STABILIZATION OF PENDULUM OSCILLATIONS AROUND UPPER POSITION [1]

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Abstract: The paper addresses problems of control of pendulum oscillations under high-frequency vertical vibration of the pivot. By using a model of oscillations and virtual energy concept, an energy-based feedback control law with the observer of slow state variables is designed to provide desired stable oscillations of the pendulum with respect to the upright position.

Keywords: nonlinear control, pendular systems, virtual energy, oscillation stabilization.

1. INTRODUCTION

Problems of analysis and control of pendular systems and nonlinear oscillations permanently evoke an interest of the researchers and are a subject of numerous publications of the last century (see References). Having, as a rule, a minor direct technical application, such problems often represent a considerable interest as a benchmarks for many natural, technological and physical phenomena from biological processes and vibrational technologies to problems of anti-gravity. The known researches were concentrated around the problems of upright position stabilization, swinging-up and control of periodic motion of the pendulum by using energy-based control techniques (Chung and Hauser, 1995; Miroshnik and Olkhovskaya, 2003), speed-gradient method (Andrievsky *et al.*, 1996; Fradkov, 1996; Fradkov and Pogromsky, 1998; Shiriaev *et al.*, 1998; Shiriaev *et al.*, 1999) and geometric approaches (Miroshnik and Bobtzov, 2000; Aracil *et al.*, 2002).

The desired stabilization or oscillating motions are usually obtained due to horizontal movement of the pivot of the pendulum. Nevertheless the problem of control of periodic motion of the pendulum in the neighborhood of the lower position can be also solved by using vertical pivot movement (Fradkov *et al.*, 1999; Miroshnik and Olkhovskaya, 2003). Moreover, under an appropriate high-frequency vertical excitation of the pendulum support, a complex periodic motion of the pendulum in the vicinity of the upright position is observed. This is referred to as *induced or vibrational stability* and studied in a great number of scientific publications (see (Stephenson, 1908; Kapitza, 1951; Bogolyubov and Mitropolsky, 1962; Belman *et al.*, 1986; Blekhman, 1988; Yabuno *et al.*, 2002; Odinets and Levidova, 2004)).

In this paper we make an attempt to solve the problem of stabilization of a given periodic motion of a pendulum around the upright position by using vertical vibration of the pivot and an auxiliary stabilizing control action. The problem is reduced to stabilization of the *virtual energy* of the pendulum which is associated with the energy of slow motion of an inverted pendular system with

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opposite gravity direction, and is solved on the basis of standard techniques of energy-based control developed for "normal" pendula with vertically moved support (Fradkov *et al.*, 1999; Miroshnik and Olkhovskaya, 2003). In order to separate the slow variables of the system and the current values of the virtual energy, a nonlinear observer of the vibrating pendulum is designed.

The paper is organized as follows. The procedure of the design of energy-based control laws for pendulums with mobile supports is considered in Section 2, and motion of the pendulum with vertical vibration of the pivot is analyzed in Section 3. In Section 4, the problems of the control of the inverted pendulum oscillation and observing the slow motion variables are solved by using the concept of virtual energy.

2. ENERGY-BASED CONTROL OF OSCILLATIONS

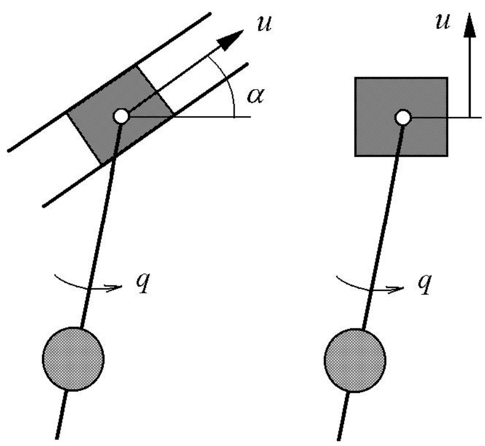

Fig. 1. Pendula with mobile supports

The general model of a pendulum on a mobile support is described by Lagrangian equation

$$J\ddot{q} + \frac{\partial \Pi}{\partial q} \;=\; G(q, \alpha)u, \qquad (1)$$

where q is the generalized coordinate (angle), u is the control action, $J = ml^2$, $\Pi(q)$ is the potential energy, $G(q)$ is a function which depends on the current angular orientation of the support α (see Fig. 1,a). The total energy (Hamiltonian function) of the unforced pendulum is computed as

$$E(q,p) = \Pi(q) + T(p) = \Pi(q) + \frac{1}{2J}p^2, \quad (2)$$

where $T(p)$ is the kinetic energy, $p = J\dot{q}$ is the momentum. Equation (1) can be rewritten in the Hamiltonian form

$$J\dot{q} = p, \quad \dot{p} = -\frac{\partial \Pi}{\partial q} + G(q,\alpha)u. \qquad (3)$$

Consider the problem of control of oscillations which is associated with keeping up a required mode of the undamped periodic pendulum motion. Taking into account that the mode of oscillations is connected with a certain level of pendulum internal energy (Andrievsky *et al.*, 1996; Fradkov and Pogromsky, 1998; Fradkov *et al.*, 1999), the problems is usually reduced to those of energy stabilization. The latter is a standard nonlinear problem of partial stabilization of a dynamical system, or stabilization with respect to the function $E = E(q,p)$ (Fradkov *et al.*, 1999).

Let us set a desired energy level E^* and introduce the energy error (deviation)

$$\xi \;=\; E(q,p) - E^*. \qquad (4)$$

After simple manipulations, we obtain the error model

$$\dot{\xi} \;=\; \frac{1}{J}pG\,u. \qquad (5)$$

A stable solution of the problem is given by different control laws of the form

$$u \;=\; -J\,U(p,G)\,k\xi, \qquad (6)$$

where $k > 0$ is a feedback gain. Substituting (6) into (5), one obtains

$$\dot{\xi} = -pGU(p,G)\,k\xi. \qquad (7)$$

We can conclude that the system is asymptotically stable with respect to the given partial equilibrium point $E = E^*$ when the function $U(p,G)$, for all $t > 0$, satisfies the inequality

$$\int_0^t pGU(p,G)d\tau \;\geq\; \lambda t, \qquad (8)$$

where $\lambda > 0$. It is easily seen, for instance, that the problem is solved by the control laws

$$u = -J\;sign(pG)\;k\xi,$$
$$u = -JpG\;k\xi.$$

Now consider a special case of the problem when the pivot accomplishes vertical motion (see Fig.1,b). Here

$$\Pi(q) = mgl(1-\cos q),$$
$$G = -ml\sin q,$$

and equation (3) takes the form

$$J\dot{q} \;=\; p, \quad \dot{p} = -mgl\sin q - ml\sin q\;u. \quad (9)$$

It is worth to note that the unforced pendulum (9) has two equilibrium points. The first one

$(q, p) = (0, 0)$ corresponds to the lower position of the pendulum and is asymptotically stable. The other points $(q, p) = (\pm\pi, 0)$ associated with the upright position, are unstable and, in the case considered, cannot be stabilized by using standard control techniques. The same situation is observed in pendulum oscillations. Energy based control (7), or, for instance, the control

$$u = sign(p \ \sin q) \ k\xi,$$

provides stable oscillation around the lower point corresponding the given energy level

$$E^* \ < \ E_m = 2mgl.$$

(see Fig. 2). If $E^* \ > \ E_m$, the pendulum demonstrates proportional rotation around the pivot. Oscillations of the pendulum around the upper position are impossible without special pivot excitation.

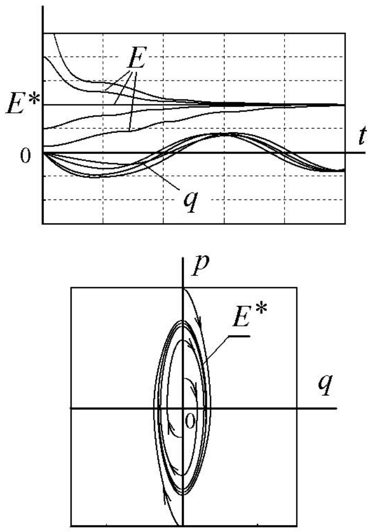

Fig. 2. Stabilization of oscillations around lower position

3. PIVOT EXCITATION AND KAPITZA PENDULUM

High-frequency vertical vibration of the support (see Fig. 3) essentially changes the properties of the pendulum (Stephenson, 1908; Kapitza, 1951; Bogolyubov and Mitropolsky, 1962; Belman *et al.*, 1986; Blekhman, 1988; Yabuno *et al.*, 2002). Under the relevant conditions, the open loop pendular system known as *Kapitza pendulum* becomes stable (or asymptotically stable) with respect to the upper equilibriums $(q, p) = (\pm\pi, 0)$. This property is a crucial point to the problem of the control of pendulum oscillations around the upright position.

Consider a pendulum under a high-frequency vertical excitation of the pivot without an additional control actions. Let the vibrational motion of the support be described by equation

$$\ddot{s} + \omega^2 s \ = \ 0, \qquad (10)$$

where s is coordinate of the pivot, $s(0) = s_0$, $\dot{s}(0) = \dot{s}_0$, ω is a frequency of vibration, and therefore

$$s(t) \ = \ A \sin(\omega t + \varphi),$$

where $A = A(s_0, \dot{s}_0)$, $\varphi = \varphi(s_0, \dot{s}_0)$. Setting $u = \omega^2 s$ in the model (9), one can write

$$J\dot{q} = p, \quad \dot{p} = -ml(g + \omega^2 s)\sin q. \qquad (11)$$

If the frequency ω is large enough, the solution of the system (11) is approximately represented by a two-frequency signal of the form (Bogolyubov and Mitropolsky, 1962; Blekhman, 1988)

$$q \ \cong \ \overline{q}(1 + \widetilde{s}).$$

Here

$$\widetilde{s}(t) \ = \ s - s_0$$

is a fast component of the oscillations, and $\overline{q}$ is a slow component, being a solution of the equation

$$J\dot{\overline{q}} = \overline{p}, \quad \dot{\overline{p}} = -m(gl + \frac{\theta\omega^2 A^2}{2})\sin\overline{q}, \quad (12)$$

where

$$\theta = 1 \ \text{ if } \ |\overline{q}| < \frac{\pi}{2}, \quad \theta = -1 \ \text{ if } \ \frac{\pi}{2} < |\overline{q}| < \frac{3\pi}{2}.$$

When

$$\omega^2 A^2 \ > \ 2gl \qquad (13)$$

equation of slow motion (12) can be rewritten as

$$J\dot{\overline{q}} = \overline{p}, \quad \dot{\overline{p}} = -\theta m l^2 \overline{\omega}^2 \sin\overline{q}, \qquad (14)$$

where

$$\overline{\omega} \ = \ \sqrt{\frac{2g\theta l + \omega^2 A^2}{2l^2}}, \qquad (15)$$

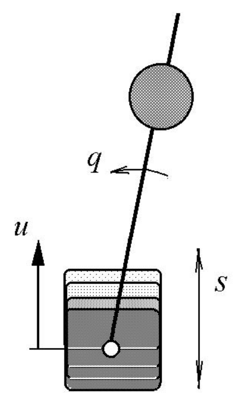

Fig. 3. Inverted pendulum with vibrating support

or in the form

$$\ddot{\bar{q}} + \theta\varpi^2 \sin \bar{q} \;=\; 0. \qquad (16)$$

The latter shows that the system acquires two stable equilibriums $(q, p) = (0, 0)$ and $(q, p) = (\pm\pi, 0)$, and the pendulum can accomplish "slow" oscillation around the upper (see Fig. 4) or lower positions.

Consider the pendulum motion in the neighborhood of the upright position, where $\pi/2 < |\bar{q}| < 3\pi/2$ and $\theta = -1$. Introduce *virtual gravity acceleration*

$$\bar{g} \;=\; \frac{\omega^2 A^2}{2l} - g \;>\; 0,$$

turned to the opposite direction with respect to the gravity, and the *virtual potential energy* of the slow motion

$$\bar{\Pi}(\bar{q}) \;=\; m\bar{g}l(1 + \cos \bar{q})). \qquad (17)$$

Note that $\bar{\Pi}(\bar{q}) \geq 0$ in the neighborhood of the upright position and $\bar{\Pi}(\pm\pi) = 0$. Then the model of the pendulum system (14) can be rewritten as

$$J\dot{\bar{q}} = \bar{p}, \quad \dot{\bar{p}} = -\frac{\partial \bar{\Pi}}{\partial \bar{q}}. \qquad (18)$$

The latter is equivalent to a description of the free motion of the ordinary pendulum considered in Section 2 (see equation (3)).

Thus, under the high-frequency excitation of the pivot, the equilibrium points $(q, p) = (\pm\pi, 0)$ become stable that enable one, by using an appropriate control actions, to provide the required stable slow oscillations of the pendulum around the upright position.

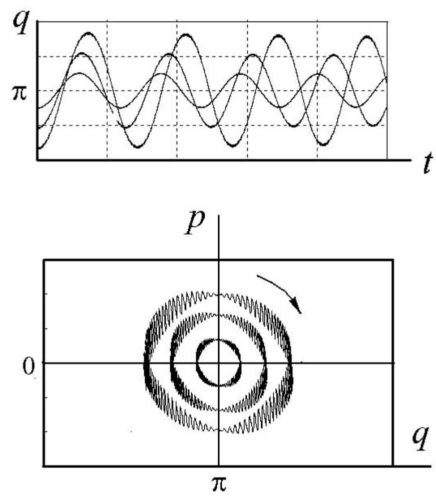

Fig. 4. Free oscillations around upper position

4. STABILIZATION OF OSCILLATIONS OF INVERTED PENDULUM

Consider the controlled pendulum motion with respect to the upright position, supposing that condition (13) is satisfied. Choose the control

$$u = \omega^2 s + \bar{u},$$

where $\bar{u}$ is the stabilizing signal. The model of slow motion of the pendulum in the neighborhood of the upright position $\pi/2 < |q| < 3\pi/2$ takes the form

$$J\dot{\bar{q}} = \bar{p}, \quad \dot{\bar{p}} = ml\bar{g}\sin\bar{q} - ml\sin\overline{qu}, \quad (19)$$

where $\bar{g}$ is a virtual gravity acceleration (see Section 3). Let us introduce the *virtual energy* of slow motion as

$$\bar{E}(\bar{q}, \bar{p}) \;=\; \bar{\Pi}(\bar{q}) + \frac{1}{2J}\bar{p}^2. \qquad (20)$$

Then the model of the inverted pendulum system (19) can be rewritten as

$$J\dot{\bar{q}} = \bar{p}, \quad \dot{\bar{p}} = -\frac{\partial \bar{\Pi}}{\partial \bar{q}} + G(\bar{q})\bar{u}. \qquad (21)$$

where

$$G(\bar{q}) \;=\; -ml\sin\bar{q}.$$

Such as a desired mode of pendulum oscillations is associated with a certain level of its virtual energy, the control problem is reduced to that of energy stabilization considered in Section 2.

We define the required energy level $\bar{E}^*$, introduce the error (virtual energy deviation) as

$$\bar{\xi} \;=\; \bar{E}(q, p) - \bar{E}^* \qquad (22)$$

and find the error model

$$\dot{\bar{\xi}} = \frac{1}{J}\bar{p}G(\bar{q})\;\bar{u}. \qquad (23)$$

A stable solution of the problem is given by control laws of the form

$$\bar{u} \;=\; J\,U(\bar{p}, G)\;k\bar{\xi}. \qquad (24)$$

The system is asymptotically stable with respect to the given level $\bar{E}^*$ when the function U, for all $t > 0$, satisfies inequality (8).

The main difficulty of realization of the control (24) is associated with separation of the signals $\bar{q}$ and $\bar{p}$, corresponding to the slow motion of the pendulum, from the measurable two-frequency signal

$$q \;=\; \bar{q}(1 + \tilde{s}), \qquad (25)$$

where $\tilde{s} = s - s_0$. This is overcame by using the following nonlinear observer (Odinets and

Levidova, 2004), corresponding the structure of the system (21), (25):

$$\begin{aligned} J\dot{\widehat{q}} &= \widehat{p} + \widetilde{u}_1, \\ \dot{\widehat{p}} &= -\frac{\partial \overline{\Pi}(\widehat{q})}{\partial \widehat{q}} + G(\widehat{q})\overline{u} + \widetilde{u}_2, \\ \widehat{q} &= \widehat{q}(1+\widetilde{s}), \end{aligned} \tag{26}$$

where $\widehat{q}$, $\widehat{p}$, $\widehat{q}$ are estimates of the relevant variables,

$$\widetilde{u}_1 = k_1\,(q-\widehat{q}), \quad \widetilde{u}_2 = k_2\,(q-\widehat{q}) \tag{27}$$

are the observer feedback signals, $k_1 > 0$, $k_2 > 0$.

The validity of the result is confirmed by simulation. The pendulum with parameters $m = 0.01$, $l = 0.1$ and pivot vibration $s = 0.002 \sin 2000t$ was considered. The control law

$$\overline{u} = J\ sign(\widehat{p}\sin\widehat{q})k\widehat{\xi},$$

where

$$\widehat{\xi} = \overline{E}(\widehat{q},\widehat{p}) - \overline{E}^*, \tag{28}$$

provides stabilization of the virtual energy $\overline{E}$ at the levels $\overline{E}^*$ from 0 to 0.056, which corresponds to stable oscillations of the pendulum around the upper position at the amplitudes up to 1.2 rad. Fig. 5 illustrates the convergence of the processes for the case $\overline{E}^* = 0.025$ when the oscillation amplitude is 0.45 rad.

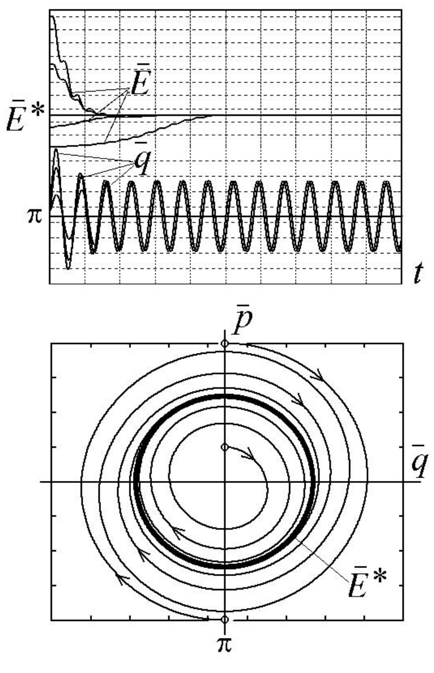

Fig. 5. Stabilization of oscillations around upper position

5. CONCLUSION

The problem of stabilization of a given periodic motion of the inverted pendulum around the upright position was solved by using vertical vibration of the pivot and an auxiliary stabilizing control action. The problem was reduced to stabilization of the virtual energy of the pendulum and standard techniques of energy-based control developed for pendula with vertically moved support. In order to estimate the slow variables of the system and its virtual energy, a nonlinear observer of the vibrating pendulum was designed. The simulation confirmed the validity of the results.

REFERENCES

Andrievsky, B.R., P.Yu. Guzenko and A.L. Fradkov (1996). Control of nonlinear oscillations of mechanical systems by speed-gradient method. *Automation and Remote Control* **4**, 4–17.

Aracil, J., F. Gordillo and J.A. Acosta (2002). Stabilization of oscillations in the inverted pendulum. *15 IFAC World Congress*, Barselona.

Belman, R.E., J. Bentsman and S.M.Meerkov (1986). Vibrational control of a class of nonlinear systems: Vibrational srabilization. *IEEE Trans. Aut. Control* **32**(8), 710–716.

Blekhman, I. (1988). *Synchronization in Science and Technology.* ASME Press. New York.

Bogolyubov, N. and Ju.A. Mitropolsky (1962). *Asymptotic methods in theory of nonlinear oscillations.* Pfismatgiz. Moscow (in Russian).

Chung, C.C. and J. Hauser (1995). Nonlinear control of a swinging pendulum. *Automatica* **31**, 851–862.

Fradkov, A.L. (1996). Swinging control of nonlinear oscillations. *International Journal of Control* **64**(6), 1189–1202.

Fradkov, A.L. and A. Pogromsky (1998). *Introduction to control of oscillations and chaos.* World Scientific. Singapore.

Fradkov, A.L., I.V. Miroshnik and V.O. Nikiforov (1999). *Nonlinear and Adaptive Control of Complex Systems.* Kluwer Acad. Pub. Dordrecht.

Kapitza, P.L. (1951). *Dynamical stability of a pendulum when its point of suspension vibrates and pendulum with a vibrating suspension.* In Collected papers of P.L. Kapitza 2D, Tar Harr, Pergamon Press Ltd. London.

Miroshnik, I. and A. Bobtzov (2000). Stabilization of motions of multipendulum systems. *2nd Int. Conf. on Control of oscillation and chaos*, St.Petersburg **1**, 22–25.

Miroshnik, I.V and E. Olkhovskaya (2003). Spatial problems of nonlinear dynamics. motiva-

tion and analysis.. *Int. Conference Physics and Control (PhisCon 2003)*, St.Petersburg.
Odinets, N. and N. Levidova (2004). Kapitsa pendulum: Modelling and observing. *Preprints of 10-th Baltic Olympiad on Automatic Control*, SPbSU ITMO, St.Petersburg.
Shiriaev, A.S., H. Ludvigsen, O. Egeland and A.L. Fradkov (1999). Swinging up of non-affine in control pendulum. *Proc. American Control Conference*, San Diego, California.
Shiriaev, A.S., O. Egeland and H. Ludvigsen (1998). Global stabilization of unstable equilibrium point of pendulum. *Proc. 37th CDC*, Tampa, pp. 4584–4585.
Stephenson, A. (1908). On induced stabily. *Phil. Mag.* **15**, 233–236.
Yabuno, H., K. Goto and N. Aoshima (2002). Swing-up and stabilization of an inverted pendulum without feedback control - application of a bifurcation control to a pendulum with two degrees of rotational freedom. *SICE Annual Conference*, TEA 09-3.