}, q^{12\cdot}, \cdots\cdots, q^{lk\cdot}, \ldots\}$ *be the column of generalized velocities composed from the columns of generalized velocities of the system kinematic couples – see (5.62);*
3. M *be a block–diagonal* $(6n \times 6n)$*–dimensional matrix of transformation from the generalized velocities to the quasi–velocities of the system kinematic couples (5.62) with* 6×6*–dimensional blocks* M_f^{lkc} *(see (5.46))*

$$M = \mathrm{diag}\{\ldots, M_f^{lkc}, \ldots\} \tag{6.7}$$

4. $\|f\|$ *be* $(6n \times \dim\ q^{st})$*–dimensional matrix with* $(6 \times \dim\ q^{lk})$*–dimensional blocks* $\|f^{lk}\|$ *composed from unit orthogonal vectors* f_i^{lk} *directed along with the movability axes* $(i = \overline{1,6})$

$$\|f\| = \mathrm{diag}\{\ldots, \|f^{lk}\|, \ldots\} \tag{6.8}$$

Then

$$(\mathbf{V}) = M\|f\|q^{\cdot} \tag{6.9}$$

Proposition 6.4 *Let: 1.* $(10)_+$ *be a tree–like multibody system;*

2. 1×6*–row or* $(\dim q^{st} \times 6)$*–dimensional matrices (operators) (see (5.74) and (5.76))*

$$s_{lk}^{st-\alpha} = f_\alpha^{st,T} M_f^{stc,T} L_{lk}^{st}, \ s_{lk}^{st-\alpha} = \begin{bmatrix} \cdots\cdots\cdots\cdots\cdots \\ f_\alpha^{st,T} M_f^{stc,T} L_{lk}^{st} \\ \cdots\cdots\cdots\cdots \end{bmatrix} \tag{6.10}$$

which project the screw X_{lk}^{st} *(defined in* $\mathbf{E}_{lk}$*) of any origination on* α*–axis (axes) of functional motion of* (st)*–element in* $(i, t-1; st)$*–kinematic couple,* $\alpha = \overline{1,6}$*;*

3. $\mathcal{S}$ *be a* $(\dim\ q \times 6n)$*–dimensional upper triangular block matrix with the blocks (6.10) on the intersection of* $(\dim q^{st} \times 6n)$*–dimensional–(matrix) rows and* $(\dim q \times 6)$*–dimensional–(matrix) columns, if* $(lk) \in (\mathbf{st})_+$*, and* 6×6*–dimensional zero blocks, if* $(lk) \notin (\mathbf{st})_+$*.*

Then: 1. the kinematic equation has the following form

$$\mathbf{V} = \mathcal{S}^T q^{\cdot}, \quad \mathcal{S} = \|f\|^T M^T L \tag{6.11}$$

2. the matrix S *is called parastrophic one of the tree–like system.*

Comments

1. The parastrophic matrix plays the central role in the worked out formalism of mechanics. It contents whole information about the system structure:

   – about the system internal configuration and the structure of system kinematic scheme (system graph) – the matrix L;

   – about transform from quasi–coordinates to generalized ones – the matrix M;

   – about the motion nature w.r.t. every freedom degree – the matrix $\|f\|$.

   The physical sense of $\mathcal{S}$ is clear: it projects the $(6n)$–dimensional vector $P = \mathrm{col}\,\{\cdots\ldots P_{lk}^{lk}, \ldots\}$ of screws (of any origination) on the movability axes. The matrix $\mathcal{S}^T$ performs the opposite action, in particular, it transforms generalized velocities to vectors of quasi–velocities (in system kinematic screws).

2. Elements of $\mathcal{S}$ differ from zero only in the case where $(lk) \in (\mathbf{st})_+$. That is why this matrix defines the influence of (lk)–element motion on (st)–one. *E.g.*, in the classical scheme of 'hedgehog' – a body with attached rotating dynamically unbalanced some bodies (5.213) – there is no influence of one rotating body on an other.

6.2. Motion equation for multibody system

6.2.1. *Motion equation in quasi–velocities and generalized ones*

Notation Henceforth:

1. the body $\mathbf{B}_{lk}$ is an element of some tree–like system;
2. $\mathbf{E}_{10}$ is an inertial frame and $\mathbf{E}_{lk}$ is a frame attached to $\mathbf{B}_{lk}$;
3. $V_{lk}^{10,lk} = \mathrm{col}\{v_{lk}^{10,lk}, \omega_{lk}^{10,lk}\}$ is the vector of quasi–velocities of $\mathbf{B}_{lk}$ with respect to $\mathbf{E}_{10}$ in $\mathbf{E}_{lk}$;
4. Θ_{lk}^{lk} is the Mises matrix of $\mathbf{B}_{lk}$ in $\mathbf{E}_{lk}$ (the constant matrix (5.171) is calculated at the initial point o_{lk} and in the basis $\mathbf{e}^{lk}$ of the frame attached to the body);
5. $R_{lk}^{lk}(i,k-1;lk)$ and $R_{j,k+1}^{j,k+1}(lk;j,k+1)$ are wrenches of reactions of the body $\mathbf{B}_{i,k-1} \in (\mathbf{lk})^-$ to the body $\mathbf{B}_{lk}$, and of the body $\mathbf{B}_{lk}$ to the body $\mathbf{B}_{j,k+1} \in (\mathbf{lk})^+$, respectively,
$$R_{lk}^{lk} = R_{lk}^{lk}(i,k-1;lk) - \sum_j L_{j,k+1}^{lk} R_{j,k+1}^{j,k+1}(lk;j,k+1)$$
6. $N_{lk}^{lk}(i,k-1;lk)$ and $N_{j,k+1}^{j,k+1}(lk;j,k+1)$ are the wrenches of friction of $\mathbf{B}_{i,k-1} \in (\mathbf{lk})^-$ about $\mathbf{B}_{lk}$, and of $\mathbf{B}_{lk}$ about $\mathbf{B}_{j,k+1} \in (\mathbf{lk})^+$
$$N_{lk}^{lk} = N_{lk}^{lk}(i,k-1;lk) - \sum_j L_{j,k+1}^{lk} N_{j,k+1}^{j,k+1}(lk;j,k+1)$$
7. $U_{lk}^{lk}(i,k-1;lk)$ and $U_{j,k+1}^{j,k+1}(lk;j,k+1)$ are the wrenches of control of motion of the body $\mathbf{B}_{lk}$ with respect to the body $\mathbf{B}_{i,k-1} \in (\mathbf{lk})^-$ and of $\mathbf{B}_{j,k+1} \in (\mathbf{lk})^+$ with respect to $\mathbf{B}_{lk}$, respectively,
$$U_{lk}^{lk} = U_{lk}^{lk}(i,k-1;lk) - \sum_j L_{j,k+1}^{lk} U_{j,k+1}^{j,k+1}(lk;j,k+1)$$
8. P_{lk}^{lk} is the aero-hydrodynamic wrench of the body $\mathbf{B}_{lk}$ in $\mathbf{E}_{lk}$;
9. G_{lk}^{lk} is the wrench of gravity of the body $\mathbf{B}_{lk}$ in $\mathbf{E}_{lk}$;
10. $\Phi_{lk}^{10,lk}$ is the matrix of quasi–velocities of the body $\mathbf{B}_{lk}$ of the (5.183)–kind;
$$Z_{lk}^{lk} = R_{lk}^{lk} + U_{lk}^{lk} + N_{lk}^{lk}, \quad H_{lk}^{lk} = P_{lk}^{lk} + G_{lk}^{lk}$$
11. * means the operation of differentiation in $\mathbf{E}_{lk}$.

Proposition 6.5 *The equation of motion of the body $\mathbf{B}_{lk}$ as an element of the tree–like system of rigid bodies has the following form (Konoplev 1985 and 1987b)*
$$\Theta_{lk}^{lk}(V_{lk}^{10,lk})^* + \Phi_{lk}^{10,lk}\Theta_{lk}^{lk}V_{lk}^{10,lk} = Z_{lk}^{lk} + H_{lk}^{lk}$$

Notation Henceforth:

– $\mathcal{F} = \mathrm{col}\{F_{10}^{10}, F_{12}^{12}, \cdots, F_{lk}^{lk}, \cdots\}$ is $6n$–dimensional vector of the wrenches
$$\begin{aligned} F_{lk}^{lk} &= Z_{lk}^{lk} + H_{lk}^{lk} + T_{lk}^{lk} \\ Z_{lk}^{lk} &= R_{lk}^{lk} + U_{lk}^{lk} + N_{lk}^{lk}, H_{lk}^{lk} = P_{lk}^{lk} + G_{lk}^{lk} \end{aligned} \tag{6.12}$$

– $\mathcal{R}$ is the vector of wrenches of inner reactions of the system

$$\mathcal{R} = \text{col}\{R_{10}^{10}, R_{12}^{12}, \cdots, R_{lk}^{lk}, \cdots\} \tag{6.13}$$

– $\mathcal{U}$ is $6n$–dimensional vector of control wrenches of the system, $\mathfrak{u}$ is the vector–column of control forces $u(\mu, k-1; lk)$ along the movability axes of the $(\mu, k-1; lk)$–kinematic couples from the system, dim $u(\mu, k-1; lk) = \sum_{lk} \text{dim } q_{lk}^{s,k-1}$

$$\begin{aligned} \mathcal{U} &= \text{col}\{U_{10}^{10}, U_{12}^{12}, \cdots, U_{lk}^{lk}, \cdots\} && (6.14) \\ \mathfrak{u} &= \text{col}\{u(10,11), u(11,12), \ldots, u(\mu, k-1; lk), \ldots\} && (6.15) \end{aligned}$$

(among the components of $u(\mu, k-1; lk)$ there can be zero ones, this fact meaning that a free motion is realized in direction of these generalized coordinates);

– $\mathcal{N}$ is the vector of wrenches of friction along the axes of motion of the system, $\mathfrak{n}$ is the vector–column of the forces of friction $n(\mu, k-1; lk)$–kinematic couples along the axes of motion of the kinematic couples $(\mu, k-1; lk)$, dim $n(\mu, k-1; lk) = \sum_{lk} \text{dim } q_{lk}^{s,k-1}$

$$\begin{aligned} \mathcal{N} &= \text{col}\{N_{10}^{10}, N_{12}^{12}, \cdots, N_{lk}^{lk}, \cdots\} && (6.16) \\ \mathfrak{n} &= \text{col}\{n(10,11), n(11,12), \ldots, n(\mu, k-1; lk), \ldots\} && (6.17) \end{aligned}$$

Proposition 6.6 – $\mathcal{R} \in \text{Ker } \mathcal{S}$, *i.e.,*

$$\mathcal{S}\mathcal{R} = 0 \tag{6.18}$$

– *the parastrophic matrix* $\mathcal{S}$ *extracts the control force* $\mathfrak{u}$ *and the friction force* $\mathfrak{n}$ *from the wrenches* $\mathcal{U}$ *and* $\mathcal{N}$, *respectively,*

$$\mathfrak{u} = \mathcal{S}\mathcal{U}, \;\; \mathfrak{n} = \mathcal{S}\mathcal{N} \tag{6.19}$$

Notation Henceforth

– $\mathcal{R}_{lk}^{\mu,k-1} = \text{col}\{R_{lkc}^{\mu,k-1}, \;\; R_{lk}^{\;lkc}\}$ is the configuration set of $\mathbf{B}_{lk}$;

– $\mathcal{V} = \text{col}\{V_{11}^{10,11}, V_{12}^{10,12}, \cdots V_{lk}^{10,lk}, \cdots\}$ is $6n$–dimensional vector of quasi–velocities of bodies from the system (see (5.49));

– $\mathcal{A} = \text{diag}\,\{A_{lk}^{lk}\}$ is the block constant inertia $6n \times 6n$–dimensional matrix (see (5.199));

– $\Phi = \text{diag}\,\{\Phi_{11}^{10,11}, \Phi_{12}^{10,12}, \cdots \Phi_{lk}^{10,lk}, \cdots\}$ is the matrix of quasi–velocities with 6×6–dimensional blocks (5.183);

– $\mathcal{H}+\mathcal{T} = \text{col}\{(H+T)_{10}^{10}, (H+T)_{12}^{12}, \cdots, (H+T)_{lk}^{lk}, \cdots\}$ is $6n$–dimensional vector of wrenches of reactions of the external environment $H_{lk}^{lk} = P_{lk}^{lk} + G_{lk}^{lk}$ and the action of rotating bodies T_{lk}^{lk} (supported by the system) on the system of bodies.

Proposition 6.7 *There are the following equation of motion of the multibody system with tree–like structure:*

– *in terms of quasi–velocities*

$$\mathcal{S}\mathcal{A}\mathcal{V}^{*} + \mathcal{S}\mathcal{A}\mathcal{V} = \mathcal{S}(\mathcal{H} + \mathcal{T}) + \mathfrak{u} + \mathfrak{n} \tag{6.20}$$

– *in terms of generalized velocities*

$$\mathcal{A}(q)q^{\cdot\cdot} + \mathcal{B}(q,q^{\cdot})q^{\cdot} = \mathcal{Q} \tag{6.21}$$

where

$$\mathcal{A}(q) = \mathcal{SAS}^T,\ \mathcal{B}(q,q^{\cdot}) = \mathcal{SBS}^T + \mathcal{SAS}^{T\cdot},\ \mathcal{Q} = \mathcal{S}(\mathcal{H}+\mathcal{T}) + \mathfrak{u} + \mathfrak{n}$$

Proposition 6.8 *The derivatives of the rows $s_{lk}^{st-\alpha}$ of $\mathcal{S}$ from equality (6.21) are computed by the following recurrent relations*

$$\begin{array}{lcl} s_{lk}^{st-\alpha\cdot} & = & s_{n,k-1}^{st-\alpha\cdot} L_{lk}^{n,k-1} + s_{n,k-1}^{st-\alpha} L_{lk}^{n,k-1} \Phi_{lk}^{n,k-1;lk} \\ s_{st}^{st-\alpha\cdot} & = & f_\alpha^{st,n} M_{st}^{stc,n\cdot},\ V_{lk}^{n,k-1;lk} = M_{lk}^{lkc} f_\alpha^{lk} q^{lk\cdot} \end{array} \tag{6.22}$$

where, due to (6.3)and (6.4), for the entries of $M_{st}^{stc,T}$ there are

$$\begin{array}{lcl} c_{st}^{stc\cdot} & = & c_{st}^{stc} < \omega_{st}^{stc} >^{st} \\ \epsilon_{st}^{stc\cdot} & = & (\ -< e_3^{st} > c_3^T(\theta_6^{st}) c_2^T(\theta_5^{st}) \theta_6^{st\cdot} - c_3^T(\theta_6^{st}) c_2^T(\theta_5^{st}) < e_2^{st} > \theta_5^{st\cdot}) e_1^{st}, \\ & & < e_3^{st} > c_3^T(\theta_6^{st}) \theta_6^{st\cdot} e_2^{st},\quad \mathbf{0}\ ) \end{array} \tag{6.23}$$

6.2.2. *Equations of Hooke–elastic body system motion*

Notation Henceforth

- $\{B_{lk}\}$ is the system of elastic elements for a given system of bodies;
- $C_{lk}^{lk}(i,k-1;lk)$ is the wrench of action of $(i,k-1)$–th body on lk–th body that arises in result of the elasticity of $(i,k-1)$–th body;
- $C(i,k-1)$ is 6×6–dimensional matrix of stiffness of $(i,k-1)$–th body, non-zero rows and columns of which have indexes corresponding to the indexes of the axes of kinematic motion for $(i,k-1;lk)$–th kinematic couple (see subsection 5.4.6);
- $\mathcal{C}$ is the block–diagonal matrix with blocks $\|f^{lk}\|^T C(i,k-1)\|f^{lk}\|$ at the diagonal.

Proposition 6.9 *Suppose that the generalized forces $M_{lk}^{lkc,T} C_{lk}^{lk}(i,k-1;lk)$ of elasticity are linear transformation of the columns of the generalized coordinates $\|f^{lk}\|q^{lk}$ of the $(i,k-1;lk)$–kinematic couple (we assume here that the rigid bodies obey to Hooke elasticity law),*

$$M_{lk}^{lkc,T} C_{lk}^{lk}(i,k-1;lk) = C(i,k-1)\|f^{lk}\|q^{lk} \tag{6.24}$$

Then there are:

– *the motion equation of lk–th element of the system of bodies*

$$A_{lk}^{lk} V_{lk}^{10,lk*} + B_{lk}^{lk} V_{lk}^{10,lk} = Z_{lk}^{lk} + H_{lk}^{lk} + T_{lk}^{lk} + C_{lk}^{lk} \tag{6.25}$$

where

$$C_{lk}^{lk} = C_{lk}^{lk}(i,k-1;lk) - \sum_j L_{j,k+1}^{lk} C_{j,k+1}^{j,k+1}(lk;j,k+1) \tag{6.26}$$

– *the equation of motion for the elastic bodies that obey to the Hooke law (Konoplev et al. 1991)*

$$\mathcal{A}(q)q^{\cdot\cdot} + \mathcal{B}(q,q^{\cdot})q^{\cdot} + \mathcal{C}q = \mathcal{S}(\mathcal{H}+\mathcal{T}) + \mathfrak{u} + \mathfrak{n} \tag{6.27}$$

where $\mathcal{A}(q) = \mathcal{S}\mathcal{A}\mathcal{S}^T,\ \mathcal{B}(q,q^{\cdot}) = \mathcal{S}\mathcal{B}\mathcal{S}^T + \mathcal{S}\mathcal{A}\mathcal{S}^{T\cdot}$.

Notation Henceforth

– $\mathcal{P}$ is the operation of permutation of rows and columns of the matrices $\mathcal{A}(q)$, $\mathcal{B}(q,q^{\cdot}),\ \mathcal{C}$ from relation (6.27) such that

$$\begin{aligned}\mathcal{P}\mathcal{A}(q) &= \begin{bmatrix} \mathcal{A}_+^+(q) & \mathcal{A}_+^-(q) \\ \mathcal{A}_-^+(q) & \mathcal{A}_-^-(q)\end{bmatrix},\ \mathcal{P}\mathcal{C} = \begin{bmatrix} O & O \\ O & \mathcal{C}_-^- \end{bmatrix}\\ \mathcal{P}\mathcal{B}(q,q^{\cdot}) &= \begin{bmatrix} \mathcal{B}_+^+(q,q^{\cdot}) & \mathcal{B}_+^-(q,q^{\cdot}) \\ \mathcal{B}_-^+(q,q^{\cdot}) & \mathcal{B}_-^-(q,q^{\cdot})\end{bmatrix}\end{aligned}$$

where $\mathcal{D}_+^+(q),\ \mathcal{D}_+^-(q),\ \mathcal{D}_-^+(q)$ and $\mathcal{D}_-^-(q)$ are the matrices of contribution of one motion (upper index) into another motion (lower index) for any matrix symbol D, $+$ stands for non–elastic ('slow') motion, and $-$ stands for elastic ('fast') motion;

– $q = \mathrm{col}\{q_+, q_-\},\ q_+,\ q_-$ are the generalized coordinates of non–elastic ('slow') and elastic ('fast') motions;

– $\|f_+\|$ and $\|f_-\|$ are the matrices of 6–dimensional unit vectors of non–elastic and elastic motions, respectively.

Proposition 6.10 *There are the equations of motion for the system of bodies with elastic elements that obey to the Hooke law*

$$\begin{aligned}&\mathcal{A}_+^+(q)q_+^{\cdot\cdot} + \mathcal{B}_+^+(q,q^{\cdot})q_+^{\cdot} + \mathcal{A}_+^-(q)q_-^{\cdot\cdot} + \mathcal{B}_+^-(q,q^{\cdot})q_-^{\cdot} = \\ &\qquad \|f_+\|^T\mathcal{P}[\mathcal{S}(\mathcal{H}+\mathcal{T}) + \mathfrak{u} + \mathfrak{n}] \\ &\mathcal{A}_-^-(q)q_-^{\cdot\cdot} + \mathcal{B}_-^-(q,q^{\cdot})q_-^{\cdot} + \mathcal{A}_-^+(q)q_+^{\cdot\cdot} + \mathcal{B}_-^+(q,q^{\cdot})q_+^{\cdot} + \mathcal{C}_-^-q_- \\ &\qquad \|f_-\|^T\mathcal{P}[\mathcal{S}(\mathcal{H}+\mathcal{T})]\end{aligned} \tag{6.28}$$

6.2.3. $O(n^3)$*–operation algorithms for constructing entries of inertia matrices*

Notation Henceforth

– $A_p^p,\ B_p^p$ are 6×6–dimensional inertia matrices from equations (5.216) of motion for p–th element of the system of bodies, $p \equiv lk$;

– $(\mathbf{st},\mathbf{lk})_+ = (\mathbf{st})_+ \cap (\mathbf{lk})_+$ is the intersection set of the sets of accessibility of st–th and lk–th elements of the system, respectively;

– $A_{lk-\beta}^{st-\alpha},\ B_{lk-\beta}^{st-\alpha}$ are the entries of the matrices $\mathcal{A}(q)$ and $\mathcal{B}(q,q^{\cdot})$, respectively, which are situated at intersection of $(st-\alpha)$–th rows and $(lk-\beta)$–th columns;

– $s_p^{st-\alpha} = f_\alpha^{st,T} M_{st}^{stc,T} L_p^{st},\ p \in (\mathbf{st})_+$ are the rows of the parastrophic matrix $\mathcal{S}$ for the system of bodies, the derivative $s_p^{st-\alpha\cdot}$ is computed with the help of the recurrent relation (6.22).

Proposition 6.11 *The equations of motion for the given tree–like system of rigid bodies can be presented in the form (Konoplev et al. 2001)*

$$\mathcal{A}(q)q^{\cdot\cdot} + \mathcal{B}(q, q^{\cdot})q^{\cdot} = \mathcal{S}(\mathcal{H} + \mathcal{T}) + \mathfrak{u} + \mathfrak{n} \tag{6.29}$$

where for $p \in (\mathbf{st}, \mathbf{lk})_+$

$$A^{st-\alpha}_{lk-\beta} = \sum_{p\in(\mathbf{st},\mathbf{lk})_+} s^{st-\alpha}_p A^p_p s^{lk-\beta,T}_p \tag{6.30}$$

$$B^{st-\alpha}_{lk-\beta} = \sum_{p\in(\mathbf{st},\mathbf{lk})_+} (s^{st-\alpha}_p B^p_p s^{lk-\beta,T}_p + s^{st-\alpha}_p A^p_p s^{lk-\beta,T\cdot}_p) \tag{6.31}$$

and for $p \notin (\mathbf{st}, \mathbf{lk})_+$

$$A^{st-\alpha}_{lk-\beta} = B^{st-\alpha}_{lk-\beta} = 0 \tag{6.32}$$

Comments

1. From relations (6.31) and (6.32) follows that the kinematic equations may content the kinematic characteristics of (st)–bodies and contrarily only in the case where their sets of accessibility are not empty. *E.g.*, in the classical scheme of 'hedgehog' – in principal, equations of any rotating body do not contain generalized coordinates of other bodies (rotation angles w.r.t. the base body) as well as their velocities and accelerations.
2. From the analysis of equations (6.29) follows that in the case of $(\mathbf{st}, \mathbf{lk})_+ = \emptyset$ the mutual influence of motions of bodies is possible only from the p–bodies of the system if $(\mathbf{p}, \mathbf{lk})_+ = \emptyset$ and $(\mathbf{st}, \mathbf{p})_+ = \emptyset$. *E.g.*, for systems of the type of 'hedgehog' the mutual influence of motions of rotating bodies realize only through the motion of the support. Thus any effects that the above influence causes are defined by terms of inertial origin that depend on the product of linear and angular velocities of separate bodies and the support body (this can be easily seen from the visual analysis of the motion equations (6.29). When the support body motion is absent it is clear that this influence is absent, too.

6.2.4. *$O(n^2)$–operation algorithms for constructing matrices of the kinetic energy*

Notation Henceforth

- $\mathcal{V} = \mathrm{col}\{\ldots, V^{10,lk}_{lk}, \ldots\}$ is the vector of quasi–velocities of the system of bodies with tree–like structure (see (5.49));
- $q^{\cdot} = \mathrm{col}\{\ldots, q^{lk\cdot}, \ldots\}$ is the vector of generalized velocities of the system;
- A is the block diagonal matrix with blocks A^{lk}_{lk} and $\mathcal{S}$ is the parastrophic one.

Definition 6.2 *The quadratic form*

$$\mathcal{T} = \frac{1}{2}\mathcal{V}^T \mathcal{A}\mathcal{V} = \frac{1}{2}(q^{\cdot})^T \mathcal{A}(q)q^{\cdot} \tag{6.33}$$

with properties of measure on σ^μ_3 is called kinetic energy of the system of bodies with tree–like structure, the matrices $\mathcal{A}$ and $\mathcal{A}(q)$ being called matrices of the kinetic energy of the system of bodies in quasi–velocities and in generalized velocities, respectively.

Comment In the first variant of representation (6.33) the kinetic energy is the sum of kinetic energies of bodies that form the system but in the second one it is not.

Proposition 6.12 *There is the following relation (Konoplev 1986a)*

$$A_+^{lk} = A_{lk}^{lk} + \sum_{st \in (\mathbf{lk})^+} L_{st}^{lk} A_+^{st} L_{st}^{lk,T} \tag{6.34}$$

where

- $A_+^{lk},\ A_+^{st}$ *are* 6×6*–dimensional inertia matrices defined for subtrees* $(\mathbf{lk})_+$ *and* $(\mathbf{st})_+$ *of the main tree with branches (lk) and (st), respectively;*
- $L_{st}^{lk} \in \mathcal{L}(\mathcal{R}, 6)$ *is the matrix of transformation of* E_{st} *in* E_{lk}*,* $(st) \in (\mathbf{lk})_+$*;*
- $(\mathbf{lk})^+$ *is the set of right incidence of lk–th element of the system (see D 6.1).*

Proposition 6.13 *Let* β_+^{lk} *be* 6*–dimensional row of the kind*

$$\beta_+^{lk} = A_+^{lk} M_{lk}^{lkc} f_\beta^{lk} \tag{6.35}$$

where

$$A_+^{lk} = A_{lk}^{lk} + \sum_{st \in (\mathbf{lk})^+} L_{st}^{lk} A_+^{st} L_{st}^{lk,T} \tag{6.36}$$

Then: 1. to compute or construct analytically the element $A_{lk-\beta}^{st-\alpha}$ *from the right–upper part of the matrix of kinetic energy of the system –* $O(n^2)$*–operation algorithm can be used*

$$A_{lk-\beta}^{st-\alpha} = \begin{cases} s_{lk}^{st-\alpha} \beta_+^{lk} & \text{if } (lk) \in (\mathbf{st})_+ \\ 0 & \text{otherwise} \end{cases} \tag{6.37}$$

2. for the right–upper part of the matrix of kinetic energy – the matrix representation is fulfilled

$$\mathcal{A}^+(q) = \mathcal{S}\mathcal{A}_+\mathcal{M}\|f\| \tag{6.38}$$

6.2.5. *$O(n)$–operation algorithm for constructing $\mathfrak{b}(q, q^\cdot)$*

The straightforward calculation or analytic construction of the column $\mathfrak{b}(q, q^\cdot) = \mathcal{B}(q, q^\cdot)q^\cdot$ in relation (6.29) can be done by using the $O(n^3)$–operation algorithm (6.21). Belkov (1992) suggested a more effective, linear in n, algorithm for constructing this column (Belkov *et al.* 1997, Konoplev *et al.* 2001).

Proposition 6.14 *Let the algorithm for numerical or analytical constructing of the vector* $\mathfrak{b}(q, q^\cdot) = \mathcal{B}(q, q^\cdot)q^\cdot$*, due to relation (6.21), be taken in the form*

$$\mathfrak{b}(q, q^\cdot) = \mathcal{S}(\mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{V}),\ \mathcal{X} = \mathcal{S}^{T\cdot} q^\cdot \tag{6.39}$$

Then there is the following recurrent (from the root of the tree (11) *root to the kidneys)* $O(n)$*–operation algorithm for constructing the vector* $\mathcal{X}$

$$\begin{aligned} X_{lk} &= L_{lk}^{\mu,k-1,T} X_{\mu,k-1} + L_{lk}^{\mu,k-1,T\bullet} V_{\mu,k-1}^{10;\mu,k-1} \\ X_{11} &= M_{11}^{11c\bullet} \| f^{11} \| q^{11\bullet} \end{aligned} \tag{6.40}$$

where the matrices $L_{lk}^{\mu,k-1,T\bullet}$ *and* $M_{11}^{11c\bullet}$ *are constructed (in numerical or analytical form) by using algorithms (6.23) and (7.11), while vector* $V_{\mu,k-1}^{10;\mu,k-1}$ *is constructed by using the kinematic equations for system (6.11) and taking into account that the matrix* $\mathcal{S}$ *has been constructed at the step of constructing the matrix* $\mathcal{A}(q)$*.*

Proposition 6.15 *Let:*

– *the algorithm for numerical or analytical constructing of the vector* $\mathfrak{b}(q, q^{\bullet})$ *be presented, with using (6.39), in the form*

$$\mathfrak{b}(q, q^{\bullet}) = \|f\|^T \mathcal{M}^T \mathcal{Z},\ \mathcal{Z} = \mathcal{L}\mathcal{Y},\ \mathcal{Y} = (\mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{V}),\ \mathcal{X} = \mathcal{S}^{T\bullet} q^{\bullet} \tag{6.41}$$

– *the vector* $\mathcal{X} = \mathcal{S}^{T\bullet} q^{\bullet}$ *be computed by algorithm (6.41);*

– (cd) *be a tree bud.*

Then there is the recurrent (from the buds (cd) *to the root of the tree* (11)*)* $O(n)$*–operation algorithm for constructing the vector* $\mathcal{Z}$

$$Z_{\mu,k-1} = Y_{\mu,k-1} + \sum_{(\boldsymbol{\mu},\mathbf{k-1})^+} L_{lk}^{\mu,k-1} Z_{lk},\ Z_{cd} = Y_{cd} \tag{6.42}$$

Chapter 7

Foundations of algebraic screw theory

The full theory variant is worked in (Konoplev 1987a, Konoplev *et al.* 2001). In this chapter there are its basic results for giving references in the main book text.

7.1. Sliding vector set

Notation Henceforth

- $\mathbf{A}_3$ is 3-dimensional affine-vector space; $\mathbf{E}_s = (o_s, [\mathbf{e}^s])$ is a frame; $x \in \mathbf{V}_3$ is a free vector; x^s is its coordinate column in the basis $[\mathbf{e}^s]$;
- $a \in \mathbf{D}_3$ is a point, r^s_a is its radius vector in $\mathbf{E}_s$; r^{ss}_a is its coordinate column in the basis $[\mathbf{e}^s]$; $< r^s_a >^s$ is the skew-symmetric matrix generated by r^s_a with the help of the relation
$$< r^s_a >^s = \begin{bmatrix} 0 & -r^s_{a3} & r^s_{a2} \\ r^s_{a3} & 0 & -r^s_{a1} \\ -r^s_{a2} & r^s_{a1} & 0 \end{bmatrix}$$
- $\mathbf{l}^x_a$ is the line (*i.e.*, the straight line passing through the point a in parallel to the vector x): the set of points $b \in \mathbf{D}_3$ such that $b = a + kx$ where $k \in \mathbf{R}_1$.

Definition 7.1 *The* 6 *functions defined by the relations*
$$l^{xs}_s = R^s_{as}\,\mathrm{col}\{x^s, x^s\} \tag{7.1}$$
$$\mathrm{col}\{x^s, x^s\} \in \mathbf{R}_6 = \mathbf{R}_3 \times \mathbf{R}_3,\ R^s_{as} = \begin{bmatrix} E & 0 \\ 0 & < r^s_a >^s \end{bmatrix}$$
is called a sliding vector, generated by the free vector x and the line $\mathbf{l}^x_a$ in the frame $\mathbf{E}_s$(reduced to the point o_s – the subscript – and calculated in the basis $[\mathbf{e}^s]$ – superscript).

Proposition 7.1 *The sliding vector (7.1) does not depend on the choice of $a \in \mathbf{l}^x_a$.*

Proof Let us choose any point $b \neq a$, $b \in \mathbf{l}^x_a$. Then $R^s_{as}\,\mathrm{col}\{x^s, x^s\} - R^s_{bs}\,\mathrm{col}\{x^s, x^s\} = 0$ as $r^s_a - r^s_b = kx^s$ ($k \in \mathbf{R}_1$) and, therefore, we have $(< r^s_a >^s - < r^s_b >^s)x^s = 0$. What is why we shall omit the index a in the notation of R^s_{as}.

Proposition 7.2 *The set $\mathbf{I} = \{l^{xs}_s,\ x \in \mathbf{V}_3\}$ of sliding vectors is not a linear space.*

Proof Indeed, let us take a free vector $y \in \mathbf{V}_3 \neq 0$ and two points a and $b \in \mathbf{D}_3$ such that $a^s - b^s \neq kx^s$ $(k \in \mathbf{R}_1)$. Then we may define two sliding vectors $l_s^{xs} = R_{as}^s \operatorname{col}\{x^s, x^s\}$ and $l_s^{-xs} = R_{bs}^s \operatorname{col}\{-x^s, -x^s\}$, their sum having the form $\operatorname{col}\{0, 0, 0, (< a^s >^s - < b^s >^s)x^s\} \notin \mathbf{I}$ since $< a^s - b^s >^s x^s = - < x^s >^s (a^s - b^s) \neq 0$.

Comment The six functions (7.1) is linearly dependent as $\det\{< r_a^s >^s\} = 0$ as there is the relation $x^{s,T} < r_a^s >^s x^s = 0$.

Proposition 7.3 *A sliding vector l_s^{xs} is equal zero if and only if the vector x^s is zero, too:*

$$l_s^{xs} = 0 \Longleftrightarrow x^s = 0 \tag{7.2}$$

Proof The necessity: $l_s^{xs} = R_{as}^s \operatorname{col}\{x^s, x^s\} = 0 \rightarrow l_s^{xs} = \operatorname{col}\{x^s, < r_a^s >^s x^s\} = 0 \rightarrow x^s = 0$. The sufficiency: $x^s = 0 \rightarrow l_s^{xs} = 0$.

7.2. Screw vector space

Henceforth $\mathbf{I}^j = \{l_{sk}^{xs} = R_s^s \operatorname{col}\{x_k^s, x_k^s\},\ x_k^s \in \mathbf{R}_3,\ k \in \mathbf{K}(j)\}$ is (j)–system of sliding vectors in the frame $\mathbf{E}_s$; $\mathbf{K}(j)$ is the index set of (j)–system; j is the system index.

Definition 7.2 *The six functions generated by the relation*

$$H_s^s = \sum_{k \in \mathbf{K}(j)} l_{sk}^{xs} \tag{7.3}$$

are called a screw generated by (j)–system of sliding vectors in the frame $\mathbf{E}_s$.

Proposition 7.4 *The set $\mathbf{H}$ of screws in any frame is a 6–dimensional linear space.*

Definition 7.3 *The first 3 screw coordinates x_k^s is called the main vector of the screw. It is also noted as*

$$mv\, x_k^s \tag{7.4}$$

Let us take a vector $x \in \mathbf{V}_3$ and define two sliding vectors l_a^{xs} and l_b^{-xs} where a and $b \in \mathbf{D}_3$ are arbitrary points.

Definition 7.4 *The screw*

$$H_s^s = l_s^{xs} + l_s^{-xs} \tag{7.5}$$

is called a couple, and the distance $h = \min \parallel r_y^s - r_z^s \parallel$ between the lines $\mathbf{l}_a^x$ and $\mathbf{l}_b^x$ is called the couple arm (here r_y^s and r_z^s are the radius-vectors of points $y \in \mathbf{l}_a^x$ and $z \in \mathbf{l}_b^x$).

Definition 7.5 *The sliding vector and the screw such that*

$$l_s^{xs} = \mathrm{col}\,\{x,\, 0_3\}, \quad H_s^s = \mathrm{col}\,\{\sum_{k\in\mathbf{K}(\alpha)} x_k,\, 0_3\}, \quad x \neq 0, \sum x_k \neq 0 \tag{7.6}$$

are called degenerate.

Comments Note that

- a sliding vector l_s^x is degenerate if its action line $\mathbf{l}_a^x$ goes through the center $o_s \in \mathbf{D}_3$;
- a sliding vector l_s^x being degenerate in the frame $\mathbf{E}_s$ is not degenerate in a frame $\mathbf{E}_\sigma$ if $o_s \neq o_\sigma$;
- it is not necessary for a degenerate screw to be always a sum of degenerate sliding vectors;
- the set of degenerate screws is a 3–dimensional subspace of $\mathbf{V}_3 \times \mathbf{V}_3$.

Proposition 7.5 *Couple (7.5) and the degenerate screw (7.6) are invariant w.r.t. choice of the reduction center o_s. They are generated by the coordinate column of $x \in \mathbf{V}_3$, the first one depending on its arm.*

7.3. Group of motions

The following results give a simple, easily realizable in computers, algebraic method of transforming screw coordinates when one frame is changed with another.

Notation 1. $\mathbf{E}_s = (o_s, [\mathbf{e}^s])$ and $\mathbf{E}_t = (o_t, [\mathbf{e}^t)]$ are Cartesian frames in $\mathbf{A}_3$ with origins in points o_s and $o_t \in \mathbf{D}_3$, orthonormal bases $[\mathbf{e}^s] = \{e_1^s,\, e_2^s,\, e_3^s\}$ and $[\mathbf{e}^t] = \{e_1^t,\, e_2^t,\, e_3^t\} \in \mathbf{V}_3$, respectively;

2. $c_t^s \in \mathcal{SO}(\mathcal{R}, 3)$ is the transformation matrix from the basis $[\mathbf{e}^s]$ to $[\mathbf{e}^t]$, $[\mathbf{e}^t] = c_t^s[\mathbf{e}^s]$ to $[\mathbf{e}^t]$;

3. o_t^s is the vector of translation of $\mathbf{E}_t$ to $\mathbf{E}_s$, o_t^{ss} is its coordinate column in the basis $[\mathbf{e}^s]$,

4. H_s^s and H_t^t are screws calculated in the frames $\mathbf{E}_s$ and $\mathbf{E}_t$ and generated by the same free vector $x \in \mathbf{V}_3$.

Proposition 7.6 *The motion $L_t^s : \mathbf{H}_t^t \rightarrow \mathbf{H}_s^s$ is defined as*

$$H_s^s \ = L_t^s H_t^t, \quad L_t^s = T_t^{ss}[c_t^s] \tag{7.7}$$

where $\mathbf{H}_t^t$ and $\mathbf{H}_s^s$ are screw spaces generated by H_s^s and H_t^t, respectively;

$$T_t^{ss} = \begin{bmatrix} E & O \\ < o_t^s >^s & E \end{bmatrix}, \qquad [c_t^s] = \mathrm{diag}\{c_t^s,\, c_t^s\} \tag{7.8}$$

are matrices of shift and rotation (induced by the shift with the vector o_t^s and the rotation c_t^s), respectively; O is 3×3–dimensional null matrix.

Proof Suppose $l^{xs}_{sk} = R^s_s \operatorname{col}\{x^s_k, x^s_k\}$ is the coordinate representation of a sliding vector from relation (7.1). Then

$$\begin{aligned} L^s_t\, l^{xt}_{tk} = L^s_t R^t_t \operatorname{col}\{x^t_k, x^t_k\} &= T^{ss}_t [c^s_t] R^t_t \operatorname{col}\{x^t_k, x^t_k\} = \\ T^{ss}_t [c^s_t] R^t_t c^{s,T}_t c^s_t \operatorname{col}\{x^t_k, x^t_k\} &= T^{ss}_t R^s_t \operatorname{col}\{x^s_k, x^s_k\} = R^s_s \operatorname{col}\{x^s_k, x^s_k\} = l^{xs}_{sk} \end{aligned}$$

Taking in account that L^s_t is a linear operator and going in the both sides of this equality to sums of the (7.3)–kind, we easily obtain the desired result.

Proposition 7.7 *The set of motions*

$$L^s_t : \mathbf{H}^t_t \to \mathbf{H}^s_s \tag{7.9}$$

of the screw space is the multiplicative group

$$\mathcal{L}(\mathcal{R}, 6) = \{L^s_t : L^s_t = T^{ss}_t [c^s_t],\ s, t \in \mathbf{N}\} \tag{7.10}$$

Proof It is true because $L^s_t L^t_p = T^{ss}_t [c^s_t] T^{tt}_p [c^t_p] = T^{ss}_t [c^s_t] T^{tt}_p [c^s_t]^T [c^s_t][c^t_p] = T^{ss}_t T^{ts}_p [c^s_p] = T^{ss}_p [c^s_p] = L^s_p$ and $(L^s_t)^{-1} = (T^{ss}_t [c^s_t])^{-1} = [c^s_t]^T (T^{ss}_t)^{-1} = [c^t_s] T^{ts}_s [c^t_s]^T [c^t_s] = T^{tt}_s [c^t_s] = L^t_s \in \mathcal{L}_t(\mathcal{R}, 6)$.

7.4. Kinematics equation on group $\mathcal{L}_t(\mathcal{R}, 6)$

Notation $\mathcal{L}_t(\mathcal{R}, 6) = \{L^s_t(t) : L^s_t(t) = T^{ss}_t(t)[c^s_t(t)], s, t \in \mathbf{N}\}$ is one–parametric group of transforming screws when there is passage from one frame to another.

Proposition 7.8 *The equation of kinematics on the one–parameter group $\mathcal{L}_t(\mathcal{R}, 6)$ has the following form*

$$L^{s\cdot}_t(t) = L^s_t(t)\Phi^{st}_t(t), \qquad \Phi^{st}_t(t) = \begin{bmatrix} <\omega^s_t>^t & O \\ <v^s_t>^t & <\omega^s_t>^t \end{bmatrix} \tag{7.11}$$

where $<v^s_t>^t = <o^{s\cdot}_t(t)>^t$ *and* $<\omega^s_t>^t = c^{s\cdot}_t(t) c^{s,T}_t(t)$ *are skew–symmetric matrices generated by the coordinate columns* v^{st}_t *and* w^{st}_t.

Proof For the sake of brevity we shall omit argument t. Then

$$L^{s\cdot}_t = (T^{ss}_t [c^s_t])^{\cdot} = T^{ss\cdot}_t [c^s_t] + T^{ss}_t [c^s_t]^{\cdot} = (T^{ss\cdot}_t + T^{ss}_t [c^s_t]^{\cdot} [c^s_t]^T)[c^s_t]$$

It is easy to check that $T^{ss\cdot}_t = T^{ss}_t T^{ss\cdot}_t$. That is why

$$L^{s\cdot}_t = T^{ss}_t (T^{ss\cdot}_t + [c^s_t]^{\cdot}[c^s_t]^T)[c^s_t] = T^{ss}_t (T^{ss\cdot}_t + [<\omega^s_t>^t])[c^s_t]$$

where $[<\omega^s_t>^t] = \operatorname{diag}\{<\omega^s_t>^t, <\omega^s_t>^t\}$. Hence

$$L^{s\cdot}_t = T^{ss}_t \Phi^{ss}_t [c^s_t] = T^{ss}_t [c^s_t][c^s_t]^T \Phi^{ss}_t [c^s_t] = T^{ss}_t [c^s_t] \Phi^{st}_t = L^s_t \Phi^{st}_t$$

Definition 7.6 $W^{st}_{tt} = W^{st}_t = \operatorname{col}\{w^{st}_t, v^{st}_t\}$ *is called the kinematic screw of* $\mathbf{E}_t$*–motion w.r.t.* $\mathbf{E}_s$ *in* $\mathbf{E}_t$.

Comment Equation (7.11) permits us to introduce the operation of differentiation in various screw spaces that can be easily realized numerically.

Proposition 7.9 *The differentiation operations in the (moving and immovable) frames* $\mathbf{E}_t$ *and* $\mathbf{E}_s$*, respectively, are connected by the following relation*

$$H_s^{s\cdot} = L_t^s(H_t^{t*} + \Phi_t^{st} H_t^t) \tag{7.12}$$

Proof From $H_s^s = L_t^s H_t^t$ follows that $H_s^{s\cdot} = L_t^s H_t^{t*} + L_t^{s\cdot} H_t^t = L_t^s H_t^{t*} + L_t^s \Phi_t^{st} H_t^t$.

Comment The symbol * is used here only to underline the fact that the differentiation is produced in the moving frame $\mathbf{E}_t$. In fact the combination of symbols t and * is sufficient one to see this.

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