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Description of mechanism of thermal conduction and internal damping by means of two component Cosserat continuum

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Abstract A new approach to derivation of the theory of thermoviscoelasticity is proposed. Neither the hypothesis of fading memory nor the rheological models are used within the framework of this approach. The proposed approach is based on the mechanical model of a one-rotor gyrostat continuum. In special cases, the mathematical description of this model is proved to reduce to the equations of the coupled problem of thermoelasticity, the self-diffusion equation, and the equation describing the flow of viscous incompressible fluid. In the context of this model, we consider the original treatment of physical nature of the mechanism of thermal conduction and internal damping. The first part of the paper contains the aforesaid theoretical results. The second part of the paper is devoted to the determination of some parameters of the model. On the base of the proposed theory we obtain the dependence of the acoustic wave attenuation factor on a signal frequency. This dependence is in close agreement with the classical dependence in the low-frequency range and agrees with the dependence obtained on the base of the phonon theory in the hypersonic frequency range. We discuss some ways of determination of the volume and shear viscosities and the heat flow relaxation timescale by using known values of the sound velocity and the acoustic wave attenuation factor. The obtained values of the heat flow relaxation timescale are compared with the values derived from the phonon theory.

Part 1: Theory

1 Introduction

Accounts of the different theories of viscoelasticity and thermoviscoelasticity can be found, for example, in [1], [2], [3], [4], [5], [6], [7], [8], [9]. Many models describing the properties of the real materials have been proposed. A great number of the specific problems have been solved, see [1], [4], [7]. The general approaches and some models are included in the textbooks on continuum mechanics (see, for example, [3], [4], [5], [6]) and the books meant for design engineers (see, for example, [8]). However, not all of the theoretical problems concerned with dissipative processes in a continuous medium are solved within the framework of the classical mechanics. In particular, it is generally agreed that in the hypersonic frequency range, the dependence of the acoustic wave attenuation factor on a signal frequency can be described only by using the quantum-mechanical approach which is based on the idea of an interaction of sound and thermal phonons [10], [11]. By using the fundamental laws of classical mechanics, we evolve the theory of thermoviscoelasticity which allows us to

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obtain the dependence of the acoustic wave attenuation factor on a signal frequency which is in agreement with the dependence obtained on the base of the phonon theory.

At present two conventional approaches to construct the mathematical models of thermoviscoelastic bodies are evolved.

The first and the most general approach is based on the theory of media with fading memory (see, for example, [2], [4]). The main point of this theory is the statement of the fact that the stresses depend not only on the values of strains and temperature at the given time instant but also on their values at the previous time instants, and the material “remembers” the recent past better than remote past. The equations of the linear theory of thermoviscoelasticity obtained on the base of the hypothesis of fading memory have the form [4]:

$$\nabla\cdot\boldsymbol{\tau}+\rho\mathbf{f}=\rho\,\frac{d^2\mathbf{u}}{dt^2},\qquad \boldsymbol{\varepsilon}=\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right),\qquad \varepsilon=\operatorname{tr}\boldsymbol{\varepsilon},$$
$$\boldsymbol{\tau}=\int\limits_0^t G_1(t-\tau)\,\frac{\partial\boldsymbol{\varepsilon}(\tau)}{\partial\tau}\,d\tau+\left[\int\limits_0^t G_2(t-\tau)\,\frac{\partial\varepsilon(\tau)}{\partial\tau}\,d\tau-\int\limits_0^t\varphi(t-\tau)\,\frac{\partial\tilde{T}_a(\tau)}{\partial\tau}\,d\tau\right]\mathbf{E},$$
$$\frac{k}{T_a^*}\,\Delta\tilde{T}_a=\frac{d}{dt}\int\limits_0^t m(t-\tau)\,\frac{\partial\tilde{T}_a(\tau)}{\partial\tau}\,d\tau+\frac{d}{dt}\int\limits_0^t\varphi(t-\tau)\,\frac{\partial\varepsilon(\tau)}{\partial\tau}\,d\tau\,. \qquad (1)$$

Here, $\boldsymbol{\tau}$ is the stress tensor, $\mathbf{f}$ is the mass density of external forces, $\mathbf{u}$ is the displacement vector, $\boldsymbol{\varepsilon}$ is the strain tensor, $\tilde{T}_a$ is derivation of the temperature from its reference value T_a^*. The quantities ρ, $G_1(t)$, $G_2(t)$, $\varphi(t)$, k and $m(t)$ characterize the mechanical and thermodynamical properties of the material. The first equation in (1) is the equation of motion, the second and third equations introduce the strain characteristics, the fourth one is the constitutive equation, and the fifth one is the heat conduction equation. Certainly, the relaxation functions $G_1(t)$, $G_2(t)$, $\varphi(t)$ and $m(t)$ have to satisfy the restrictions which follow from the second law of thermodynamics and the hypothesis of fading memory, see [4].

The second approach to derivation of the equations of thermoviscoelasticity is based on using the rheological models (see, for example, [6], [7]). Different versions of the linear theory of thermoviscoelasticity obtained on the base of the rheological models differ from each other and from the coupled problem of thermoelasticity only by the constitutive equations. We give, as an example, the constitutive equations for the Kelvin–Voigt solid [7]

$$\boldsymbol{\tau}=2G\,\boldsymbol{\varepsilon}+2G\tau_k\,\frac{d\boldsymbol{\varepsilon}}{dt}+\left[\left(K-\frac{2}{3}G\right)\varepsilon-\frac{2}{3}G\tau_k\,\frac{d\varepsilon}{dt}-\alpha K\tilde{T}_a\right]\mathbf{E} \qquad (2)$$

and for the Maxwell solid [7]

$$\frac{d\boldsymbol{\tau}}{dt}+\tau_M^{-1}\boldsymbol{\tau}=2G\frac{d\boldsymbol{\varepsilon}}{dt}+\left[\left(K-\frac{2}{3}G\right)\frac{d\varepsilon}{dt}+K\tau_M^{-1}\varepsilon-\alpha K\left(\frac{d\tilde{T}_a}{dt}+\tau_M^{-1}\tilde{T}_a\right)\right]\mathbf{E}. \qquad (3)$$

Here, K is the bulk modulus, G is the shear modulus, α is the volume coefficient of thermal expression, τ_k is the strain relaxation time, and τ_M is the stress relaxation time.

Neither the hypothesis of fading memory nor the rheological models are used further. We develop an original method for describing the internal damping. Our model allows us to describe the internal damping inherent to even the materials that are usually considered to be non-viscous, for example, the ideal crystals. So we believe that the mechanism of internal damping inherent in our model has a physical nature different from the physical nature of internal damping modeled by the traditional methods. We show that due to the aforesaid mechanism, the attenuation of acoustic waves takes place, and some part of mechanical energy is transformed into heat. Based on these facts, we can assert that we describe some kind of internal damping.

The physical object under consideration is a conventional isotropic homogeneous material without microstructure, inclusions, etc. This material has elastic, viscous, and thermodynamic properties. In order to describe the thermodynamic processes in the material by means of a mechanical model without using statistical methods, we introduce the continuum possessing internal rotational degrees of freedom. The internal degrees of freedom are used for modeling the thermal processes. Motions associated with the internal degrees of freedom have no relation to the real motions of the material particles. Characteristics of the motions associated with the internal degrees of freedom as well as characteristics of the interactions associated with the internal degrees of freedom should be considered as analogs of thermodynamic quantities. The main ideas of the proposed theory consist in the following:

1. To model a material medium, we use the one-rotor gyrostat continuum (continuum possessing the internal rotational degrees of freedom). This continuum is considered to be elastic. The interaction of carrier bodies of the gyrostats is charged with the mechanical processes. The interaction of rotors models the thermal processes. The interference of carrier bodies and rotors provides the interplay of mechanical and thermal processes.
2. Particles of the material medium are considered to be embedded into some medium having infinite extent. This medium represents the "physical vacuum", the "field", the "ether," or something like that. In what follows, it will be called the "thermal ether." The rotors of gyrostats interact with the "thermal ether" by means of elastic moments concerned with the rotational degrees of freedom. We suppose the heat conduction mechanism to be provided just due to the interaction between the rotors and the "thermal ether".
3. The motion of the rotors of gyrostats causes the appearance of waves in the "thermal ether." As a result, certain part of energy of the material particles is spent on the formation of these waves. We suppose the internal damping mechanism to be provided due to the material medium energy dissipation into the "thermal ether." Certainly, we do not rule out the existence of other internal damping mechanisms (in particular, that are considered in the classical books on viscoelasticity).

Now we explain the physical meaning of the proposed model. We start with a discussion of modern views on atoms and the simplest models of atoms which are used in physics and mechanics. According to the concepts of modern physics, atoms have a very complex internal structure, so that they can be in different energy states and possess the ability to radiate and absorb the energy quanta and elementary particles. However, in statistical physics, atoms are considered as mass points. Similarly, when modeling the crystal lattices, very simple models of atoms are used; to be exact, atoms are assumed to be mass points or infinitesimal rigid bodies. With the help of long-wave approximation, one can pass from the discrete model of a crystal lattice to the continuous model. If the atoms are modeled by mass points, then one obtains the equations of classical momentless continuum, and if the atoms are modeled by infinitesimal rigid bodies, then one obtains the equations of the Cosserat continuum. In both cases, one obtains a model possessing only mechanical properties. Suppose that after passing from a discrete model to continual theory, we would like to have a continuum which possesses not only mechanical properties but also some physical properties described by the partial differential equations. We mean the continuous medium with electric, magnetic, and thermal properties which is able to radiate into the surrounding space both the acoustic waves and waves of another physical nature. In order to construct a discrete model of the medium with such properties, we should consider atoms to be complex particles with internal structure and internal degrees of freedom rather than mass points or infinitesimal rigid bodies.

It is obvious that there are two types of particles with internal structure: the particles with internal translational degrees of freedom and the particles with internal rotational degrees of freedom. Particles of the first type are able to deform. Continua consisting of such particles possess two kinds of strains and stresses: the strains and stresses associated with the distance between the particles and moreover strains of the particles themselves (the local strains) and the local stresses associated with the local strains. Such continua are called micromorphic continua. Particles of the second type are the quasi-rigid bodies which are also called the multi-spin particles or gyrostats. These particles consist of rotating rotors. The term "quasi-rigid body" means that the distances between any two rotors are kept unchanged under arbitrary motions of the quasi-rigid body but each rotor can rotate independently of rotations of other rotors. Continua consisting of such particles are called micropolar continua. The peculiarity of these continua is the fact that each of the points has three translational and a few rotational degrees of freedom the number of which is determined by the number of rotors in multi-spin particles. All additional strains and stresses in micropolar continua are associated with the rotational degrees of freedom. In principle, both deformable particles and multi-spin ones can be used to model atoms and, consequently, both micromorphic and micropolar continua can be used to model the media possessing some non-mechanical properties. Let us consider a micromorphic continuum and assume the possibility of large strains associated with internal degrees of freedom as well as the possibility of large relative velocities and accelerations. In this case, we are confronted with the difficulties associated with keeping the atoms as a whole and holding their characteristic sizes. These difficulties do not arise in the case of micropolar continua. Therefore, in view of the fact that under certain conditions any physical processes are described by nonlinear equations, it can be argued that micropolar continua are better suited for modeling media which possess not only mechanical properties.

Thus, for modeling media with a combination of various physical properties, we propose to consider atoms as complex particles with internal rotational degrees of freedom. Number of rotors inside the particles depends on the number of physical properties of the substance that should be taken into account. If only one non-mechanical property of the substance (e. g., the ability to heat conduction) should be taken into account, then one internal rotor is enough. When a continuum theory is considered, the radius-vector determining position of some point of the medium, in fact, determines position of the mass center of a representative volume which contains billions of atoms. The continuous characteristics of rotational motions represent the quantities averaged over all atoms in the representative volume.

This paper is a continuation and evolution of [12], [13]. A new method of description of the rotational molecular spectra lying in the infrared range (i. e., concerned with the thermal radiation) is proposed in [12]. This method is based on using the continual mechanical model possessing the rotational degrees of freedom. The idea of description of thermoelastic and thermoviscoelastic processes by means of the mechanical model of one-rotor gyrostat continuum was first stated in [13]. The derivation of the equations of one-rotor gyrostat continuum can be found in the aforesaid paper. The concepts of temperature, entropy, and heat flow which are introduced in the context of the proposed model are also contained in [13]. Unlike [13], in what follows we consider a more complex mathematical model which describes not only the volume viscosity but also the shear viscosity. Furthermore, proceeding from some theoretical considerations (a concept of the “thermal ether”) and the analysis of model problems, we discuss the physical nature of the mechanism of thermal conduction and internal damping. In contrast to [13] where the volume viscosity was assumed to coincide with the conventional one, and the heat flow relaxation timescale was determined on the basis of quantum-mechanical consideration; further, we determine these quantities by using known values of the acoustic wave attenuation factor.

In what follows, the direct tensor calculus is used. A summary of the basic rules and formulas of the direct tensor calculus can be found in the appendices to books [2], [14]. The comprehensive description of the subject is presented in [15].

2 Linear theory of the moment elastic continuum

Constructing a model of moment elastic medium, we will use a body-point (a one-spin particle) as the base material object. The body-point is the material object occupying zero volume in space. The position of a body-point is considered to be determined if the position vector $\mathbf{R}(t)$ and the rotation tensor $\mathbf{P}(t)$ are assigned. The rotation tensor is a properly orthogonal tensor which represents the solution of the equations

$$\mathbf{P}\cdot\mathbf{P}^T = \mathbf{P}^T\cdot\mathbf{P} = \mathbf{E}, \qquad \det\mathbf{P} = 1. \tag{4}$$

Here $\mathbf{E}$ is the unit tensor. The translational and angular velocities of a body-point are calculated by the formulas

$$\mathbf{v}(t) = \frac{d\mathbf{R}(t)}{dt}, \qquad \boldsymbol{\omega}(t) = -\frac{1}{2}\left(\frac{d\mathbf{P}(t)}{dt}\cdot\mathbf{P}^T(t)\right)_{\times}, \tag{5}$$

where “$(\,)_\times$” denotes the vector invariant of a tensor.

Let us consider a body-point whose inertia tensors are the spherical part of tensors and the kinetic energy has the form

$$K = m_*\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \hat{B}\,\mathbf{v}\cdot\boldsymbol{\omega} + \frac{1}{2}\hat{J}\,\boldsymbol{\omega}\cdot\boldsymbol{\omega}\right). \tag{6}$$

Here m_* is the mass of a body-point, $\hat{B}$ and $\hat{J}$ are the moments of inertia. The momentum and the proper angular momentum of a body-point are

$$\mathbf{K}_1 = m_*\,(\mathbf{v} + \hat{B}\,\boldsymbol{\omega}), \qquad \mathbf{K}_2 = m_*\,(\hat{B}\,\mathbf{v} + \hat{J}\,\boldsymbol{\omega}). \tag{7}$$

In the moment theories of continua (such as the rod theory, the shell theory, the 3D theory of elasticity), the elementary volume of a continuum is considered to be infinitesimal rigid body. Thus, inertia tensors in the continuum mechanics have the same structure as the inertia tensors of macroscopic rigid bodies. It is important to note that the body-points (6) and (7) differ from the infinitesimal rigid body by the additional parameter $\hat{B}$ which equals to zero in the case of rigid body. For the first time, the body-points (6) and (7) have been introduced by Zhilin, see [16], [17], [18], [19]. A substantiation of the model can be found in [20].

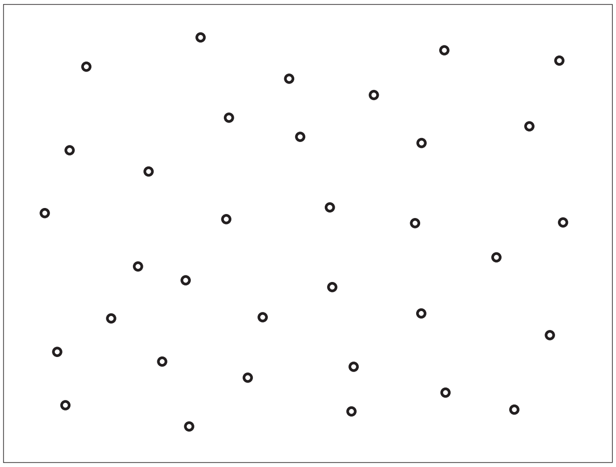

Fig. 1 Elementary volume of continuum consisting of body-points

The material medium (see Fig. 1) consisting of body-points (6) and (7) is considered. Now we formulate the basic equations of the linear theory of the continuum. We assume that in the reference configuration, the tensor $\mathbf{P}(\mathbf{r},t)$ (rotation tensor of body-points) is equal to the unit tensor. Therefore, upon the linearization near the reference position, it takes the form

$$\mathbf{P}(\mathbf{r},t) = \mathbf{E} + \boldsymbol{\theta}(\mathbf{r},t) \times \mathbf{E}, \tag{8}$$

where $\boldsymbol{\theta}(\mathbf{r},t)$ is the rotation vector field of body-points. Kinematic relations in the linear approximation are

$$\mathbf{v} = \frac{d\mathbf{u}}{dt}, \qquad \boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt}. \tag{9}$$

Here $\mathbf{u}(\mathbf{r},t)$ is the displacement vector field of body-points.

The mass balance equation has the form

$$\frac{d\hat{\rho}}{dt} + \hat{\rho}\,\nabla\cdot\mathbf{v} = 0, \tag{10}$$

where $\hat{\rho}$ is the mass density in the actual configuration. Solving Eq. (10), we obtain relation between the mass density in the actual configuration and the volume strain $\nabla\cdot\mathbf{u}$:

$$\hat{\rho} = \tilde{\rho}\,(1 - \nabla\cdot\mathbf{u}), \tag{11}$$

where $\tilde{\rho}$ is the mass density in the reference configuration.

The equations of motion of the material continuum are

$$\nabla\cdot\boldsymbol{\tau} + \tilde{\rho}\mathbf{f} = \tilde{\rho}\frac{d}{dt}\big(\mathbf{v} + \hat{B}\boldsymbol{\omega}\big), \qquad \nabla\cdot\mathbf{T} + \boldsymbol{\tau}_\times + \tilde{\rho}\mathbf{L} = \tilde{\rho}\frac{d}{dt}\big(\hat{B}\mathbf{v} + \hat{J}\boldsymbol{\omega}\big). \tag{12}$$

Here $\boldsymbol{\tau}$ and $\mathbf{T}$ are the stress tensor and the moment stress tensor, respectively, $\mathbf{f}$ is the mass density of external forces, $\mathbf{L}$ is the mass density of external moments.

The equation of energy balance is

$$\frac{d}{dt}\left(\tilde{\rho}U_m\right) = \boldsymbol{\tau}^T\cdot\cdot\frac{d\boldsymbol{\varepsilon}}{dt} + \mathbf{T}^T\cdot\cdot\frac{d\boldsymbol{\vartheta}}{dt} + \nabla\cdot\mathbf{H} + \tilde{\rho}Q. \tag{13}$$

Here the symbol “$\cdot\cdot$” has the following sense: $\mathbf{ab}\cdot\cdot\,\mathbf{cd} = (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})$ and is called double scalar product; U_m is the internal energy density per unit mass; $\boldsymbol{\varepsilon}$ and $\boldsymbol{\vartheta}$ are the strain tensors; Q is the rate of the energy supply in volume; $\mathbf{H}$ is the energy flux vector. The strain tensors are determined by the formulas

$$\boldsymbol{\varepsilon} = \nabla\mathbf{u} + \mathbf{E}\times\boldsymbol{\theta}, \qquad \boldsymbol{\vartheta} = \nabla\boldsymbol{\theta}. \tag{14}$$

If the supply of energy of “non-mechanical nature” is ignored, i. e., the body is considered to be isolated, then Eq. (13) takes a more simple form:

$$\frac{d}{dt}\left(\tilde{\rho}U_m\right) = \boldsymbol{\tau}^T \cdot\cdot \frac{d\boldsymbol{\varepsilon}}{dt} + \mathbf{T}^T \cdot\cdot \frac{d\boldsymbol{\vartheta}}{dt}. \tag{15}$$

In what follows, we consider only isolated elastic bodies. For the elastic material, the Cauchy–Green relations follow from the energy balance equation (15):

$$\boldsymbol{\tau} = \frac{\partial(\tilde{\rho}U_m)}{\partial\boldsymbol{\varepsilon}}, \qquad \mathbf{T} = \frac{\partial(\tilde{\rho}U_m)}{\partial\boldsymbol{\vartheta}}. \tag{16}$$

We represent the density of internal energy in the form:

$$\tilde{\rho}U_m = \boldsymbol{\tau}_0 \cdot\cdot \boldsymbol{\varepsilon} + \mathbf{T}_0 \cdot\cdot \boldsymbol{\vartheta} + \frac{1}{2}\boldsymbol{\varepsilon} \cdot\cdot {}^4\tilde{\mathbf{C}}_1 \cdot\cdot \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \cdot\cdot {}^4\tilde{\mathbf{C}}_2 \cdot\cdot \boldsymbol{\vartheta} + \frac{1}{2}\boldsymbol{\vartheta} \cdot\cdot {}^4\tilde{\mathbf{C}}_3 \cdot\cdot \boldsymbol{\vartheta}. \tag{17}$$

The coefficients of the quadratic form (17) are called the stiffness tensors; the coefficients $\boldsymbol{\tau}_0$ and $\mathbf{T}_0$ are the initial stresses. After substituting the expression for the density of internal energy (17) into the Cauchy–Green relations (16), we obtain the constitutive equations:

$$\boldsymbol{\tau}^T = \boldsymbol{\tau}_0^T + {}^4\tilde{\mathbf{C}}_1 \cdot\cdot \boldsymbol{\varepsilon} + {}^4\tilde{\mathbf{C}}_2 \cdot\cdot \boldsymbol{\vartheta}, \qquad \mathbf{T}^T = \mathbf{T}_0^T + \boldsymbol{\varepsilon} \cdot\cdot {}^4\tilde{\mathbf{C}}_2 + {}^4\tilde{\mathbf{C}}_3 \cdot\cdot \boldsymbol{\vartheta}. \tag{18}$$

The moment theory of the elastic continuum is formulated above. The equations of a moment continuum and the method of derivation of these equations are well known. The only difference between the proposed model and the known one is in the fact that the inertia properties of the continuum under consideration are characterized by the additional parameter $\hat{B}$.

3 Linear model of the “thermal ether.” Interaction of a body-point with the “thermal ether”

Accepting three important hypotheses, we consider a special case of the theory stated above.

Hypothesis 1 There are no the external forces and the force interaction between the particles of the medium:

$$\mathbf{f} \equiv \mathbf{0}, \qquad \boldsymbol{\tau} \equiv \mathbf{0}. \tag{19}$$

Hypothesis 2 The moment stress tensor $\mathbf{T}$ is the spherical part of tensor:

$$\mathbf{T} = T\mathbf{E}. \tag{20}$$

Hypothesis 3 The external moments and the initial moment stresses are absent:

$$\mathbf{L} \equiv \mathbf{0}, \qquad \mathbf{T}_0 \equiv \mathbf{0}. \tag{21}$$

We will call the model of elastic continuum satisfying the hypotheses (19)–(21) the “thermal ether.” We note two important properties of the medium under consideration. First, if a body is positioned in the medium then the medium influences upon the body by moments and does not influence by forces. Second, a body in the medium dissipates energy into the medium due to the moment interactions.

In view of assumptions (19)–(21), the equations of motion (12) take the form:

$$\tilde{\rho}\frac{d}{dt}\big(\mathbf{v} + \hat{B}\boldsymbol{\omega}\big) = \mathbf{0}, \qquad \nabla T = \tilde{\rho}\frac{d}{dt}\big(\hat{B}\mathbf{v} + \hat{J}\boldsymbol{\omega}\big). \tag{22}$$

In view of assumption (20), the last term on the right-hand side of the energy balance equation (15) can be reduced as follows:

$$\mathbf{T}^T \cdot\cdot \frac{d\boldsymbol{\vartheta}}{dt} = T\mathbf{E} \cdot\cdot \frac{d\boldsymbol{\vartheta}}{dt} = T\,\frac{d(\mathbf{E} \cdot\cdot \boldsymbol{\vartheta})}{dt} = T\,\frac{d(\mathrm{tr}\,\boldsymbol{\vartheta})}{dt}. \tag{23}$$

By using the notation

$$\vartheta = \mathrm{tr}\,\boldsymbol{\vartheta} \equiv \nabla \cdot \boldsymbol{\theta} \tag{24}$$

and Eqs. (23) and (19), the energy balance equation (15) is written as

$$\frac{d}{dt}(\tilde\rho U_m) = T\,\frac{d\vartheta}{dt}. \tag{25}$$

The material medium being elastic, we obtain the Cauchy–Green relation which is analogous to the second relation in (16) but has a simpler form

$$T = \frac{\partial(\tilde\rho U_m)}{\partial\vartheta}. \tag{26}$$

It is obvious from Eq. (25) that the density of internal energy is a function of single variable ϑ. Let us specify the density of internal energy in the simplest form

$$\tilde\rho U_m = \frac{1}{2}\tilde k\,\vartheta^2, \tag{27}$$

where $\tilde k$ is the torsional stiffness. Then, the constitutive equation takes the form

$$T = \tilde k\,\vartheta. \tag{28}$$

As follows from Eqs. (22), (9), (24), and (28), the “thermal ether” is described by the wave equation

$$\Delta\vartheta - \frac{\tilde\rho(\hat J - \hat B^2)}{\tilde k}\,\frac{d^2\vartheta}{dt^2} = 0, \tag{29}$$

and the translational and angular velocities are calculated by the formulas

$$\mathbf{v} = -\frac{\tilde k\hat B}{\tilde\rho(\hat J - \hat B^2)}\int\nabla\vartheta\,dt, \qquad \boldsymbol{\omega} = \frac{\tilde k}{\tilde\rho(\hat J - \hat B^2)}\int\nabla\vartheta\,dt. \tag{30}$$

It is obvious from the first equation in (22) that the displacement vector and the rotation vector are related with each other by

$$\mathbf{u} = -\hat B\boldsymbol{\theta} + (\mathbf{v}_0 + \hat B\boldsymbol{\omega}_0)t + \mathbf{u}_0 + \hat B\boldsymbol{\theta}_0, \tag{31}$$

where $\mathbf{v}_0$, $\boldsymbol{\omega}_0$ are the initial translational and angular velocities, $\mathbf{u}_0$, $\boldsymbol{\theta}_0$ are the initial displacement vector and the initial rotation vector, respectively.

Let us discuss the influence of the “thermal ether” on a particle imbedded in it. Further, we consider two model problems, but we start with some opening remarks. The problem of motion of an oscillator on an elastic waveguide has been solved in [21]. The statement of problem is: a semi-infinite inertial rod is connected with a point mass by means of an inertialess spring; at the initial instant of time, the point mass possessed an initial velocity and was removed from the static equilibrium position. It has been proved that the mathematical description of motion of the system consisting of the inertia and elastic elements only can be reduced to the equation of motion of a linear oscillator with viscous damping. The coefficient of damping depends on stiffness of the spring and the quantities characterizing elastic and inertia properties of the rod. The problem of interaction of a body-point with a one-dimensional semi-infinite continuum of body-points and the problem mentioned above are the same except the fact that in the former case, the energy dissipates through the rotational degrees of freedom, and the equation of motion of an oscillator contains a damping moment instead of a damping force. To prove aforesaid and determine the structure of damping moment, we consider the problem being one-dimensional analog to the problem of the interaction of a body-point with the “thermal ether.”

Now we consider a semi-infinite inertial rod (see Fig. 2), consisting of the body-points (6) and (7). The rod is connected with the analogous body-point by means of an inertialess spring working in torsion (rotation about the axis of the rod). The inertia of the rod is characterized by the moments of inertia $\hat B$, $\hat J$ and the linear density of mass $\sigma\tilde\rho$, where σ is “the area of rod section” and $\tilde\rho$ is the volume density of mass. The elastic properties of the rod are characterized by the torsional stiffness $\sigma\tilde k$, where the coefficient σ is introduced in order that stiffness $\tilde k$ possesses the dimension in 3D problem. The inertia of the body-point is characterized by the mass m and the moments of inertia B, J. The torsional stiffness of the spring connecting the body-point with the rod is equal to $\sigma k_*/r_0$, where r_0 is “the length” of the spring. The coefficients σ and r_0 are introduced

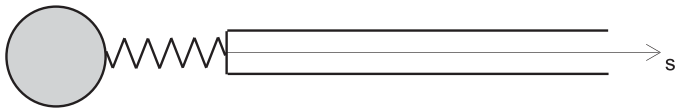

Fig. 2 Interaction of the body-point with the one-dimensional semi-infinite continuum

in order that stiffness k_* possesses the dimension like $\tilde{k}$. We suppose that the particles of the rod interact only by the moments. The force interaction of the rod particles is assumed to be zero. At the initial instant of time, the displacements and the rotation angles as well as the translational and angular velocities of the rod particles are equal to zero. The body-point possesses a non-zero initial angular velocity directed along the axis of the rod and a non-zero initial angle of rotation about the axis of the rod. It is evident that under such initial condition, the system will be in motion which is the longitudinal–torsional oscillations.

It is proved that after elimination of variables characterizing the motion of the rod, the problem is reduced to the set of equations [20]

$$m(B\ddot{y} + J\ddot{\psi}) + m\beta(B\dot{y} + J\dot{\psi}) + \frac{\sigma k_*}{r_0}\psi = m\beta(Bv_0 + J\omega_0), \qquad m(\ddot{y} + B\ddot{\psi}) = F, \tag{32}$$

where $y(t)$ is the displacement of the body-point along the axis of the rod, $\psi(t)$ is the angle of rotation of the body-point about the axis of the rod, v_0 and ω_0 are the translational and angular velocities of the body-point at the initial instant of time. The coefficient β is calculated by the formula

$$\beta = \frac{ck_*}{r_0\tilde{k}} = \frac{k_*/r_0}{\sqrt{\tilde{k}\tilde{\rho}(\hat{J} - \hat{B}^2)}}\,. \tag{33}$$

According to Eq. (32), the moment of viscous damping characterizing the radiation of energy in the surrounding medium is proportional to the angular momentum of the body-point, i. e., it depends on both the angular velocity and the translational velocity. To be exact, the moment of viscous damping is proportional to the difference between the value of angular momentum at the present moment of time and its value at the initial time, i. e., it is equal to $-\beta[m(B\dot{y} + J\dot{\psi}) - m(Bv_0 + J\omega_0)]$. On the one hand, this means that the moment of viscous damping does not depend on the choice of the inertial reference system. Indeed, if we replace $\dot{y}$ and v_0 by $\dot{y} + V_0$ and $v_0 + V_0$, respectively, then the expression $\beta[m(B\dot{y} + J\dot{\psi}) - m(Bv_0 + J\omega_0)]$ does not change. On the other hand, the expression for moment of viscous damping, as well as any constitutive equation, must be independent on the initial conditions. Therefore, using the inertial reference system in which the particles of unperturbed rod are motionless, we can assume the moment of viscous damping to be proportional to the angular momentum $m(B\dot{y} + J\dot{\psi})$. If the inertial reference system moving relative to the undisturbed rod is used, then the expression for the moment of viscous damping should be modified by replacing the absolute velocity of the particle to the particle velocity relative to the undisturbed rod. If $B = 0$, then the dependence on the translational velocity vanishes. In this case, the problem under consideration becomes similar to the problem of the motion of an ordinary oscillator on the elastic waveguide. Analysis of formula (33) for the coefficient of damping β allows us to conclude that increasing the torsional stiffness of the spring connecting the body-point and the rod causes increasing of the radiation in the surrounding medium.

The foregoing problem is the simplest model illustrating the process of dissipation of the body-point energy into the "thermal ether." The problem of the interaction of a body-point with the "thermal ether" in the case of spherical symmetry is more complicated but a more appropriate model of the process of dissipation.

Now we consider the spherical source of radius r_0 (see Fig. 3) consisting of the body-points (6) and (7). We suppose that the source can pulsate, and the change of its radius is characterized by the variable $\xi(t)$. At the same time, the body-points of the spherical source rotate about its radius. The angles of rotation of all body-points are assumed to be the same, and they are characterized by the variable $\psi(t)$. Thus, the kinematics of the spherical source is described by the displacement vector and by the rotational vector

$$\boldsymbol{\xi} = \xi(t)\,\mathbf{e}_r, \qquad \boldsymbol{\psi} = \psi(t)\,\mathbf{e}_r, \tag{34}$$

where $\mathbf{e}_r$ is the unit vector of the spherical coordinate system. The inertia properties of the spherical source are characterized by the mass m evenly distributed on the source surface and the moments of inertia B, J. The spherical source interacts with the "thermal ether" by means of an elastic connection. The elastic connection

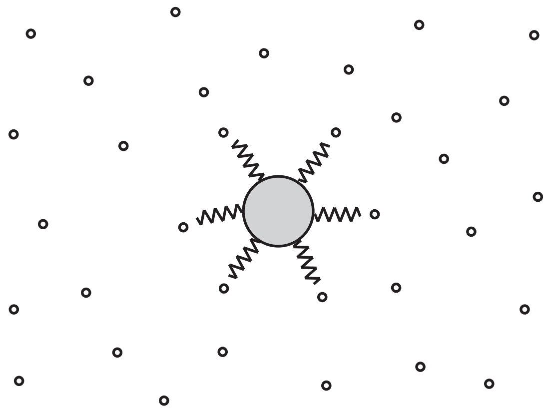

Fig. 3 Interaction of the spherical source with the “thermal ether”

constitutes the system of the identical springs working in torsion. Each of them connects the body-point of the spherical source with the body-point of the “thermal ether” (see Fig. 3). The stiffness of the connection per unit area of spherical source is characterized by the stiffness k_*/r_0 where coefficient r_0^{-1} is introduced in order to the dimension of stiffness k_* be the same as the dimension of stiffness of the “thermal ether.” At the initial instant of time, the “thermal ether” is at rest. The following initial conditions are assumed for the spherical source: $\xi(0) = \xi_0$, $\dot{\xi}(0) = v_0$, $\psi(0) = \psi_0$, $\dot{\psi}(0) = \omega_0$.

It is proved that after elimination of variables characterizing the motion of the “thermal ether,” the problem is reduced to the set of equations [22]

$$m\big(B\ddot{\xi} + J\ddot{\psi}\big) + m\beta\big(B\dot{\xi} + J\dot{\psi}\big) + \frac{mk_*}{r_0^2\tilde{\rho}(\hat{J}-\hat{B}^2)}\big(B\xi + J\psi\big) + 4\pi r_0 k_* \psi = m\beta\big(Bv_0 + J\omega_0\big) + \\ + \frac{mk_*}{r_0^2\tilde{\rho}(\hat{J}-\hat{B}^2)}\Big[(Bv_0 + J\omega_0)t + B\xi_0 + J\psi_0\Big], \qquad m\big(\ddot{\xi} + B\ddot{\psi}\big) = 4\pi r_0^2 f, \quad (35)$$

where the coefficient β is calculated by the formula:

$$\beta = \frac{ck_*}{r_0\tilde{k}} = \frac{k_*/r_0}{\sqrt{\tilde{k}\tilde{\rho}(\hat{J}-\hat{B}^2)}}\,. \quad (36)$$

Notice that Eq. (35), as well as Eq. (32) obtained for the case of the interaction of a body-point with the one-dimension continuum, does not depend on the choice of the inertial reference system. A comparison of Eq. (35) with Eq. (32) shows that although these equations somewhat differ from each other, nevertheless, they have one important similarity. Both of them have the dissipative terms proportional to the angular momentum, and the dependence of coefficients of viscous damping on the parameters of models is the same in both cases, see Eqs. (33) and (36). Thus, the analysis of solutions of two model problems gives reason to believe that in the general case, the moment of viscous damping characterizing the radiation of energy of a particle into the “thermal ether” is proportional to the vector of angular momentum of the particle providing that the inertial reference system fixed relative to the undisturbed “thermal ether” is used. This result is important for the subsequent constructions.

4 Continuum of one-rotor gyrostats

We construct the linear theory of the elastic medium which is a continuum of one-rotor gyrostats. The one-rotor gyrostat is a complex particle consisting of the carrier body and the rotor (see Fig. 4). The rotor can rotate independently of rotation of the carrier body, but it cannot translate relative to the carrier body. The carrier body of the gyrostat is a classical body-point whose inertial properties are analogous to that of a rigid body. The rotor of the gyrostat is the body-point (6) and (7).

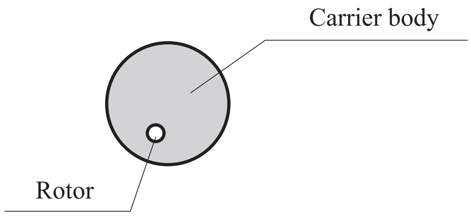

Fig. 4 One-rotor gyrostat

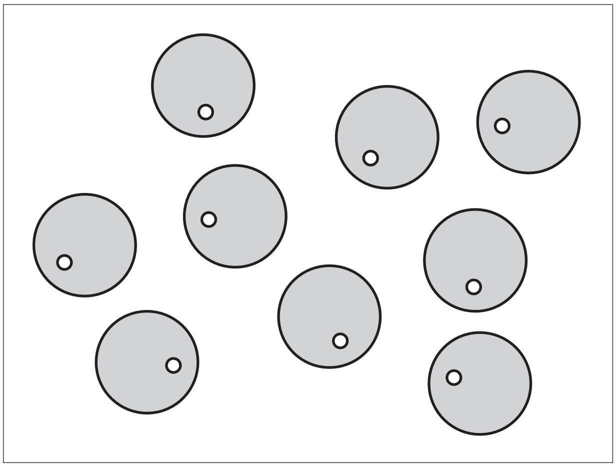

Fig. 5 Elementary volume of a medium consisting of one-rotor gyrostats

Now we consider the material medium consisting of one-rotor gyrostats (see Fig. 5). Let vector $\mathbf{r}$ determine the position of some point of space. We introduce the following notations: $\rho(\mathbf{r}, t)$ is the mass density of the material medium in a given point of space, $\mathbf{v}(\mathbf{r}, t)$ is the velocity field, $\mathbf{u}(\mathbf{r}, t)$ is the displacement field, $\tilde{\mathbf{P}}(\mathbf{r}, t)$, $\tilde{\boldsymbol{\omega}}(\mathbf{r}, t)$ are the fields of the rotation tensors and the angular velocity vectors of the carrier bodies, $\mathbf{P}(\mathbf{r}, t)$, $\boldsymbol{\omega}(\mathbf{r}, t)$ are the fields of the rotation tensors and the angular velocity vectors of the rotors. In the reference configuration, the tensors $\tilde{\mathbf{P}}(\mathbf{r}, t)$ and $\mathbf{P}(\mathbf{r}, t)$ are assumed to be equal to the unit tensor. Hence, upon the linearization near the reference position, they take the form

$$\tilde{\mathbf{P}}(\mathbf{r}, t) = \mathbf{E} + \boldsymbol{\varphi}(\mathbf{r}, t) \times \mathbf{E}, \qquad \mathbf{P}(\mathbf{r}, t) = \mathbf{E} + \boldsymbol{\theta}(\mathbf{r}, t) \times \mathbf{E}, \tag{37}$$

where $\boldsymbol{\varphi}(\mathbf{r}, t)$, $\boldsymbol{\theta}(\mathbf{r}, t)$ are the rotation vector fields of the carrier bodies and rotors, respectively. Kinematic relations in the linear approximation are

$$\mathbf{v} = \frac{d\mathbf{u}}{dt}, \qquad \tilde{\boldsymbol{\omega}} = \frac{d\boldsymbol{\varphi}}{dt}, \qquad \boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt}. \tag{38}$$

The mass balance equation has the form analogous to (10), and the solution of the mass balance equation is represented by the formula analogous to (11).

The equations of balance of linear momentum for the gyrostats and of angular momentum for the carrier bodies of gyrostats have the form

$$\nabla \cdot \boldsymbol{\tau} + \rho_* \mathbf{f} = \rho_* \frac{d}{dt}\big(\mathbf{v} + B\boldsymbol{\omega}\big), \qquad \nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_\times + \rho_* \mathbf{m} = \rho_* \frac{d}{dt}\big(\mathbf{I}_0 \cdot \tilde{\boldsymbol{\omega}}\big). \tag{39}$$

Here $\boldsymbol{\tau}$ is the stress tensor, $\boldsymbol{\mu}$ is the moment stress tensor characterizing the interaction of the carrier bodies of gyrostats, $\mathbf{f}$ is the mass density of external forces, $\mathbf{m}$ is the mass density of external moments acting on the carrier bodies of gyrostats, ρ_* is the mass density of the material in the reference configuration, $\mathbf{I}_0$ is the mass density of the inertia tensors of carrier bodies in the reference configuration.

The equation of balance of angular momentum for the rotors of gyrostats has the form

$$\nabla \cdot \mathbf{T} + \rho_* \mathbf{L} = \rho_* \frac{d}{dt}\big(B\mathbf{v} + J\boldsymbol{\omega}\big), \tag{40}$$

where $\mathbf{T}$ is the moment stress tensor characterizing the interaction of the rotors of gyrostats, $\mathbf{L}$ is the mass density of the external moments acting on the rotors. Quantities B and J in Eqs. (39) and (40) represent the mass densities of the moments of inertia of the rotors. In the expression for the kinetic energy of rotor, the moment of inertia B is the coefficient of the production of linear and angular velocities, and the moment of inertia J is the coefficient of the squared angular velocity.

The constitutive equations are

$$\begin{aligned}\boldsymbol{\tau}^T &= \boldsymbol{\tau}_0^T + {}^4\mathbf{C}_1 \cdot\cdot\, \boldsymbol{\varepsilon} + {}^4\mathbf{C}_2 \cdot\cdot\, \boldsymbol{\kappa} + {}^4\mathbf{C}_4 \cdot\cdot\, \boldsymbol{\vartheta},\\ \boldsymbol{\mu}^T &= \boldsymbol{\mu}_0^T + \boldsymbol{\varepsilon} \cdot\cdot\, {}^4\mathbf{C}_2 + {}^4\mathbf{C}_3 \cdot\cdot\, \boldsymbol{\kappa} + {}^4\mathbf{C}_5 \cdot\cdot\, \boldsymbol{\vartheta},\\ \mathbf{T}^T &= \mathbf{T}_0^T + \boldsymbol{\varepsilon} \cdot\cdot\, {}^4\mathbf{C}_4 + \boldsymbol{\kappa} \cdot\cdot\, {}^4\mathbf{C}_5 + {}^4\mathbf{C}_6 \cdot\cdot\, \boldsymbol{\vartheta}.\end{aligned} \tag{41}$$

Here $\boldsymbol{\tau}_0$, $\boldsymbol{\mu}_0$, $\mathbf{T}_0$ are the initial stresses, ${}^4\mathbf{C}_k$ are the stiffness tensors, $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$, $\boldsymbol{\vartheta}$ are the strain tensors. The strain tensors are determined by the formulas

$$\boldsymbol{\varepsilon} = \nabla \mathbf{u} + \mathbf{E} \times \boldsymbol{\varphi}, \qquad \boldsymbol{\kappa} = \nabla \boldsymbol{\varphi}, \qquad \boldsymbol{\vartheta} = \nabla \boldsymbol{\theta}. \tag{42}$$

The basic equations describing the dynamics of the elastic continuum of one-rotor gyrostats are stated above — see Eqs. (38)–(42). The detailed derivation of these equations can be found in [13].

5 Continuum of one-rotor gyrostats as a model of medium with thermoviscoelastic properties

A two-component medium is presented in Fig. 6. One component of this medium is a continuum of one-rotor gyrostats and other component is a continuum called the "thermal ether." We are not going to study the interaction of media constituting the two-component continuum. In what follows, we consider the gyrostats continuum as an object under study. The "thermal ether" positioned in space between gyrostats is an external factor with respect to continuum under study. We will model the influence of the "thermal ether" on the gyrostats by an external moment in the equation of the rotors motion (40).

Accepting three important hypotheses, we consider a special case of the linear theory of one-rotor gyrostat continuum (38)–(42).

Hypothesis 1 Vector $\mathbf{L}$ (the mass density of external actions on the rotors of gyrostats) is a sum of the moment $\mathbf{L}_h$ characterizing the external actions of all sorts and the moment of linear viscous damping

$$\mathbf{L}_f = -\beta(B\mathbf{v} + J\boldsymbol{\omega}). \tag{43}$$

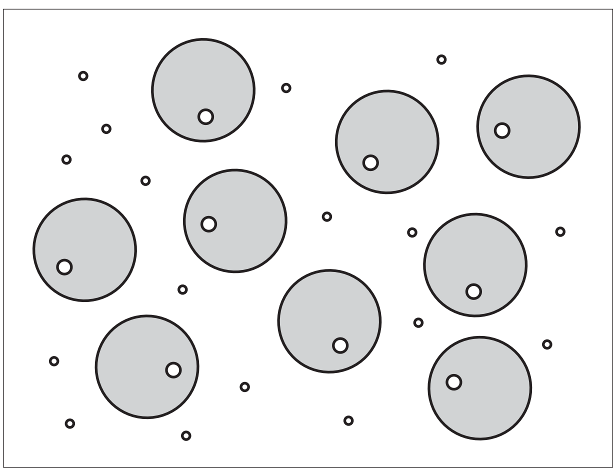

Fig. 6 Elementary volume of one-rotor gyrostat continuum deep in the "thermal ether"

The moment (43) characterizes the influence of the "thermal ether." Structure of the moment is chosen based on the analysis of two model problems considered above — see Eqs. (32) and (35). In accordance with the results of solution of the aforesaid problems, the velocity $\mathbf{v}$ in Eq. (43) is the absolute velocity of a particle in the inertial reference system fixed relative to the "thermal ether." The model of thermoviscoelastic medium is constructed in the reference system fixed relative to the Earth. Consequently, accepting Eq. (43), we suppose that the "thermal ether" does not move relative to the Earth. This is a physical hypothesis which has no relation to the choice of the inertial reference system. Rejecting this hypothesis, we have to solve more complicated model problems, and as a result, we will obtain a different expression for the moment of viscous damping. This study is beyond the scope of this paper. In what follows, we use the moment of viscous damping (43). It is important to keep in mind that Eq. (43) is true only in the reference system related with the Earth. If one wants to use the reference system moving at a constant velocity $-\mathbf{V}_0$ relative to the Earth, then Eq. (43) should be replaced by the expression

$$\mathbf{L}_f = -\beta\big[B(\mathbf{v} - \mathbf{V}_0) + J\boldsymbol{\omega}\big], \tag{44}$$

where the difference $\mathbf{v} - \mathbf{V}_0$ is the particle velocity relative to the Earth or, what is the same, relative to the "thermal ether".

Hypothesis 2 The moment interaction between the carrier bodies of gyrostats is supposed to be characterized by the antisymmetric tensor; there is no influence of external moments upon the carrier bodies of gyrostats; and the inertia tensors of the carrier bodies can be neglected

$$\boldsymbol{\mu} = -\boldsymbol{\mu}_v \times \mathbf{E}, \qquad \mathbf{m} = \mathbf{0}, \qquad \mathbf{I}_0 = \mathbf{0}. \tag{45}$$

Notice that when the moment stress tensor $\boldsymbol{\mu}$ is not equal to zero, the force stress tensor $\boldsymbol{\tau}$ can have a non-zero antisymmetric part, i. e., $\boldsymbol{\tau}_\times \neq \mathbf{0}$.

Hypothesis 3 The moment stress tensor $\mathbf{T}$ characterizing the interaction between rotors is the sum of the spherical part of tensor and the antisymmetric tensor

$$\mathbf{T} = T\mathbf{E} - \mathbf{M} \times \mathbf{E}. \tag{46}$$

Hypothesis (46) differs from supposition (20) accepted in the case of the "thermal ether" by the presence of the antisymmetric part of the moment stress tensor.

In view of assumptions (45), the motion of carrier bodies of gyrostats is described by the equations

$$\nabla \cdot \boldsymbol{\tau} + \rho_* \mathbf{f} = \rho_* \frac{d}{dt}\big(\mathbf{v} + B\boldsymbol{\omega}\big), \qquad \nabla \times \boldsymbol{\mu}_v = \boldsymbol{\tau}_\times. \tag{47}$$

Representing $\boldsymbol{\tau}$ as a sum of the symmetric and antisymmetric tensors

$$\boldsymbol{\tau} = \boldsymbol{\tau}^s - \mathbf{q} \times \mathbf{E}, \qquad \mathbf{q} = \frac{1}{2}\boldsymbol{\tau}_\times, \tag{48}$$

we rewrite Eq. (47) in the form

$$\nabla \cdot \boldsymbol{\tau}^s - \nabla \times \mathbf{q} + \rho_* \mathbf{f} = \rho_* \frac{d}{dt}\big(\mathbf{v} + B\boldsymbol{\omega}\big), \qquad \nabla \times \boldsymbol{\mu}_v = 2\mathbf{q}. \tag{49}$$

In view of assumptions (43) and (46), the equation of motion of the rotors takes the form

$$\nabla T - \nabla \times \mathbf{M} - \beta\rho_*(B\mathbf{v} + J\boldsymbol{\omega}) + \rho_* \mathbf{L}_h = \rho_* \frac{d}{dt}\big(B\mathbf{v} + J\boldsymbol{\omega}\big). \tag{50}$$

The energy balance equation for the elastic continuum of one-rotor gyrostats is written as

$$\frac{d(\rho_* U_m)}{dt} = \boldsymbol{\tau}^T \cdot\cdot \frac{d\boldsymbol{\varepsilon}}{dt} + \boldsymbol{\mu}^T \cdot\cdot \frac{d\boldsymbol{\kappa}}{dt} + \mathbf{T}^T \cdot\cdot \frac{d\boldsymbol{\vartheta}}{dt}, \tag{51}$$

where U_m is the internal energy density per unit mass; the strain tensors $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$, $\boldsymbol{\vartheta}$ are determined by formulas (42).

In view of Eq. (48), the first term on the right-hand side of Eq. (51) can be reduced as follows

$$\boldsymbol{\tau}^T \cdot\cdot \frac{d\boldsymbol{\varepsilon}}{dt} = \boldsymbol{\tau}^s \cdot\cdot \frac{d\boldsymbol{\varepsilon}}{dt} + (\mathbf{q}\times\mathbf{E}) \cdot\cdot \frac{d\boldsymbol{\varepsilon}}{dt} = \boldsymbol{\tau}^s \cdot\cdot \frac{d\boldsymbol{\varepsilon}^s}{dt} + \mathbf{q}\cdot\frac{d\boldsymbol{\gamma}}{dt}, \tag{52}$$

where the following notations are used

$$\boldsymbol{\varepsilon}^s = \frac{1}{2}\Big(\nabla\mathbf{u} + \nabla\mathbf{u}^T\Big), \qquad \boldsymbol{\gamma} = \nabla\times\mathbf{u} - 2\boldsymbol{\varphi}. \tag{53}$$

Let us note that the trace of $\boldsymbol{\varepsilon}$ is equal to the trace of $\boldsymbol{\varepsilon}^s$. That is why we will use the notation $\varepsilon = \mathrm{tr}\,\boldsymbol{\varepsilon} = \mathrm{tr}\,\boldsymbol{\varepsilon}^s$.

In view of assumption (45), the second term on the right-hand side of Eq. (51) takes the form

$$\boldsymbol{\mu}^T \cdot\cdot \frac{d\boldsymbol{\kappa}}{dt} = (\boldsymbol{\mu}_v\times\mathbf{E}) \cdot\cdot \frac{d\boldsymbol{\kappa}}{dt} = \boldsymbol{\mu}_v \cdot \frac{d\boldsymbol{\kappa}_\times}{dt}, \qquad \boldsymbol{\kappa}_\times = \nabla\times\boldsymbol{\varphi}. \tag{54}$$

We suppose that the strain vector $\boldsymbol{\kappa}_\times$, on which the moment vector $\boldsymbol{\mu}_v$ works, is equal to zero

$$\nabla\times\boldsymbol{\varphi} = 0. \tag{55}$$

However, the moment vector $\boldsymbol{\mu}_v$ has the finite value. It is possible if the corresponding stiffness tends to infinity. In that case, the constitutive equation becomes indeterminate, and vector $\boldsymbol{\mu}_v$ is found as a result of solution of Eq. (49).

In view of assumption (46), the last term on the right-hand side of Eq. (51) can be reduced as follows

$$\mathbf{T}^T \cdot\cdot \frac{d\boldsymbol{\vartheta}}{dt} = T\mathbf{E} \cdot\cdot \frac{d\boldsymbol{\vartheta}}{dt} + (\mathbf{M}\times\mathbf{E}) \cdot\cdot \frac{d\boldsymbol{\vartheta}}{dt} = T\,\frac{d(\mathrm{tr}\,\boldsymbol{\vartheta})}{dt} + \mathbf{M}\cdot\frac{d\boldsymbol{\vartheta}_\times}{dt}. \tag{56}$$

Using the results of transformations (52), (54), and (56) and taking into account assumption (55), we write down the energy balance equation (51) in the form

$$\frac{d(\rho_* U_m)}{dt} = \boldsymbol{\tau}^s \cdot\cdot \frac{d\boldsymbol{\varepsilon}^s}{dt} + \mathbf{q}\cdot\frac{d\boldsymbol{\gamma}}{dt} + T\,\frac{d\vartheta}{dt} + \mathbf{M}\cdot\frac{d\boldsymbol{\psi}}{dt}\,, \tag{57}$$

where the following notations are used

$$\vartheta = \mathrm{tr}\,\boldsymbol{\vartheta}, \qquad \boldsymbol{\psi} = \boldsymbol{\vartheta}_\times, \qquad \boldsymbol{\vartheta} = \nabla\boldsymbol{\theta}. \tag{58}$$

In view of elasticity of the medium under consideration, we obtain the Cauchy–Green relations

$$\boldsymbol{\tau}^s = \frac{\partial(\rho_* U_m)}{\partial\boldsymbol{\varepsilon}^s}, \qquad \mathbf{q} = \frac{\partial(\rho_* U_m)}{\partial\boldsymbol{\gamma}}, \qquad T = \frac{\partial(\rho_* U_m)}{\partial\vartheta}, \qquad \mathbf{M} = \frac{\partial(\rho_* U_m)}{\partial\boldsymbol{\psi}}. \tag{59}$$

According to the energy balance equation (57), the energy density is the function of four independent variables: $\boldsymbol{\varepsilon}^s$, $\boldsymbol{\gamma}$, ϑ, and $\boldsymbol{\psi}$. Let us define the energy density as

$$\begin{aligned}\rho_* U_m = \boldsymbol{\tau}_0 \cdot\cdot \boldsymbol{\varepsilon}^s + \mathbf{q}_0\cdot\boldsymbol{\gamma} + T_*\vartheta + \mathbf{M}_*\cdot\boldsymbol{\psi} + G\,\mathrm{dev}\,\boldsymbol{\varepsilon}^s \cdot\cdot \mathrm{dev}\,\boldsymbol{\varepsilon}^s +& \\ + \frac{1}{2}K_{ad}\,\varepsilon^2 + \varUpsilon\,\varepsilon\,\vartheta + \frac{1}{2}K\vartheta^2 + \frac{1}{2}\,A\,\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} + D\,\boldsymbol{\gamma}\cdot\boldsymbol{\psi} + \frac{1}{2}\,\varGamma\,\boldsymbol{\psi}\cdot\boldsymbol{\psi}\,.&\end{aligned} \tag{60}$$

Here $\boldsymbol{\tau}_0$, $\mathbf{q}_0$, T_*, $\mathbf{M}_*$ are the initial stresses, K_{ad} is the adiabatic modulus of compression, G is the shear modulus, $\varUpsilon$, K, A, D and $\varGamma$ are constants whose physical meaning will be discussed further. The notation “dev” is used for the deviator part of tensor.

Substituting Eq. (60) into the Cauchy–Green relations (59), we obtain

$$\begin{aligned}\boldsymbol{\tau}^s = \boldsymbol{\tau}_0 + K_{ad}\,\varepsilon\mathbf{E} + 2G\,\mathrm{dev}\,\boldsymbol{\varepsilon} + \varUpsilon\vartheta\mathbf{E}, \qquad \mathbf{q} = \mathbf{q}_0 + A\,\boldsymbol{\gamma} + D\,\boldsymbol{\psi}, \\ T = T_* + \varUpsilon\,\varepsilon + K\vartheta, \qquad \mathbf{M} = \mathbf{M}_* + D\,\boldsymbol{\gamma} + \varGamma\,\boldsymbol{\psi}\,.\end{aligned} \tag{61}$$

Further, we show that the one-rotor gyrostat continuum whose dynamics is described by Eqs. (38), (49), (50), (53), (55), (58), and (61) can be interpreted as a model of thermoviscoelastic medium.

6 Thermodynamic analogy

6.1 Temperature and entropy

Suppose that the model constructed above describes behavior of the classical medium which possesses not only elastic properties but also the viscous and thermic properties. Now for a simplicity sake, we assume that $\mathbf{q} = \mathbf{0}$ and $\mathbf{M} = \mathbf{0}$. In this case, the energy balance equation (57) takes the form

$$\frac{d(\rho_* U_m)}{dt} = \boldsymbol{\tau}^s \cdot\cdot \frac{d\boldsymbol{\varepsilon}^s}{dt} + T\,\frac{d\vartheta}{dt}. \tag{62}$$

If we consider Eq. (62) to be the energy balance equation for the classical medium, then we interpret the last term on the right-hand side of this equation as thermodynamical one. In this case, the quantity T acquires meaning of temperature and the quantity ϑ acquires meaning of volume density of entropy.

Dimensions of the temperature and the entropy introduced in the framework of the proposed model are different from those in classical thermodynamics. This problem can be solved by introduction of a normalization factor a

$$T = aT_a, \qquad \vartheta = \frac{1}{a}\,\vartheta_a. \tag{63}$$

Here T_a is the absolute temperature measured by a thermometer, ϑ_a is volume density of the absolute entropy. Let us introduce the similar relations for the remaining variables

$$\boldsymbol{\theta} = \frac{1}{a}\,\boldsymbol{\theta}_a, \qquad \boldsymbol{\omega} = \frac{1}{a}\,\boldsymbol{\omega}_a, \qquad \mathbf{M} = a\mathbf{M}_a, \qquad \boldsymbol{\psi} = \frac{1}{a}\,\boldsymbol{\psi}_a, \qquad \mathbf{L}_h = a\mathbf{L}_h^a. \tag{64}$$

By introducing new parameters

$$B_a = \frac{B}{a}, \quad J_a = \frac{J}{a^2}, \quad \varUpsilon_a = \frac{\varUpsilon}{a}, \quad K_a = \frac{K}{a^2}, \quad D_a = \frac{D}{a}, \quad \varGamma_a = \frac{\varGamma}{a^2}, \tag{65}$$

the normalization factor a can be eliminated from all equations.

6.2 Hyperbolic type thermoelasticity and classical thermoelasticity

Now using the notations (63)–(65), we resume the set of equations describing the dynamics of the one-rotor gyrostat continuum.

According to the accepted suppositions, the stress tensor and the moment stress tensor have the form

$$\boldsymbol{\tau} = \boldsymbol{\tau}^s - \mathbf{q} \times \mathbf{E}, \qquad \boldsymbol{\mu} = -\boldsymbol{\mu}_v \times \mathbf{E}, \qquad \mathbf{T}_a = T_a\,\mathbf{E} - \mathbf{M}_a \times \mathbf{E}. \tag{66}$$

It is well known that applying the linear theory is admissible in certain range of temperatures and entropy densities. That is why we introduce deviations of the quantities introduced above from their reference values T_a^*, $\mathbf{M}_a^*$ (which are not zero) and ϑ_a^*, $\boldsymbol{\psi}_a^*$ (which can be considered to be zero without loss of generality):

$$T_a = T_a^* + \tilde{T}_a, \qquad \mathbf{M}_a = \mathbf{M}_a^* + \tilde{\mathbf{M}}_a, \qquad \vartheta_a = \vartheta_a^* + \tilde{\vartheta}_a, \qquad \boldsymbol{\psi}_a = \boldsymbol{\psi}_a^* + \tilde{\boldsymbol{\psi}}_a. \tag{67}$$

In view of Eq. (67), the equations of motion (49) and (50) are rewritten as

$$\begin{gathered}\nabla \cdot \boldsymbol{\tau}^s - \nabla \times \mathbf{q} + \rho_*\mathbf{f} = \rho_*\frac{d}{dt}\big(\mathbf{v} + B_a\boldsymbol{\omega}_a\big), \qquad \nabla \times \boldsymbol{\mu}_v = 2\mathbf{q},\\ \nabla\tilde{T}_a - \nabla \times \tilde{\mathbf{M}}_a - \rho_*\beta\,(B_a\mathbf{v} + J_a\boldsymbol{\omega}_a) + \rho_*\mathbf{L}_h^a = \rho_*\frac{d}{dt}\big(B_a\mathbf{v} + J_a\boldsymbol{\omega}_a\big).\end{gathered} \tag{68}$$

Kinematical and geometrical relations (38), (53), (55), and (58) take the form

$$\begin{gathered}\mathbf{v} = \frac{d\mathbf{u}}{dt}, \qquad \boldsymbol{\omega}_a = \frac{d\boldsymbol{\theta}_a}{dt}, \qquad \boldsymbol{\varepsilon}^s = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \qquad \varepsilon = \operatorname{tr}\boldsymbol{\varepsilon}^s,\\ \boldsymbol{\gamma} = \nabla \times \mathbf{u} - 2\boldsymbol{\varphi}, \qquad \nabla \times \boldsymbol{\varphi} = 0, \qquad \vartheta_a = \nabla \cdot \boldsymbol{\theta}_a, \qquad \boldsymbol{\psi}_a = \nabla \times \boldsymbol{\theta}_a.\end{gathered} \tag{69}$$

In view of Eq. (67) and the simplifying assumptions $\boldsymbol{\tau}_0 = \mathbf{0}$ and $\mathbf{q}_0 = \mathbf{0}$, the constitutive equations (61) are reduced to the form

$$\begin{gathered} \boldsymbol{\tau}^s = \Bigl(K_{ad} - \frac{2}{3}\,G\Bigr)\varepsilon\mathbf{E} + 2\,G\,\boldsymbol{\varepsilon}^s + \varUpsilon_a\,\tilde{\vartheta}_a\,\mathbf{E}, \qquad \mathbf{q} = A\,\boldsymbol{\gamma} + D_a\,\tilde{\boldsymbol{\psi}}_a, \\ \tilde{T}_a = \varUpsilon_a\,\varepsilon + K_a\,\tilde{\vartheta}_a, \qquad \tilde{\mathbf{M}}_a = D_a\,\boldsymbol{\gamma} + \varGamma_a\,\tilde{\boldsymbol{\psi}}_a. \end{gathered} \tag{70}$$

The mass density in the actual configuration is determined by the formula: $\rho = \rho_*\,(1 - \varepsilon)$.

Now we consider a special case when the parameters A, D_a, $\varGamma_a$, B_a are equal to zero, and the remaining parameters are calculated by the formulas

$$\beta J_a = \frac{T_a^*}{\rho_* \lambda}, \qquad K_a = \frac{T_a^*}{\rho_* c_v}, \qquad \varUpsilon_a = -\frac{\alpha K_{iz} T_a^*}{\rho_* c_v}, \tag{71}$$

where c_v is the specific heat at constant volume, λ is the heat conduction coefficient, K_{iz} is the isothermal modulus of compression, α is the volume coefficient of thermal expansion,

$$K_{ad} = K_{iz}\frac{c_p}{c_v}, \qquad c_p - c_v = \frac{\alpha^2 K_{iz} T_a^*}{\rho_*} \qquad \Rightarrow \qquad K_{ad} = K_{iz} + \frac{\alpha^2 K_{iz}^2 T_a^*}{\rho_* c_v}, \tag{72}$$

where c_p is the specific heat at constant pressure.

In [13], it is shown that the special case under consideration is described by the set of equations

$$\begin{gathered} \nabla\cdot\boldsymbol{\tau}^s + \rho_*\mathbf{f} = \rho_*\frac{d^2\mathbf{u}}{dt^2}, \qquad \boldsymbol{\tau}^s = \Bigl(K_{iz} - \frac{2}{3}\,G\Bigr)\varepsilon\mathbf{E} + 2G\,\boldsymbol{\varepsilon}^s - \alpha K_{iz}\tilde{T}_a\,\mathbf{E}, \\ \varDelta\tilde{T}_a - \frac{\rho_* c_v}{\lambda}\left(\frac{d\tilde{T}_a}{dt} + \frac{1}{\beta}\,\frac{d^2\tilde{T}_a}{dt^2}\right) = \frac{\alpha K_{iz} T_a^*}{\lambda}\left(\frac{d\varepsilon}{dt} + \frac{1}{\beta}\,\frac{d^2\varepsilon}{dt^2}\right) - \rho_*\nabla\cdot\mathbf{L}_h^a, \\ \boldsymbol{\varepsilon}^s = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \qquad \varepsilon = \operatorname{tr}\boldsymbol{\varepsilon}^s. \end{gathered} \tag{73}$$

Equation (73) follows from Eqs. (66)–(70). The second equation in (73) is obtained from Eq. (70) by eliminating $\tilde{\vartheta}_a$. The third equation in (73) is obtained by taking the divergence of both sides of the third equation in (68) and eliminating $\nabla\cdot\boldsymbol{\omega}_a$ by means of Eq. (69). The quantity $\rho_*\nabla\cdot\mathbf{L}_h^a$ in the third equation in (73) plays role of the source term characterizing the heat energy supply.

It is easy to see that if $\beta^{-1} = 0$, then the set of equations (73) is equivalent to the classical statement of coupled problem of thermoelasticity (see, for example, [23]). If the parameter β^{-1} (which is usually called the heat flow relaxation timescale) is not equal to zero, then Eq. (73) is the statement of problem of the hyperbolic type thermoelasticity (see, for example, [24]). Two ways of determination of the parameter β^{-1} based on comparison with the phonon theory are considered in [13]. The first way is based on the data taken from [25] (p. 141). According to these data, the velocity c_r of heat wave propagation in the crystal solids is related to the velocity c_a of longitudinal acoustic wave propagation by $c_r = c_a/\sqrt{3}$. The second way is based on the data taken from [10] (p. 239). According to these data, the velocity of phonon motion is equal to the group velocity of acoustic wave in the crystal lattice: $c_r = c_a$. In [13], one can find two formulas for calculating β^{-1} in the case of solids, the methods of extrapolation of these formulas in the case of liquids and gases, and the results of calculations for a number of substances. Estimations of the heat flow relaxation timescale can also be found in [26] for metals, in [27] for some liquids, and in [24] for gases.

The asymptotic analysis of Eq. (73) shows that if the macroscopic objects and not very high frequencies are considered, then the terms containing the second time derivatives can be ignored in the heat conduction equation. In this case, the solution possesses the properties analogous to the properties of solution of the classical problem of thermoelasticity. When the acoustic wave propagation is studied, we can find the asymptotically principal term of the solution by neglecting the term $\varDelta\tilde{T}_a$ in the heat conduction equation. When the process of heat conduction is studied in order to find the asymptotically principal term of the solution, we can solve the heat conduction equation neglecting the volume expansion ε and then find the stress–strain state as a solution of the quasi-static problem where the temperature is assumed to be known. If the nano-objects and very high frequencies (lying in the gigahertz frequency range) are considered, then the terms containing the second time derivatives cannot be ignored in the heat conduction equation. In this case, the asymptotic solutions are meaningless and the coupled problem of hyperbolic type thermoelasticity (73) must be solved in the exact statement.

Usually, the heat conduction equation is obtained from the equation of energy balance. We derive the heat conduction equation from the equation of motion of rotors (50). This fact should be explained. First of all, we note that the aforesaid peculiarity of our model is a consequence of the proposed mechanical interpretation of temperature. In our model, the spherical part of moment stress tensor characterizing the interaction of rotors is analog of the absolute temperature. In accordance with the classical ideas, the absolute temperature has an energetic sense, namely the absolute temperature is proportional to the average kinetic energy of chaotic motion of molecules. Analysis of the situation leads us to the question: what is the temperature as the average kinetic energy? Is this a reflection of the physical reality or is it just a mathematical model? To answer this question, we discuss the available experimental data. First, we note that the temperature cannot be measured directly. In order to measure the temperature, we should choose a physical quantity whose change is a sign of changes in temperature and measure this physical quantity. Then, using the formula which relates the change in chosen physical quantity with the change in temperature and taking into account the fixed point of temperature scale, we should calculate the temperature. Consequently, when we measure the temperature, we do not measure the average kinetic energy of the chaotic motion of molecules. Hence, the interpretation of temperature accepted in the kinetic theory is the mathematical model and nothing more. That is why any alternative model of thermal processes whose mathematical description is reduced to the known equations can be considered. We believe that any mechanical interpretations of the temperature and other thermodynamic quantities are of interest if they can be useful for description of thermal processes within the framework of continuum mechanics and by using the methods of continuum mechanics.

6.3 Description of thermal processes near the zero temperature

Above we set out a linear theory of thermoelasticity that is correct in a certain temperature range near T_a^* provided that T_a^* is far from the zero temperature. Under these conditions, the entropy varies near the value ϑ_a^* which is not close to zero and corresponds to the temperature T_a^*. The aforesaid theory can be extended over a more wide temperature range if the linear constitutive equations (70) will be replaced by nonlinear ones. For example, in order to describe the thermal processes in a non-deformable body in the case of wide temperature range down to the zero temperature, we can assign the following expression to the energy density:

$$\rho_* U_m = A_U \big[\sinh(\varkappa\,\vartheta_a)\big]^{4/3}, \tag{74}$$

where A_U and $\varkappa$ are constant. Substituting Eq. (74) into the third equation in (59) and taking into account Eq. (63), we obtain

$$T_a = A_U \frac{4\varkappa}{3}\,\big[\sinh(\varkappa\,\vartheta_a)\big]^{1/3} \cosh(\varkappa\,\vartheta_a). \tag{75}$$

From Eq. (75), it follows that

$$\big[\sinh(\varkappa\,\vartheta_a)\big]^{2/3} dT_a = A_U \frac{4\varkappa^2}{9}\,\big(1 + 4\sinh^2(\varkappa\,\vartheta_a)\big)\, d\vartheta_a. \tag{76}$$

As seen from Eqs. (75) and (76), if the absolute temperature tends to zero, then the entropy and its increment also tend to zero:

$$\vartheta_a \xrightarrow{T_a \to 0} 0, \qquad d\vartheta_a \xrightarrow{T_a \to 0} 0. \tag{77}$$

It is obvious that Eq. (77) is in agreement with the Nernst principle.

By definition the specific heat at constant volume C_v is calculated as

$$C_v = \frac{\partial U_m}{\partial T_a}. \tag{78}$$

In view of the fact that $T_a = T_a(\vartheta_a)$, the expression (78) can be rewritten in the form

$$C_v = \frac{\partial U_m}{\partial \vartheta_a}\frac{\partial \vartheta_a}{\partial T_a} = \frac{T_a}{\rho_*}\left(\frac{\partial T_a}{\partial \vartheta_a}\right)^{-1}. \tag{79}$$

Substituting Eq. (75) into Eq. (79), we obtain

$$C_v = \frac{3\sinh(\varkappa\vartheta_a)\cosh(\varkappa\vartheta_a)}{\rho_*\varkappa\left[\cosh^2(\varkappa\vartheta_a)+3\sinh^2(\varkappa\vartheta_a)\right]}. \tag{80}$$

It is easy to show that in the case of ϑ_a close to zero, Eqs. (75) and (80) take the following approximate form

$$T_a \approx A_U\frac{4\sqrt[3]{\varkappa^4\vartheta_a}}{3}, \qquad C_v \approx \frac{3\vartheta_a}{\rho_*}. \tag{81}$$

According to Eq. (81) for temperatures near absolute zero, we have

$$C_v \approx \frac{81\,T_a^3}{64\,\rho_*\varkappa^4 A_U^3}. \tag{82}$$

For large values of ϑ_a, the expressions (75) and (80) can be approximated by the formulas

$$T_a \approx A_U\frac{\sqrt[3]{4}\,\varkappa}{3}\exp\left(\frac{4\varkappa\vartheta_a}{3}\right), \qquad C_v \approx \frac{3}{4\,\rho_*\varkappa}. \tag{83}$$

Consequently, at temperatures close to absolute zero, the specific heat C_v is proportional to T_a^3, whereas at sufficiently high temperatures, C_v is constant. This fact gives reason to assert that the expression for the specific heat (80) is in agreement with the Debye law.

7 Model of internal damping

7.1 Classical models of viscous damping

Now we put a stop to discussion of the proposed model and consider the motion of a viscous fluid in which the pressure obeys the Stokes law

$$p = \eta_v^{cl}\,\frac{d\varepsilon}{dt}, \tag{84}$$

where η_v^{cl} is the volume (acoustic) viscosity whose values for the different substances can be found, for example, in [28]. By using the constitutive Eq. (84), we obtain the well-known self-diffusion equation

$$\eta_v^{cl}\,\Delta\varepsilon - \rho_*\frac{d\varepsilon}{dt} = -\rho_*\Psi, \tag{85}$$

where $\rho_*\Psi$ is the source term.

Let us consider the equations of motion of an incompressible Newtonian viscous fluid

$$\nabla\cdot\boldsymbol{\tau} + \rho_*\mathbf{f} = \rho_*\frac{d\mathbf{v}}{dt}, \qquad \operatorname{dev}\boldsymbol{\tau} = \eta_s^{cl}\operatorname{dev}\left(\nabla\mathbf{v}+\nabla\mathbf{v}^T\right), \qquad \varepsilon = 0. \tag{86}$$

Here η_s^{cl} is the shear viscosity whose values for the different substances can be found in almost every handbook (see, for example, [29]). The notation “dev” is used for the deviator part of tensor. In view of the second equation in (86) and the assumption of potentiality of external forces by taking the curl operator of both sides of the first equation in (86), we obtain the following equation of vortex motion of a viscous fluid

$$\eta_s^{cl}\Delta\nabla\times\mathbf{v} = \rho_*\frac{d}{dt}\nabla\times\mathbf{v}. \tag{87}$$

Thus, the coefficients of the derivatives with respect to spatial coordinates in Eqs. (85) and (87) represent, respectively, the coefficients of volume viscosity and shear viscosity in their classical treatment. Certainly, the use of different rheological models leads to other viscous characteristics. However, discussion of these characteristics is beyond the scope of the paper.

7.2 Mechanism of volume viscosity

Let us return to the model proposed above. Now we abandon the assumption that the parameters A, D_a, Γ_a, B_a are equal to zero. We consider the model of internal damping based on the supposition that the terms in Eqs. (66)–(70) containing parameter B_a (inertia parameter responsible for the interference of the translational and rotational motion of rotors) are concerned with the internal damping mechanism. In accordance with our hypothesis, the mechanism of internal damping consists in the following. The energy of translational motion turns into the energy of rotational motion of rotors, and the dissipation of energy on the rotational degrees of freedom is the result of interaction of the rotors with the "thermal ether."

An isentropic process is considered, i. e., the volume density of entropy is assumed to be constant

$$\vartheta_a = \vartheta_a^* = \text{const} \qquad \Rightarrow \qquad \tilde{\vartheta}_a = 0 \qquad \Rightarrow \qquad \tilde{T}_a = \Upsilon_a\, \varepsilon. \tag{88}$$

We take the divergence of both sides of the third equation in (68) and transform the obtained equation taking into account Eqs. (69) and (70). As a result, we have

$$\Delta \tilde{T}_a - \rho_* \beta \left(B_a \frac{d\varepsilon}{dt} + J_a \frac{d\tilde{\vartheta}_a}{dt} \right) - \rho_* \left(B_a \frac{d^2\varepsilon}{dt^2} + J_a \frac{d^2\tilde{\vartheta}_a}{dt^2} \right) = -\rho_* \nabla \cdot \mathbf{L}_h^a. \tag{89}$$

Reducing Eq. (89) in view of the condition of isentropy (88), we obtain

$$\Upsilon_a\, \Delta\varepsilon - \rho_* \beta B_a \frac{d\varepsilon}{dt} - \rho_* B_a \frac{d^2\varepsilon}{dt^2} = -\rho_*\, \nabla \cdot \mathbf{L}_h^a. \tag{90}$$

Equation (90) contains the dissipative term $\rho_* \beta B_a \dfrac{d\varepsilon}{dt}$ which is in no way concerned with the heat conduction phenomena. In what follows, we show that the entropy balance equation contains the non-negative term proportional to the kinetic energy of translational motion, namely the term $\dfrac{\rho_* \beta B_a^2}{4 J_a T_a} \mathbf{v} \cdot \mathbf{v}$. This term represents the entropy production due to the dissipative process, i. e., it characterizes the transfer of mechanical energy into heat. It is easy to see that the coefficient of this term is proportional to βB_a. Hence, if the parameter βB_a is equal to zero, then the internal dissipation is ignored. That is why we consider the term in Eq. (90) which contains the coefficient βB_a to be the dissipative term.

In order to clarify the physical meaning of the coefficients in Eq. (90), we compare this equation with the self-diffusion equation (85). It is easy to see that these two equations are equivalent with the only difference that the former contains the inertial term. We introduce the following notation

$$\eta_v = \frac{\Upsilon_a}{\beta B_a}. \tag{91}$$

Eliminating Υ_a from Eq. (91) by means of the third equation in (71), we get

$$\beta B_a = -\frac{\alpha K_{iz} T_a^*}{\rho_* c_v \eta_v}. \tag{92}$$

As evident from Eq. (92), parameter B_a is negative for finite positive values of the volume viscosity η_v and is equal to zero when $\eta_v \to \infty$. In view of Eqs. (72), (91) and (92), we rewrite Eq. (90) in the form

$$\eta_v\, \Delta\varepsilon - \rho_* \frac{d\varepsilon}{dt} - \beta^{-1} \rho_* \frac{d^2\varepsilon}{dt^2} = \rho_* \Psi_v, \qquad \Psi_v = \frac{\alpha\, c_v \eta_v}{c_p - c_v} \nabla \cdot \mathbf{L}_h^a. \tag{93}$$

It is evident that the parameter η_v has the sense of volume viscosity. Since this parameter is the coefficient in Eq. (93) describing the isentropic process, it will be called the isentropic volume viscosity.

Now we consider an alternative approach to derivation of the self-diffusion equation in the framework of the proposed model. An isobaric process is considered, i. e., the quantity p_e determining the deviation of elastic pressure from its equilibrium value is assumed to be equal to zero

$$p_e = 0 \qquad \Rightarrow \qquad \tilde{\vartheta}_a = -\frac{K_{ad}}{\Upsilon_a}\, \varepsilon \qquad \Rightarrow \qquad \tilde{T}_a = \left(\Upsilon_a - \frac{K_a K_{ad}}{\Upsilon_a} \right) \varepsilon. \tag{94}$$

We transform Eq. (89) in view of the isobaric condition (94) and the expressions for parameters of the model (71), (72), and (92). As a result, we obtain

$$\Delta\varepsilon - \rho_* \left(\frac{c_p}{\lambda} - \frac{c_p - c_v}{c_v \eta_v} \right) \frac{d\varepsilon}{dt} - \beta^{-1} \rho_* \left(\frac{c_p}{\lambda} - \frac{c_p - c_v}{c_v \eta_v} \right) \frac{d^2\varepsilon}{dt^2} = -\rho_* \alpha \nabla \cdot \mathbf{L}_h^a. \tag{95}$$

By using the notations

$$\frac{1}{\eta_p} = \frac{c_p}{\lambda} - \frac{c_p - c_v}{c_v \eta_v}, \qquad \Psi_p = \eta_p \, \alpha \, \nabla \cdot \mathbf{L}_h^a, \tag{96}$$

we rewrite Eq. (95) in the form

$$\eta_p \, \Delta\varepsilon - \rho_* \frac{d\varepsilon}{dt} - \beta^{-1} \rho_* \frac{d^2\varepsilon}{dt^2} = -\rho_* \Psi_p. \tag{97}$$

For concreteness, η_p will be called the isobaric volume viscosity. The structure of Eq. (97) is the same as the structure of the self-diffusion Eq. (93). However, the expressions for the volume viscosity and the source term obtained under the condition of isentropy differ from the expressions (96) obtained under the isobaric condition. The relation of the isobaric volume viscosity to the isentropic volume viscosity can be represented in the form

$$\frac{1}{\eta_p} - \frac{1}{\eta_v} = c_p \left(\frac{1}{\lambda} - \frac{1}{c_v \eta_v} \right). \tag{98}$$

Now it is hard to say whether or not one of these viscosities can be identified with the volume viscosity η_v^{cl} whose values is given in handbooks. The answer this question is in what follows.

7.3 Mechanism of shear viscosity

Now we pass on to the discussion of a mechanism of shear viscosity. We assume $\boldsymbol{\psi}_a$ to be constant that is counterpart of the condition of isentropy

$$\boldsymbol{\psi}_a = \boldsymbol{\psi}_a^* = \text{const} \qquad \Rightarrow \qquad \tilde{\boldsymbol{\psi}}_a = \mathbf{0} \qquad \Rightarrow \qquad \tilde{\mathbf{M}}_a = D_a \, \boldsymbol{\gamma}. \tag{99}$$

We take the curl operator of both sides of the third equation in (68) and transform the obtained equation taking into account Eqs. (69) and (70). As a result, we get

$$\nabla \times \left(\nabla \times \tilde{\mathbf{M}}_a \right) - \rho_* \beta \left(B_a \nabla \times \mathbf{v} + J_a \frac{d\boldsymbol{\psi}_a}{dt} \right) + \rho_* \nabla \times \mathbf{L}_h^a = \rho_* \left(B_a \frac{d \nabla \times \mathbf{v}}{dt} + J_a \frac{d\boldsymbol{\psi}_a}{dt} \right). \tag{100}$$

In view of condition (99) and expression (69) for vector $\boldsymbol{\gamma}$, we reduce Eq. (100) to the form

$$D_a \, \Delta \nabla \times \mathbf{u} = \rho_* \beta B_a \, \nabla \times \mathbf{v} + \rho_* B_a \, \frac{d}{dt} \nabla \times \mathbf{v} - \rho_* \, \nabla \times \mathbf{L}_h^a. \tag{101}$$

Next, we introduce the notation

$$\eta_s = \frac{D_a}{\beta B_a}. \tag{102}$$

Neglecting the inertia term and the term containing external moment $\mathbf{L}_h^a$ in Eq. (101) and differentiating the obtained equation with respect to time in view of notation (102), we get

$$\eta_s \, \Delta \nabla \times \mathbf{v} = \rho_* \frac{d}{dt} \nabla \times \mathbf{v}. \tag{103}$$

Comparing Eq. (103) with the equation of vortex motion of a viscous fluid (87), we reveal that these equations are the same. Consequently, the parameter η_s has the sense of shear viscosity.

Now we discuss an alternative approach to derivation of the equation of vortex motion of a viscous fluid. Let us consider the process such that the antisymmetric part of the stress tensor remains equal to zero

$$\mathbf{q}=\mathbf{0} \quad \Rightarrow \quad \tilde{\boldsymbol{\psi}}_a=-\frac{A}{D_a}\boldsymbol{\gamma} \quad \Rightarrow \quad \tilde{\mathbf{M}}_a=\left(D_a-\frac{A\Gamma_a}{D_a}\right)\boldsymbol{\gamma}. \tag{104}$$

We reduce Eq. (100) taking into account Eq. (104). Next, we neglect the inertia term and the term containing external moment $\mathbf{L}_h^a$ and differentiate the obtained equation with respect to time. As a result, we get

$$\eta_q\,\Delta\nabla\times\mathbf{v}=\rho_*\frac{d}{dt}\nabla\times\mathbf{v}+\rho_*\boldsymbol{\Psi}_q, \tag{105}$$

where the following notations are used

$$\eta_q=\frac{D_a^2-A\Gamma_a}{\beta(D_aB_a-AJ_a)}\,, \qquad \boldsymbol{\Psi}_q=\frac{2AJ_a}{D_aB_a-AJ_a}\,\frac{d^2\boldsymbol{\varphi}}{dt^2}\,. \tag{106}$$

Vector $\boldsymbol{\Psi}_q$ in Eq. (105) plays role of the source term. Parameter η_q represents the shear viscosity whose value, generally, differs from the value of η_s.

By using Eqs. (71), (92), (102), and (106) the parameters D_a and Γ_a can be expressed in terms of the volume and shear viscosities η_v, η_s, η_q and the known mechanical and thermodynamical constants. The parameter D_a [see Eqs. (92) and (102)] does not depend on the unknown elastic modulus A, while the parameter Γ_a depends on it

$$\Gamma_a=\frac{T_a^*}{\rho_*}\left[\frac{\eta_q}{\lambda}-\frac{\eta_s(\eta_q-\eta_s)(K_{ad}-K_{iz})}{c_v\,\eta_v^2\,A}\right]. \tag{107}$$

Let us choose the elastic modulus A as follows:

$$A=\frac{\lambda\,(\eta_q-\eta_s)(K_{ad}-K_{iz})}{c_v\,\eta_v^2}\,. \tag{108}$$

Finally, by using Eqs. (92), (102), (107), and (108), we obtain the expressions for D_a and Γ_a:

$$D_a=-\frac{\alpha K_{iz}T_a^*\eta_s}{\rho_*c_v\eta_v}\,, \qquad \Gamma_a=\frac{(\eta_q-\eta_s)\,T_a^*}{\lambda\,\rho_*}\,. \tag{109}$$

In what follows, we discuss the methods of determination of the shear viscosities η_s and η_q for solids. Now we consider fluids and gases.

Suppose that in the case of fluids and gases, the shear viscosities η_q and η_s are equal to each other and equal to the shear viscosity η_s^{cl}. Then, the parameters A, D_a and Γ_a are determined by Eqs. (108) and (109) where $\eta_q=\eta_s=\eta_s^{cl}$:

$$A=0, \qquad D_a=-\frac{\alpha K_{iz}T_a^*\,\eta_s^{cl}}{\rho_*c_v\eta_v}\,, \qquad \Gamma_a=0. \tag{110}$$

Let us take the curl operator of both sides of the first and third equations in (68) and transform the obtained equations taking into account the kinematical relations (69) and the constitutive equations (70). Eliminating the external moments and taking into account the fact that for fluids and gases $G=0$, $A=0$ and $\Gamma_a=0$ we obtain

$$\begin{gathered} D_a\,\Delta\boldsymbol{\psi}_a=\rho_*\left(\frac{d^2\nabla\times\mathbf{u}}{dt^2}+B_a\frac{d^2\boldsymbol{\psi}_a}{dt^2}\right), \\ D_a\,\Delta\nabla\times\mathbf{u}-\rho_*\beta\left(B_a\frac{d\nabla\times\mathbf{u}}{dt}+J_a\frac{d\boldsymbol{\psi}_a}{dt}\right)=\rho_*\left(B_a\frac{d^2\nabla\times\mathbf{u}}{dt^2}+J_a\frac{d^2\boldsymbol{\psi}_a}{dt^2}\right). \end{gathered} \tag{111}$$

Eliminating variable ψ_a from the system (111), we get

$$\frac{D_a^2}{\rho_* \beta J_a} \Delta\Delta\nabla \times \mathbf{u} - \frac{B_a D_a}{J_a} \frac{d}{dt} \Delta\nabla \times \mathbf{u} - \frac{2B_a D_a}{\beta J_a} \frac{d^2}{dt^2} \Delta\nabla \times \mathbf{u} - \\ - \rho_* \left(1 - \frac{B_a^2}{J_a}\right) \frac{d^3}{dt^3} \nabla \times \mathbf{u} - \frac{\rho_*}{\beta} \left(1 - \frac{B_a^2}{J_a}\right) \frac{d^4}{dt^4} \nabla \times \mathbf{u} = \mathbf{0}. \quad (112)$$

In the case of a slow process, the asymptotically leading term of Eq. (112) is determined by the first two terms. In view of the foregoing formulas for the model parameters, the asymptotically leading term of Eq. (112) can be represented as

$$\Delta \left[\eta_s^{cl} \, \Delta\nabla \times \mathbf{u} - \rho_* \frac{d}{dt} \nabla \times \mathbf{u} \right] = \mathbf{0}. \quad (113)$$

It is evident that Eq. (113) is the counterpart of Eq. (103).

In the case of a fast process, the asymptotically leading term of Eq. (112) is determined by the terms containing the odd time derivatives. The terms containing the even time derivatives are small due to the presence of coefficient β^{-1} which is of the order 10^{-10} s for gases and 10^{-13} s for fluids. These terms should be taken into account only when the terahertz frequency range is considered. Thus, a fast process is described by the equation which (in view of the foregoing formulas for the model parameters) takes the form

$$\frac{d}{dt} \left[k \, \Delta\nabla \times \mathbf{u} + \rho_* \frac{d^2}{dt^2} \nabla \times \mathbf{u} \right] = \mathbf{0}, \qquad k = \frac{\lambda(K_{ad} - K_{iz})\eta_s^{cl}}{c_v \eta_v^2 - \beta^{-1} \lambda(K_{ad} - K_{iz})}. \quad (114)$$

In what follows, we show that the coefficient k is positive. Therefore, Eq. (114) describes the time-increasing processes rather than oscillating ones.

A distinctive feature of the foregoing model of internal damping is the fact that the shear viscosity does not influence on the dynamical processes concerned with the volume change. In accordance with the classical theory, this is not the case. In the case of fluids and gases, the formula for the sound attenuation factor is well known. According to this formula, the sound attenuation is determined by three factors: the shear viscosity, the volume (acoustic) viscosity, and the heat conductivity. This formula is usually used to determine the volume viscosity. The proposed treatment of internal damping leads to the theory of thermoviscoelasticity which allows one to calculate the sound attenuation factor by another formula. This formula differs from the classical one, particularly, by the fact that it does not depend on the shear viscosity. That is why, in general, the values of volume viscosity taken from the handbooks have to coincide neither with η_v nor with η_p.

8 Entropy balance equation and second law of thermodynamics

We start the discussion of second law of thermodynamics with the introduction of energy characteristics of thermal and dissipative processes in the framework of proposed model. For this purpose, we consider the equation of energy balance for the material medium consisting of one-rotor gyrostats

$$\frac{d}{dt} \int\limits_{(V)} \rho_* (K_m + U_m) dV = \int\limits_{(V)} \rho_* \big(\mathbf{f} \cdot \mathbf{v} + \mathbf{m} \cdot \tilde{\boldsymbol{\omega}} + \mathbf{L} \cdot \boldsymbol{\omega} + Q \big) dV + \\ + \int\limits_{(S)} \big(\boldsymbol{\tau}_n \cdot \mathbf{v} + \boldsymbol{\mu}_n \cdot \tilde{\boldsymbol{\omega}} + \mathbf{T}_n \cdot \boldsymbol{\omega} + H_n \big) dS. \quad (115)$$

Here K_m is the kinetic energy of gyrostats per unit mass; Q and H_n are the rates of "non-mechanical nature" energy supply in control volume V and through surface S, respectively. Vectors $\boldsymbol{\tau}_n$, $\boldsymbol{\mu}_n$, $\mathbf{T}_n$ are the force and moments acting on a unit area of the surface S. These vectors are related with the stress tensor and moment stress tensors by the standard formulas

$$\boldsymbol{\tau}_n = \mathbf{n} \cdot \boldsymbol{\tau}, \qquad \boldsymbol{\mu}_n = \mathbf{n} \cdot \boldsymbol{\mu}, \qquad \mathbf{T}_n = \mathbf{n} \cdot \mathbf{T}. \quad (116)$$

The remaining quantities in Eq. (115) have been introduced above.

In accordance with our approach, the quantities associated with carrier bodies describe the mechanical processes, whereas the quantities associated with rotors describe the thermal processes and processes connected with transfer of mechanical energy into heat. Thus, the terms $\mathbf{f} \cdot \mathbf{v}$ and $\mathbf{m} \cdot \tilde{\boldsymbol{\omega}}$ characterize the energy supply in control volume V due to the power of forces and moments, whereas the term $\mathbf{L} \cdot \boldsymbol{\omega}$ can be interpreted as the rate of thermal energy supply per unit mass. Then, the quantity Q has the sense of the rate of supply of energy of "non-mechanical and nonthermal nature." Similarly, the terms $\boldsymbol{\tau}_n \cdot \mathbf{v}$ and $\boldsymbol{\mu}_n \cdot \tilde{\boldsymbol{\omega}}$ represent the mechanical energy flow through the surface S, whereas the term $\mathbf{T}_n \cdot \boldsymbol{\omega}$ characterizes the thermal energy flow through this surface. Correspondingly, the quantity H_n has the sense of flow of energy of "non-mechanical and nonthermal nature" through the surface S.

Since we consider the continuum of one-rotor gyrostats to be isolated body, we assume the supply of energy of "non-mechanical and nonthermal nature" to be equal to zero, i. e.,

$$Q = 0, \qquad H_n = 0. \tag{117}$$

Notice that by standard line of reasoning in view of Eq. (117), one can obtain the local form of energy balance equation (51). Now we introduce the notations

$$Q_L = \mathbf{L} \cdot \boldsymbol{\omega}, \qquad h_n = \mathbf{T}_n \cdot \boldsymbol{\omega}. \tag{118}$$

In view of Eqs. (117) and (118), the energy balance equation (115) takes the form

$$\frac{d}{dt} \int\limits_{(V)} \rho_* (K_m + U_m) dV = \int\limits_{(V)} \rho_* \big(\mathbf{f} \cdot \mathbf{v} + \mathbf{m} \cdot \tilde{\boldsymbol{\omega}} + Q_L\big) dV + \int\limits_{(S)} \big(\boldsymbol{\tau}_n \cdot \mathbf{v} + \boldsymbol{\mu}_n \cdot \tilde{\boldsymbol{\omega}} + h_n\big) dS. \tag{119}$$

If we consider Eq. (119) as the energy balance equation for the classical medium, then we interpret Q_L and h_n as the rates of thermal energy supply in control volume V and through the surface S, respectively.

Taking into account the foregoing assumption that $\mathbf{L} = \mathbf{L}_f + \mathbf{L}_h$ where $\mathbf{L}_f$ is the moment of viscous damping given by Eq. (43), we get

$$Q_L = -\beta (B\mathbf{v} + J\boldsymbol{\omega}) \cdot \boldsymbol{\omega} + \mathbf{L}_h \cdot \boldsymbol{\omega}. \tag{120}$$

Using standard line of reasoning, we introduce the vector of thermal energy flow $\mathbf{h}$, namely

$$h_n = \mathbf{n} \cdot \mathbf{h}. \tag{121}$$

According to Eqs. (116), (118), and (121), we have

$$\mathbf{h} = \mathbf{T} \cdot \boldsymbol{\omega}. \tag{122}$$

In view of Eq. (46) for tensor $\mathbf{T}$ and Eqs. (63) and (64) relating the mechanical and thermodynamic quantities, the vector of thermal energy flow $\mathbf{h}$ can be represented as a sum

$$\mathbf{h} = \mathbf{h}_T + \mathbf{h}_M, \qquad \mathbf{h}_T = T\boldsymbol{\omega} \equiv T_a \boldsymbol{\omega}_a, \qquad \mathbf{h}_M = -\mathbf{M} \times \boldsymbol{\omega} \equiv -\mathbf{M}_a \times \boldsymbol{\omega}_a. \tag{123}$$

Here vector $\mathbf{h}_T$ has the sense of the heat flow vector, $\mathbf{h}_M$ is that part of thermal energy flow which should be taken into account if we aim to describe the attenuation of transverse waves and the transfer of mechanical energy of shear vibrations into heat.

After the introduction of the energy characteristics of thermal and dissipative processes, we turn to the derivation of the entropy balance equation. We start with the consideration of an approximate form of Eq. (50) which describes the motion of rotors

$$\nabla T - \nabla \times \mathbf{M} - \beta \rho_* (B\mathbf{v} + J\boldsymbol{\omega}) + \rho_* \mathbf{L}_h = \mathbf{0}. \tag{124}$$

Notice that Eq. (124) is obtained under the assumption that the inertia terms in Eq. (50) can be neglected that corresponds to the parabolic heat conduction equation. Taking the scalar product of both sides of Eq. (124) on the angular velocity of rotors $\boldsymbol{\omega}$ and performing simple transformations, we obtain

$$\nabla \cdot (T\boldsymbol{\omega}) - T(\nabla \cdot \boldsymbol{\omega}) - \nabla \cdot (\mathbf{M} \times \boldsymbol{\omega}) + \mathbf{M} \cdot (\nabla \times \boldsymbol{\omega}) - \beta \rho_* (B\mathbf{v} + J\boldsymbol{\omega}) \cdot \boldsymbol{\omega} + \rho_* \mathbf{L}_h \cdot \boldsymbol{\omega} = 0. \tag{125}$$

Taking into account Eqs. (38), (58), (63), (64), (120), and (123), we rewrite Eq. (125) in the form

$$T_a \frac{d\vartheta_a}{dt} = \nabla \cdot \mathbf{h}_T + \nabla \cdot \mathbf{h}_M + \mathbf{M}_a \cdot \frac{d\boldsymbol{\psi}_a}{dt} + \rho_* Q_L. \tag{126}$$

Let us introduce the entropy flow $\mathbf{h}_\vartheta$ as

$$\mathbf{h}_\vartheta = \frac{\mathbf{h}_T + \mathbf{h}_M}{T_a}\,. \tag{127}$$

By simple transformations in view of Eqs. (38), (58), (63), (123), and (127), the first three terms on the right-hand side of Eq. (126) can be rewritten as follows:

$$\nabla \cdot \mathbf{h}_T + \nabla \cdot \mathbf{h}_M + \mathbf{M}_a \cdot \frac{d\boldsymbol{\psi}_a}{dt} = T_a \nabla \cdot \mathbf{h}_\vartheta + \frac{\mathbf{h}_T \cdot \nabla T_a + \mathbf{M}_a \cdot (\nabla \times \mathbf{h}_T)}{T_a}. \tag{128}$$

Taking into account Eqs. (64) and (65), the expression (120) can be reduced to the form

$$Q_L = \frac{\beta B_a^2}{4J_a} \mathbf{v} \cdot \mathbf{v} - \beta J_a \left(\boldsymbol{\omega}_a + \frac{B_a}{2J_a} \mathbf{v} \right)^2 + \mathbf{L}_h^a \cdot \boldsymbol{\omega}_a. \tag{129}$$

Next, dividing both sides of Eq. (126) by T_a and substituting in it the expressions (128) and (129) we obtain

$$\frac{d\vartheta_a}{dt} = \nabla \cdot \mathbf{h}_\vartheta + \sigma \tag{130}$$

where σ is calculated by the formula

$$\sigma = \frac{\mathbf{h}_T \cdot \nabla T_a + \mathbf{M}_a \cdot (\nabla \times \mathbf{h}_T)}{T_a^2} + \frac{\rho_* \beta B_a^2}{4J_a T_a} \mathbf{v} \cdot \mathbf{v} - \frac{\rho_* \beta J_a}{T_a} \left(\boldsymbol{\omega}_a + \frac{B_a}{2J_a} \mathbf{v} \right)^2 + \frac{\rho_* (\mathbf{L}_h^a \cdot \boldsymbol{\omega}_a)}{T_a}. \tag{131}$$

Let us discuss the physical meaning of Eqs. (130) and (131). The first of these equations is known as the entropy balance equation, and it is one of the fundamental equations in non-equilibrium thermodynamics. The term σ has the sense of entropy production. Non-equilibrium thermodynamics is the science that considers the interplay of various physical processes. That is why the entropy balance equation often involves the terms characterizing the entropy production due to the various physical processes, for example, the dissipation, Joule heating, chemical reactions, etc. Analogously, the entropy flow can contain the components that are associated with the heat flow, the electromagnetic energy flow, the radiation flow, etc. In the proposed model, the entropy flow $\mathbf{h}_\vartheta$ consists of two parts — see Eq. (127): the component associated with the heat flow and the component associated with that part of thermal energy flow which allows us to describe the attenuation of transverse waves and the transfer of mechanical energy of shear vibrations into heat. The entropy production σ consists of four terms — see the expression (131). The first term in this expression is the entropy production due to the heat conductivity. It is easy to see that if the Fourier law of heat conduction $\mathbf{h}_T = \lambda \nabla T_a$ is used, then this term is non-negative. We do not postulate the constitutive equation for the heat flow because we derive the heat conduction equation using a different method. In accordance with our method, the heat flow vector is calculated by the second formula in Eq. (123) after the coupled problem of thermoviscoelasticity is solved. Hence, we can determine the sign of the first term on the right-hand side of Eq. (131) only when the problem is solved. The second term in the expression for σ is the entropy production due to the dissipative process. Really, this term is non-negative and it is proportional to the kinetic energy of translational motion. Therefore, it characterizes the transfer of mechanical energy into heat. It is easy to see that the coefficient of this term is proportional to B_a^2. Hence, if the parameter B_a is equal to zero that corresponds to the problem of thermoelasticity, then the internal dissipation is ignored. The third term in the expression for σ is non-positive. This term is accounted with the radiation of energy into the “thermal ether,” i. e., it has the sense of thermal radiation. One of the features of the proposed model is the account of thermal radiation that is usually ignored in continuum mechanics. The last term in the expression for σ characterizes the entropy production due to the heat supply from an external source.

Constructing the model of thermoviscoelastic medium, we suppose that the “thermal ether” in the unperturbed state is not moving relative to the Earth, and we use the inertial reference system related with the Earth. On passing to another inertial reference system (without abandoning the assumption of immobility of the "thermal ether" relative to the Earth), we should replace the expression (43) for the moment of viscous damping by the expression (44). If Eq. (44) is used instead of Eq. (43), then the expression (120) and all subsequent equations are modified so that the translational velocity $\mathbf{v}$ is replaced by the difference $\mathbf{v} - \mathbf{V}_0$. Thus, the entropy production due to the dissipative process [the second term on the right-hand part of Eq. (131)] and the radiation of energy into the "thermal ether" [the third term on the right-hand part of Eq. (131)] depend on the velocity of particles of the medium relative to the "thermal ether" and do not depend on the choice of an inertial reference system. The assumption of immobility of the "thermal ether" relative to the Earth is a physical hypothesis being the basis for the proposed model of thermoviscoelasticity. As shown above, the equations which are well known in continuum mechanics and thermodynamics follow from the aforesaid hypothesis. If we reject the assumption of immobility of the "thermal ether" relative to the Earth, then in order to obtain the constitutive equation for the moment of viscous damping, we have to solve the model problems different from the model problems considered above. It is obvious that the solution of these model problems leads to a more complicated law of viscous damping. As a result, all subsequent formulas including the expression for the entropy production (131) are modified. In this case, as in the circumstances discussed above, both the law of viscous damping and the expression for the entropy production should depend rather on the velocity relative to the "thermal ether" than on the absolute velocity.

Now we turn to a discussion of the second law of thermodynamics. First of all, we note that now there are a number of statements that express the second law of thermodynamics. For example, in [30] one can find 18 formulations of the second law of thermodynamics, and [31] contains 21 formulations. Not all statements expressing the second law of thermodynamics are equivalent. Let us discuss some formulations adopted in the non-equilibrium thermodynamics and continuum mechanics. In non-equilibrium thermodynamics, the entropy change $d\vartheta_a$ during time interval dt is divided into two parts: $d\vartheta_a = d_e\vartheta_a + d_i\vartheta_a$ where $d_e\vartheta_a$ is the entropy change due to the matter and energy exchange with the surrounding medium and $d_i\vartheta_a$ is the entropy change due to the irreversible processes in the system. The second law of thermodynamics states that $\dfrac{d_i\vartheta_a}{dt} \geq 0$. This formulation is not strictly unambiguous when it is applied to continuum because on going to the local formulation it is not clear what part of the entropy production σ causes the entropy change due to the irreversible processes in the system and what part of σ causes the entropy change due to external factors. It is obvious that the last two terms in the expression (131) are responsible for $d_e\vartheta_a$ and the second term is responsible for $d_i\vartheta_a$. However, the first term in (131) can be interpreted both as the cause of change $d_e\vartheta_a$ and as the cause of change $d_i\vartheta_a$. In the first case, the second law of thermodynamics is satisfied since the second term in (131) is non-negative. In the second case, the second law of thermodynamics is satisfied provided that

$$\frac{\mathbf{h}_T \cdot \nabla T_a + \mathbf{M}_a \cdot (\nabla \times \mathbf{h}_T)}{T_a^2} + \frac{\rho_* \beta B_a^2}{4 J_a T_a}\, \mathbf{v} \cdot \mathbf{v} \geq 0. \tag{132}$$

We can verify whether the condition (132) is satisfied only when the coupled problem of thermoviscoelasticity is solved. In continuum mechanics, the situation with the second law of thermodynamics there is similar to the situation in non-equilibrium thermodynamics. For example, in [3] (see p. 245), two different formulation of the second law of thermodynamics can be found. The first one is the Clausius–Planck inequality

$$T_a \frac{d\vartheta_a}{dt} \geq \nabla \cdot \mathbf{h} + q \tag{133}$$

where $\mathbf{h}$ is the heat flow vector, q is the heat supply from an external source per unit volume. The second formulation is the Clausius–Duhem inequality

$$T_a \frac{d\vartheta_a}{dt} \geq T_a \nabla \cdot \left(\frac{\mathbf{h}}{T_a} \right) + q \tag{134}$$

where $\mathbf{h}$ and q have the same meaning as in Eq. (133). In the proposed theory, the Clausius–Planck inequality (133) takes the form

$$\frac{d\vartheta_a}{dt} \geq \nabla \cdot \mathbf{h}_\vartheta + \frac{\mathbf{h}_T \cdot \nabla T_a + \mathbf{M}_a \cdot (\nabla \times \mathbf{h}_T)}{T_a^2} - \frac{\rho_* \beta J_a}{T_a} \left(\boldsymbol{\omega}_a + \frac{B_a}{2 J_a} \mathbf{v} \right)^2 + \frac{\rho_* (\mathbf{L}_h^a \cdot \boldsymbol{\omega}_a)}{T_a}. \tag{135}$$

Since the right-hand side of inequality (135) differs from the right-hand side of the entropy balance equation only by absence of the non-negative term — see Eqs. (130) and (131), it is obvious that the second law of thermodynamics in the form of the inequality (135) is satisfied. For the proposed model, the Clausius–Duhem inequality (134) is formulated as

$$\frac{d\vartheta_a}{dt} \geq \nabla \cdot \mathbf{h}_\vartheta - \frac{\rho_* \beta J_a}{T_a}\left(\boldsymbol{\omega}_a + \frac{B_a}{2J_a}\,\mathbf{v}\right)^2 + \frac{\rho_*(\mathbf{L}_h^a \cdot \boldsymbol{\omega}_a)}{T_a}. \tag{136}$$

In view of Eqs. (130) and (131), the inequality (136) is equivalent to the inequality (132). Hence, we can verify whether the second law of thermodynamics in the form of the inequality (136) is satisfied only when the coupled problem of thermoviscoelasticity is solved.

The foregoing reasoning holds for the case when the inertia terms in the equation of motion of rotors (50) are neglected. The statement of problem without the inertial terms in Eq. (50) leads to classical theory of thermoelasticity (73) which contains the parabolic heat conduction equation (an equation without the second time derivatives). For lack of the second time derivatives, the classical equation describes the heat propagation with infinite speed. This is not an obstacle for solving most of practical problems since the solution of parabolic heat conduction equation decays exponentially and therefore only infinitesimal quantity of heat propagates with infinite speed. However, from the theoretical point of view, the infinite speed of heat propagation is unacceptable. That is why in 1948 the hyperbolic type heat conduction equation was proposed by Cattaneo, see [32], and in 1967 the formulation of coupled problem of thermoelasticity with hyperbolic type heat conduction equation was proposed by Lord and Shulman, see [33]. Let us briefly discuss the derivation of the hyperbolic type heat conduction equation with the help of arguments standard for continuum mechanics.

In the theory of thermoelasticity, the heat conduction equation has the form

$$T_a \frac{d\vartheta_a}{dt} = \nabla \cdot \mathbf{h} + q. \tag{137}$$

Taking into account the constitutive equation

$$\vartheta_a = \vartheta_a^* + \frac{\rho_* c_v}{T_a^*}\left(T_a - T_a^*\right) + \alpha K_{iz}\varepsilon \tag{138}$$

we eliminate the entropy from Eq. (137) and linearize the obtained equation near the temperature T_a^*. As a result, we have

$$\rho_* c_v \frac{dT_a}{dt} + \alpha K_{iz} T_a^* \frac{d\varepsilon}{dt} = \nabla \cdot \mathbf{h} + q. \tag{139}$$

Next, the constitutive equation for the heat flow vector should be taken into account. The classical heat conduction equation is derived by using the Fourier law: $\mathbf{h} = \lambda \nabla T_a$. To obtain the hyperbolic type heat conduction equation, the Fourier law should be replaced by the Maxwell–Cattaneo law which has the form

$$\beta^{-1}\frac{d\mathbf{h}}{dt} + \mathbf{h} = \lambda \nabla T_a. \tag{140}$$

It is easy to see that in comparison with the Fourier law, Eq. (140) has the additional term $\beta^{-1}\dfrac{d\mathbf{h}}{dt}$. Due to this term, Eq. (140) takes into account the inertia of the heat conduction process, i. e., the fact that response to appearance of a temperature gradient does not occur instantaneously. The delay of response to appearance of a temperature gradient is determined by the coefficient of the heat flow derivative. Eliminating vector $\mathbf{h}$ from the system of Eqs. (139) and (140), we obtain

$$\rho_* c_v\left(\beta^{-1}\frac{d^2T_a}{dt^2} + \frac{dT_a}{dt}\right) + \alpha K_{iz}T_a^*\left(\beta^{-1}\frac{d^2\varepsilon}{dt^2} + \frac{d\varepsilon}{dt}\right) = \lambda \varDelta T_a + \beta^{-1}\frac{dq}{dt} + q. \tag{141}$$

It is obvious that Eq. (141) coincides with the third equation in (73). Thus, if the standard approach to derivation of the heat conduction equation is used, then the presence of the second time derivatives in this equation is the result of the presence of the time derivative of heat flow in the Maxwell–Cattaneo law (140). If the heat conduction equation is derived by the method proposed above, then the second time derivatives are presented in this equation due to the presence of inertia terms in Eq. (50). Consequently, the neglect of inertial terms in Eq. (50) is the same as leaving out the first time derivative of the heat flow in Eq. (140). If it is permissible to neglect the first time derivative of heat flow in the law of heat conduction, then it is permissible to neglect the inertial terms in Eq. (50).

The heat conduction Eq. (141) is a hyperbolic equation. According to this equation, the heat propagates at a finite speed. However, this equation has wave properties resulting in difficulties related to the second law of thermodynamics. Thus, due to the presence of second time derivatives in the heat conduction equation, the problem of infinite speed of heat propagation is solved but there is a problem with the satisfaction of the second law of thermodynamics.

The theories including the hyperbolic type heat conduction equation have long been known and studied by many authors. A review of literature on these issues can be found in [34], [35]. It is known that difficulties related to the second law of thermodynamics are taken place in all such theories, see [36], [37], [38], [39]. There are various approaches and methods to overcome these difficulties, see [27], [40]. Now there is no generally accepted approach. All approaches and methods have advantages and disadvantages, and it is difficult to prefer one of them. Notice that studying the hyperbolic heat conduction problem we are confronted with difficulties when trying to satisfy the local form of the second law of thermodynamics. This is due to an oscillatory character of solution. If we average the solution over time or spatial coordinates (using the period of oscillation as a characteristic time and wavelength as the characteristic distance), then we can show that for the average values, the second law of thermodynamics is satisfied. Consequently, the consideration of inertia of the heat conduction process does not contradict to the second law of thermodynamics in the form adopted in the equilibrium thermodynamics. There is contradiction with the local formulations of second law of thermodynamics which are adopted in continuum mechanics and non-equilibrium thermodynamics. The local formulations are more restrictive than those adopted in the equilibrium thermodynamics. In view of the fact that avoiding wave processes we cannot solve the problem of an infinite speed of heat propagation; it is quite possible that the local formulations of the second law of thermodynamics should be modified so that they would allow for existence of the wave processes in the heat conduction problem. This is what is done in all studies related to overcoming the contradictions between the second law of thermodynamics and the hyperbolic type heat conduction problem.

9 Conclusion

In the second part of the paper, we examine the coupled problem of thermoviscoelasticity formulated on the base of the proposed theory. We obtain the dependence of the acoustic wave attenuation factor on a signal frequency and show that this dependence is in close agreement with the classical dependence in the low-frequency range and agrees with the dependence obtained on the base of the phonon theory in the hypersonic frequency range. We discuss the ways of determination of some parameters of the proposed model by using known values of the sound velocity and the acoustic wave attenuation factor.

Part 2: Determination of the model parameters

10 Introduction

In the first part of the paper, we construct a new theory of thermoviscoelasticity. The method of derivation of the basic equations of this theory which describes both the mechanical and non-mechanical (thermal) processes is based on the idea of using the purely mechanical model of a continuum with internal rotational degrees of freedom (a one-rotor gyrostat continuum). We use this mechanical model to describe behavior of the ordinary material medium (medium without internal degrees of freedom) possessing not only mechanical properties but also the thermal ones. Therefore, we interpret the spherical part of the moment stress tensor characterizing the interaction between rotors as the temperature of the ordinary material medium and the corresponding deformation acquires meaning of volume density of entropy. We consider the interaction of carrier bodies of the gyrostats to be charged with the mechanical processes. Then, the interference of carrier bodies and rotors provides the interplay of mechanical and thermal properties. In the first part of the paper, we discuss the physical nature of the mechanism of thermal conduction and internal damping and show that in the context of the proposed model these concepts acquire the original treatment different from the conventional one. The volume and shear viscosities introduced in the context of the proposed model differ from the analogous quantities used in the known theories. That is why the volume and shear viscosities should be determined by means of quantities which can be found by direct measurement, for example, the acoustic wave attenuation factor. The second part of the paper is devoted to determination of the volume and shear viscosities and some other parameters of the model.

11 Thermoelastic and thermodynamic forces

Now we reduce the equations describing dynamics of the one-rotor gyrostat continuum (see the first part of the paper) to the form which is standard for the classical continuum without microstructure. It is well known that an arbitrary vector can be represented in terms of the scalar and vector Helmholtz potentials. We use this representation for dynamic term containing vector $\boldsymbol{\omega}_a$ on the right-hand side of the momentum balance equation

$$-\rho_* B_a \frac{d\boldsymbol{\omega}_a}{dt} = \nabla p - \nabla \times \mathbf{t}, \qquad \nabla \cdot \mathbf{t} = 0. \tag{142}$$

Here p is the scalar potential, $\mathbf{t}$ is the vector potential. As will be seen from further consideration, the quantities p and $\mathbf{t}$ have the sense of mechanical stresses. According to the definition (142), these quantities vanish in the case of static problems. By using the notation (142), we write down summary of the basic equations of coupled problem of thermoviscoelastisity for the Cosserat continuum without microstructure

$$\begin{gathered}
\nabla \cdot \tilde{\boldsymbol{\tau}}^s - \nabla \times \tilde{\mathbf{q}} + \rho_* \mathbf{f} = \rho_* \frac{d^2\mathbf{u}}{dt^2}, \qquad \nabla \times \tilde{\boldsymbol{\mu}}_v = 2\tilde{\mathbf{q}}, \qquad \nabla \times \boldsymbol{\varphi} = \mathbf{0}, \\
\tilde{\boldsymbol{\tau}}^s = \left[\left(K_{iz} - \frac{2}{3} G\right)\varepsilon - \alpha K_{iz} \tilde{T}_a + p\right] \mathbf{E} + 2\, G\, \boldsymbol{\varepsilon}^s, \qquad \boldsymbol{\varepsilon}^s = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right), \\
\tilde{\mathbf{q}} = \frac{\lambda\, (\eta_q - \eta_s)(K_{ad} - K_{iz})}{c_v \eta_v^2} \boldsymbol{\gamma} - \frac{\alpha K_{iz} T_a^*\, \eta_s}{\rho_* c_v \eta_v}\, \tilde{\boldsymbol{\psi}}_a + \mathbf{t}, \qquad \boldsymbol{\gamma} = \nabla \times \mathbf{u} - 2\boldsymbol{\varphi}, \\
\Delta p = \frac{\alpha K_{iz}}{\beta \eta_v} \left[\rho_* \frac{d^2 \tilde{T}_a}{dt^2} + \frac{\alpha K_{iz} T_a^*}{c_v} \frac{d^2 \varepsilon}{dt^2}\right], \qquad \Delta \mathbf{t} = \frac{\alpha K_{iz} T_a^*}{\beta c_v \eta_v} \frac{d^2 \tilde{\boldsymbol{\psi}}_a}{dt^2}, \qquad \varepsilon = \operatorname{tr} \boldsymbol{\varepsilon}^s, \\
\Delta \tilde{T}_a - \frac{\rho_* c_v}{\lambda} \left[\frac{d\tilde{T}_a}{dt} + \frac{1}{\beta} \frac{d^2 \tilde{T}_a}{dt^2}\right] = \alpha K_{iz} T_a^* \left(\frac{1}{\lambda} - \frac{1}{c_v \eta_v}\right) \left[\frac{d\varepsilon}{dt} + \frac{1}{\beta} \frac{d^2 \varepsilon}{dt^2}\right] - \rho_* \nabla \cdot \mathbf{L}_h^a, \\
(\eta_q - \eta_s) \Delta \tilde{\boldsymbol{\psi}}_a - \rho_* \left(\frac{d\tilde{\boldsymbol{\psi}}_a}{dt} + \frac{1}{\beta} \frac{d^2 \tilde{\boldsymbol{\psi}}_a}{dt^2}\right) = \\
= \frac{\lambda\, \alpha K_{iz}}{c_v \eta_v} \left[\eta_s\, \Delta \nabla \times \mathbf{u} - \rho_* \left(\frac{d \nabla \times \mathbf{u}}{dt} + \frac{1}{\beta} \frac{d^2 \nabla \times \mathbf{u}}{dt^2}\right)\right] - \frac{\lambda\, \rho_*^2}{T_a^*} \nabla \times \mathbf{L}_h^a.
\end{gathered} \tag{143}$$

The first and second equations in (143) are the dynamic equation of the Cosserat continuum consisting of classical particles whose force interaction is characterized by the symmetrical stress tensor $\tilde{\boldsymbol{\tau}}^{\,s} = \boldsymbol{\tau}^s + p\, \mathbf{E}$ and the stress vector $\tilde{\mathbf{q}} = \mathbf{q} + \mathbf{t}$. We consider the quantities p and $\mathbf{t}$ to be thermodynamic stresses. The constitutive equations for p and $\mathbf{t}$ are represented by the differential equations [the eighth and ninth equations in (143)]. The eleventh equation in (143) is the heat conduction equation. The twelfth one is an auxiliary equation which is necessary to determine vector $\mathbf{t}$. Notice that the eleventh and twelfth equations follow from the angular momentum balance equation for the rotors of gyrostats. It is easy to see that the thermodynamic stresses p and $\mathbf{t}$ vanish when $\eta_v \to \infty$. In this case, the problem of thermoviscoelasticity turns into the hyperbolic type problem of thermoelasticity.

12 Coupled problem of thermoviscoelasticity

The equations of the coupled problem of thermoviscoelasticity written down in the form corresponding to the Cosserat continuum are interesting from the theoretical point of view. For practical purposes, the original equations are more convenient. Now we write down these equations upon using the expressions (see the first part of the paper) for parameters of the model

$$
\begin{gathered}
\nabla \cdot \boldsymbol{\tau}^s - \nabla \times \mathbf{q} + \rho_* \mathbf{f} = \rho_* \frac{d^2 \mathbf{u}}{dt^2} - \frac{\alpha K_{iz} T_a^*}{\beta c_v \eta_v} \frac{d^2 \boldsymbol{\theta}_a}{dt^2}, \qquad \nabla \times \boldsymbol{\mu}_v = 2\mathbf{q}, \qquad \nabla \times \boldsymbol{\varphi} = 0, \\
\nabla \tilde{T}_a - \nabla \times \tilde{\mathbf{M}}_a + \frac{\alpha K_{iz} T_a^*}{c_v \eta_v} \left(\frac{d\mathbf{u}}{dt} + \frac{1}{\beta} \frac{d^2 \mathbf{u}}{dt^2} \right) - \frac{T_a^*}{\lambda} \left(\frac{d\boldsymbol{\theta}_a}{dt} + \frac{1}{\beta} \frac{d^2 \boldsymbol{\theta}_a}{dt^2} \right) = -\rho_* \mathbf{L}_h^a, \\
\boldsymbol{\tau}^s = \left[\left(K_{iz} - \frac{2}{3} G \right) \varepsilon - \alpha K_{iz} \tilde{T}_a \right] \mathbf{E} + 2G \, \boldsymbol{\varepsilon}^s, \qquad \boldsymbol{\varepsilon}^s = \frac{1}{2} \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right), \quad \varepsilon = \operatorname{tr} \boldsymbol{\varepsilon}^s, \qquad (144) \\
\mathbf{q} = \frac{\lambda \, (\eta_q - \eta_s)(K_{ad} - K_{iz})}{c_v \eta_v^2} \boldsymbol{\gamma} - \frac{\alpha K_{iz} T_a^* \, \eta_s}{\rho_* c_v \eta_v} \nabla \times \boldsymbol{\theta}_a, \qquad \boldsymbol{\gamma} = \nabla \times \mathbf{u} - 2\boldsymbol{\varphi}, \\
\nabla \cdot \boldsymbol{\theta}_a = \frac{\rho_* c_v}{T_a^*} \tilde{T}_a + \alpha K_{iz} \, \varepsilon, \qquad \tilde{\mathbf{M}}_a = -\frac{\alpha K_{iz} T_a^* \eta_s}{\rho_* c_v \eta_v} \boldsymbol{\gamma} + \frac{(\eta_q - \eta_s) \, T_a^*}{\lambda \, \rho_*} \nabla \times \boldsymbol{\theta}_a .
\end{gathered}
$$

Here the reference values of ϑ_a and $\boldsymbol{\psi}_a$ are assumed to be equal to zero. Consequently, $\tilde{\vartheta}_a = \nabla \cdot \boldsymbol{\theta}_a$ and $\tilde{\boldsymbol{\psi}}_a = \nabla \times \boldsymbol{\theta}_a$. The first equation in (144) is the linear momentum balance equation for the gyrostats. This equation differs from the analogous equation for the Cosserat continuum by the last term on the right-hand side of the equation. The second equation is the reduced angular momentum balance equation for the carrier bodies of gyrostats, and the third one represents the kinematic restriction related to the rotation of the carrier bodies. The fourth equation in (144) is the angular momentum balance equation for the rotors of gyrostats. It has the thermodynamical sense. If we take the divergence of both sides of this equation and eliminate $\nabla \cdot \boldsymbol{\theta}_a$ by using the tenth equation in (144), we obtain the heat conduction equation. The remaining equations in (144) are the constitutive equations and the expressions for the strain tensors.

It is well known that the classical problem of thermoelasticity can be split into two independent problems. One set of equations describes the volume thermoelastic vibrations. Another set of equations describes the shear vibrations. The proposed statement of the problem of thermoviscoelasticity possesses the same property.

1. The volume thermoviscoelastic vibrations. Now we take the divergence of both sides of the first and fourth equations in (144) and reduce obtained equations by applying the remaining equations of this system. As a result, we get the closed set of equations for unknown functions ε and $\tilde{T}_a$. This set of equations describing the volume thermoviscoelastic vibrations has the form

$$
\begin{gathered}
\left(K_{iz} + \frac{4}{3} G \right) \Delta \varepsilon - \alpha K_{iz} \, \Delta \tilde{T}_a + \rho_* \nabla \cdot \mathbf{f} = \left(\rho_* - \frac{\alpha^2 K_{iz}^2 T_a^*}{\beta c_v \eta_v} \right) \frac{d^2 \varepsilon}{dt^2} - \frac{\rho_* \alpha K_{iz}}{\beta \eta_v} \frac{d^2 \tilde{T}_a}{dt^2}, \\
\Delta \tilde{T}_a - \frac{\rho_* c_v}{\lambda} \left[\frac{d\tilde{T}_a}{dt} + \frac{1}{\beta} \frac{d^2 \tilde{T}_a}{dt^2} \right] = \alpha K_{iz} T_a^* \left(\frac{1}{\lambda} - \frac{1}{c_v \eta_v} \right) \left[\frac{d\varepsilon}{dt} + \frac{1}{\beta} \frac{d^2 \varepsilon}{dt^2} \right] - \rho_* \nabla \cdot \mathbf{L}_h^a. \qquad (145)
\end{gathered}
$$

Comparison of Eq. (145) with the classical equations of the volume thermoelastic vibrations (see, for example, [23]) reveals the following facts. The dynamic equation in (145) contains the second time derivative of the temperature. Such term is absent in the classical dynamic equation. The heat conduction equation in (145) contains the second time derivative of the temperature and the volume strain. Such terms are absent in the classical heat conduction equation. If $\beta^{-1} = 0$, then these terms vanish. In this case, the proposed formulation of the problem differs from the classical one only by the coefficients of the volume strain in the heat conduction equations. In Eq. (145), this coefficient depends on the isentropic volume viscosity, whereas in the classical heat conduction equation, it does not depend on this parameter.

2. The shear viscoelastic vibrations. Let us take the curl operator on both sides of the first and fourth equations of (144) and reduce obtained equations by using the remaining equations of this system. As a result, we obtain the closed set of equations for unknown functions $\nabla \times \mathbf{u}$ and $\nabla \times \boldsymbol{\theta}_a$. This set of equations describing the shear viscoelastic vibrations has the form

$$
\begin{gathered}
G_{ad} \, \Delta \nabla \times \mathbf{u} - \frac{\alpha K_{iz} T_a^* \eta_s}{\rho_* c_v \eta_v} \Delta \nabla \times \boldsymbol{\theta}_a + \rho_* \nabla \times \mathbf{f} = \rho_* \frac{d^2 \nabla \times \mathbf{u}}{dt^2} - \frac{\alpha K_{iz} T_a^*}{\beta c_v \eta_v} \frac{d^2 \nabla \times \boldsymbol{\theta}_a}{dt^2}, \\
(\eta_q - \eta_s) \, \Delta \nabla \times \boldsymbol{\theta}_a - \rho_* \left(\frac{d \nabla \times \boldsymbol{\theta}_a}{dt} + \frac{1}{\beta} \frac{d^2 \nabla \times \boldsymbol{\theta}_a}{dt^2} \right) = \\
= \frac{\lambda \, \alpha K_{iz}}{c_v \eta_v} \left[\eta_s \, \Delta \nabla \times \mathbf{u} - \rho_* \left(\frac{d \nabla \times \mathbf{u}}{dt} + \frac{1}{\beta} \frac{d^2 \nabla \times \mathbf{u}}{dt^2} \right) \right] - \frac{\lambda \rho_*^2}{T_a^*} \nabla \times \mathbf{L}_h^a, \qquad (146)
\end{gathered}
$$

where the parameter G_{ad} is calculated by the formula $G_{ad} = G + A$, i. e.,

$$G_{ad} = G + \frac{\lambda\,(\eta_q - \eta_s)(K_{ad} - K_{iz})}{c_v\,\eta_v^2}\,. \tag{147}$$

We call G_{ad} the adiabatic shear modulus. It is easy to see that if η_s, η_q and β^{-1} are equal to zero, then the first equation in (146) passes into the classical equation of shear vibrations.

13 Overview of classical and quantum theories

We start with the classical statement of the problem of volume thermoelastic vibrations [23]:

$$\Big(K_{iz} + \frac{4}{3}G\Big)\Delta\varepsilon - \alpha K_{iz}\,\Delta\tilde{T}_a + \rho_*\nabla\cdot\mathbf{f} = \rho_*\frac{d^2\varepsilon}{dt^2}, \qquad \Delta\tilde{T}_a - \frac{\rho_* c_v}{\lambda}\frac{d\tilde{T}_a}{dt} = \frac{\alpha K_{iz}T_a^*}{\lambda}\frac{d\varepsilon}{dt} - \frac{\rho_* q}{\lambda}. \tag{148}$$

By representing the solution of Eq. (148) in the form of

$$\varepsilon = A_\varepsilon\, e^{i\omega t - (\gamma + i\delta)s}, \qquad \tilde{T}_a = A_T\, e^{i\omega t - (\gamma + i\delta)s}, \tag{149}$$

we obtain the dispersion relation

$$\Big(K_{iz} + \frac{4}{3}G\Big)(\gamma + i\delta)^4 + \rho_*(\gamma + i\delta)^2\omega^2 - i\,\frac{\rho_* c_v}{\lambda}\left(K_{ad} + \frac{4}{3}G\right)(\gamma + i\delta)^2\omega - i\,\frac{\rho_*^2 c_v}{\lambda}\omega^3 = 0. \tag{150}$$

Let us introduce the notations

$$c = \frac{\omega}{\delta}\,, \qquad \tilde{\gamma} = \frac{\gamma}{\delta}\,. \tag{151}$$

Here c is the phase velocity, $\tilde{\gamma}$ is the dimensionless characteristic of wave attenuation. Using the notations (151) and separating Eq. (150) into the real and imaginary parts we get

$$\frac{c}{\delta}\,\frac{\rho_* c_v}{\lambda}\left[\rho_* c^2 - \left(K_{ad} + \frac{4}{3}G\right)\left(1 - \tilde{\gamma}^2\right)\right] = 2\,\tilde{\gamma}\left[\rho_* c^2 - 2\Big(K_{iz} + \frac{4}{3}G\Big)\left(1 - \tilde{\gamma}^2\right)\right],$$
$$2\,\tilde{\gamma}\,\frac{c}{\delta}\,\frac{\rho_* c_v}{\lambda}\left(K_{ad} + \frac{4}{3}G\right) - \rho_* c^2\left(1 - \tilde{\gamma}^2\right) + \Big(K_{iz} + \frac{4}{3}G\Big)\left(1 - 6\tilde{\gamma}^2 + \tilde{\gamma}^4\right) = 0. \tag{152}$$

An examination of Eq. (152) based on using numerical values of sound velocity, sound attenuation factor and thermomechanical parameters taken from the handbooks reveals that the following approximate form of Eq. (152) can be used

$$\rho_* c^2 - \left(K_{ad} + \frac{4}{3}G\right) = 0, \qquad 2\,\tilde{\gamma}\,\frac{c}{\delta}\,\frac{\rho_* c_v}{\lambda}\left(K_{ad} + \frac{4}{3}G\right) - \rho_* c^2 + \Big(K_{iz} + \frac{4}{3}G\Big) = 0. \tag{153}$$

By using the notations (151) from Eq. (153), we obtain

$$c^2 = \frac{K_{ad} + \frac{4}{3}\,G}{\rho_*}\,, \qquad \frac{2\,\gamma}{\omega^2} = \frac{\lambda(c_p - c_v)}{c^3\rho_* c_p\, c_v\left(1 + \frac{4}{3}\,GK_{ad}^{-1}\right)}\,. \tag{154}$$

According to Eq. (154), the ratio γ/ω^2 should be constant.

Let us discuss the experimental data. In the case of fluids and gases, just the ratio γ/ω^2 can be found in handbooks. However, along with a value of γ/ω^2 the frequency at which it was measured is usually indicated. From this, we can conclude that the dependence of the sound attenuation factor on frequency $\gamma/\omega^2 = \mathrm{const}$ is not accurate. Moreover, Eq. (154) cannot be used to determine the numerical values of the ratio γ/ω^2 at all

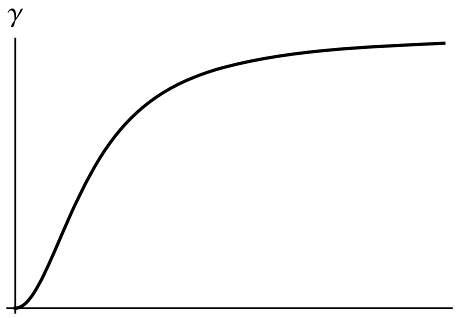

Fig. 7 Dependence of the sound attenuation factor on frequency (Akhiezer mechanism and Landau–Rumer mechanism)

since it is known that the sound attenuation factors of fluids and gases essentially depend on their viscosities. The classical formula describing the sound attenuation in fluids and gases is

$$\frac{2\,\gamma}{\omega^2} = \frac{1}{c^3\rho_*}\left(\eta_v^{cl} + \frac{4}{3}\,\eta_s^{cl} + \frac{\lambda(c_p - c_v)}{c_p\,c_v}\right), \tag{155}$$

Notice that every so often Eq. (155) is written down without the volume viscosity η_v^{cl} (see, for example, [29]).

According to quantum-mechanical ideas [10], [11] in solids, the dependence of sound attenuation factor γ on frequency ω is determined by the formula (see [41], p. 658)

$$\gamma = 1,1\,c_v\,T_a\,\Gamma_*^2\,\frac{\omega^2\tau}{c^3\rho_*(1+\omega^2\tau^2)}\,, \tag{156}$$

where Γ_* is the Grüneisen constant ($\Gamma_* = \alpha K_{ad}/c_v$), τ is the relaxation timescale (of order 10^{-11} s). The diagram of the dependence (156) of γ on ω is given in Fig. 7.

According to the quantum-mechanical concepts, we should apply different models of the phonon absorption subject to a signal frequency and the ratio of the sound wavelength to the mean free path of thermal phonons. In the range of relatively low frequencies ($\omega\tau \ll 1$), the Akhiezer mechanism of absorption takes place [11]. The essence of this mechanism consists in that the acoustic wave disturbs the equilibrium distribution of thermal phonons and, as a consequence, the irreversible process of energy dissipation occurs. In the range of hypersonic frequencies ($\omega\tau \sim 1$), the Landau–Rumer mechanism of absorption takes place [11]. In accordance with this mechanism, the sound absorption is a result of three-particle interaction of sound and thermal phonons. According to Eq. (156) in the range of relatively low frequencies, γ is proportional to ω^2 (the Akhiezer mechanism); in 10^{10} – 10^{11} Hz frequency range, γ is proportional to ω (the Landau–Rumer mechanism); and at higher frequencies, γ tends to a constant value.

14 Thermoviscoelastic volume vibrations

As a result of analysis of the problem (145) of thermoviscoelastic volume vibrations (see Appendix A) analogous to the foregoing analysis of the problem of thermoelastic volume vibrations in the case of the classical thermoelasticity, we obtain the approximate form of the dispersion relations

$$\left(1 - \frac{\lambda(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\right)\left(\frac{\rho_* c^2}{K_{ad}+4G/3} - 1\right) = \frac{\lambda(K_{ad}-K_{iz})}{c_v\eta_v^2}\left(\frac{1}{\beta} - \frac{\eta_v}{K_{ad}+4G/3}\right),$$
$$\frac{2\,\gamma}{\omega^2}\left(1 - \frac{\lambda(K_{ad}-K_{iz})}{c_v\eta_v\,(K_{ad}+4G/3)}\right) = \frac{1}{c^3\rho_*}\left(\frac{\lambda}{c_v} - \eta_v\right)\left(\frac{\rho_* c^2}{K_{ad}+4G/3} - 1\right). \tag{157}$$

According to the second formula in Eq. (157), the ratio of attenuation factor to squared frequency is a constant. This fact agrees with the classical theory of viscoelasticity [29] and the Akhiezer theory of sound absorption [11]. Let us attempt to determine the parameters of the proposed model by means of Eq. (157). We discuss several ways.

Method 1. Let us use the approximate value of sound velocity (154) whose high accuracy is well known. Then, from the first equation in the system (157), we obtain the following relation between the parameters: $\beta^{-1} = \dfrac{\eta_v}{K_{ad} + 4G/3}$, and from the second equation of the system (157), we conclude that $\gamma = 0$. Certainly, this solution is unacceptable.

Method 2. Now instead of Eq. (154), we use the experimental values of sound velocity. All data required for calculations (including the values of sound velocity c and sound attenuation characteristic γ/ω^2) were taken from [29], [42], [43], [44], [45], [46], [47]. In addition to the listed handbooks, we used [28]. By means of Eq. (157), we calculated values of the volume viscosities η_v, η_p and the heat flow relaxation timescale β^{-1}, and also all remaining parameters of the proposed model. An examination of the results shows that for many substances, the values of quantity $1 - B_a^2/J_a$ are negative. This is unacceptable from the theoretical viewpoint since positivity of $1 - B_a^2/J_a$ is the necessary condition of positive definiteness of kinetic energy. Therefore, this method of determination of the parameters should be considered as unsatisfactory.

Method 3. Assuming that the first equation in the system (157) is satisfied approximately, we exclude it from consideration. Instead of this equation, we use the formula for determination of the heat flow relaxation timescale which is obtained from the statement that the velocity of thermal wave propagation is equal to the sound velocity [13]: $\beta^{-1} \approx \dfrac{\lambda}{c_v(K_{ad} + 4G/3)}$. To determine η_v we use the second equation in the system (157) substituting the experimental values of c^2 and γ/ω^2 in it. As shown by calculations, sufficiently high accuracy of determining the value of η_v is provided by the approximate form of this equation

$$\eta_v \approx \frac{\lambda}{c_v} + \rho_* c^3 \left(\frac{2\,\gamma}{\omega^2}\right) \Bigg/ \left(1 - \frac{\rho_* c^2}{K_{ad} + 4G/3}\right). \tag{158}$$

The results of calculation of viscosity characteristics and the heat flow relaxation timescale carried out by using (157) and (158) for some solids, fluids, and gases are given in Table 1. It is easy to see that the isentropic volume viscosity can be both positive and negative. The first term on the right-hand side of Eq. (158) being small compared to the $|\eta_v|$, the sign of the second term on the right-hand side of this equation determines the sign of η_v. If $c^2 < (K_{ad} + 4G/3)/\rho_*$ then η_v is positive, otherwise it is negative. The isobaric volume viscosity η_p is always positive. The results of calculation of the parameters B_a, J_a and D_a for some fluids and gases are given in Table 2. It is easy to see that the values of $1 - B_a^2/J_a$ are positive for all substances. This fact guarantees the positive definiteness of kinetic energy. Consequently, this method of determination of the parameters does not contain any theoretical contradictions, and it can be recognized as wholly satisfactory. The last column of Table 2 contains the values of coefficient of the derivative with respect to the spatial coordinate in the equation of vortex motion of a viscous fluid in the case of a fast process (see the first part of the paper). Let us pay attention to the fact that this coefficient is positive for all substances.

Table 1 Volume viscosities and the heat flow relaxation time scale for solids, fluids and gases calculated by using the sound velocity and the sound attenuation factor

Substance	$\eta_v\ \left(\frac{\text{kg}}{\text{m}\cdot\text{s}}\right)$	$\dfrac{\lambda}{c_v\eta_v}$	$\eta_p\ \left(\frac{\text{kg}}{\text{m}\cdot\text{s}}\right)$	β^{-1} (s)
Helium (g)	$1,10\cdot10^{-2}$	$4,53\cdot10^{-3}$	$2,98\cdot10^{-5}$	$2,92\cdot10^{-10}$
Air (g)	$-1,24$	$-2,88\cdot10^{-5}$	$2,55\cdot10^{-5}$	$2,52\cdot10^{-10}$
Glycerin (f)	$8,87$	$1,50\cdot10^{-5}$	$1,18\cdot10^{-4}$	$2,57\cdot10^{-14}$
Water (f)	$-3,33\cdot10^{-1}$	$-4,31\cdot10^{-4}$	$1,43\cdot10^{-4}$	$6,50\cdot10^{-14}$
Glass (s)	$-2,78\cdot10^{3}$	$-3,55\cdot10^{-7}$	$9,87\cdot10^{-4}$	$1,37\cdot10^{-14}$
Copper (s)	$-8,62\cdot10^{3}$	$-1,22\cdot10^{-4}$	$1,03$	$5,51\cdot10^{-12}$
Aluminium (s)	$-3,61\cdot10^{3}$	$-7,76\cdot10^{-5}$	$2,69\cdot10^{-1}$	$2,69\cdot10^{-12}$
Mercury (f)	$5,90\cdot10^{-1}$	$4,12\cdot10^{-1}$	$2,23\cdot10^{-1}$	$8,12\cdot10^{-12}$
Lead (s)	$-1,42\cdot10^{4}$	$-1,97\cdot10^{-5}$	$2,67\cdot10^{-1}$	$7,00\cdot10^{-12}$
Lead (f)	$5,14$	$2,12\cdot10^{-2}$	$1,00\cdot10^{-1}$	$3,33\cdot10^{-12}$

Table 2 Values of the parameters of proposed model for fluids and gases calculated by using the sound velocity and the sound attenuation factor

Substance	$B_a\ \left(\frac{\mathrm{K}\cdot\mathrm{m}^2}{\mathrm{N}}\right)$	$J_a\ \left(\frac{\mathrm{K}^2\cdot\mathrm{m}^4}{\mathrm{N}^2}\right)$	$D_a\ (K)$	$k\ \left(\frac{\mathrm{N}}{\mathrm{m}^2}\right)$
Helium (g)	$-4,85\cdot 10^{-6}$	$2,87\cdot 10^{-6}$	$-3,14\cdot 10^{-1}$	$5,31\cdot 10^{-1}$
Air (g)	$2,21\cdot 10^{-8}$	$2,07\cdot 10^{-6}$	$1,51\cdot 10^{-3}$	$1,62\cdot 10^{-5}$
Glycerin (f)	$-7,25\cdot 10^{-13}$	$2,11\cdot 10^{-14}$	$-39,2$	$1,35\cdot 10^{3}$
Water (f)	$4,52\cdot 10^{-12}$	$3,18\cdot 10^{-14}$	$7,31\cdot 10^{-2}$	$10,4$
Mercury (f)	$-1,16\cdot 10^{-8}$	$6,03\cdot 10^{-15}$	$-2,28$	$4,48\cdot 10^{6}$
Lead (f)	$-6,78\cdot 10^{-10}$	$1,23\cdot 10^{-14}$	$-4,30\cdot 10^{-1}$	$2,38\cdot 10^{5}$

Now we consider the hypersonic frequency range. By means of the asymptotic analysis of the dispersion relations that is based on the assumptions valid for hypersonic frequencies (see Appendix B), we obtain

$$1-\frac{c_v\eta_v}{\lambda}=\frac{K_{iz}+4G/3}{\rho_*}\left(\frac{\gamma}{\omega}\right)^2+\frac{K_{iz}+4G/3}{K_{ad}+4G/3}\left(\frac{\rho_* c^2}{K_{ad}+4G/3}-1\right). \tag{159}$$

We suppose the velocity of longitudinal acoustic wave propagation to be approximated by the expression $c^2=\dfrac{K_{ad}+4G/3}{\rho_*}$ to a high degree of accuracy, i. e.,

$$\left|\frac{\rho_* c^2}{K_{ad}+4G/3}-1\right|\ll c^2\left(\frac{\gamma}{\omega}\right)^2. \tag{160}$$

In this case, the second term on the right-hand side of Eq. (159) is small compared to the first one. Hence, Eq. (159) can be simplified by retaining only the first term on the right-hand side of this equation

$$1-\frac{c_v\eta_v}{\lambda}=\frac{K_{iz}+4G/3}{\rho_*}\left(\frac{\gamma}{\omega}\right)^2. \tag{161}$$

According to Eq. (161), the ratio of sound attenuation factor to frequency is a constant. This fact agrees with the Landau–Rumer theory of sound absorption [11]. If the ratio γ/ω is known and inequality (160) is valid, then Eq. (161) allows us to determine the isentropic volume viscosity:

$$\eta_v=\frac{\lambda}{c_v}\left[1-\frac{K_{iz}+4G/3}{\rho_*}\left(\frac{\gamma}{\omega}\right)^2\right]. \tag{162}$$

To calculate the heat flow relaxation timescale β^{-1}, we can use Eq. (175) obtained in Appendix B and the approximate formula for the volume viscosity η_v based on the assumption of smallness of the second term in the expression in square brackets in Eq. (162):

$$\beta^{-1}\approx\frac{\eta_v}{K_{ad}+4G/3},\qquad \eta_v\approx\frac{\lambda}{c_v}\qquad\Rightarrow\qquad \beta^{-1}\approx\frac{\lambda}{c_v(K_{ad}+4G/3)}. \tag{163}$$

The last expression in (163) coincides with the expression for β^{-1} derived on the basis of statement that the velocity of heat wave propagation is equal to the sound velocity. Consequently, the estimation of heat flow relaxation timescale based on the asymptotic analysis of the dispersion relations of the proposed model in the hypersonic frequency range coincides with the estimation based on the quantum-mechanical considerations.

The experimental values of the ratio γ/ω for solids are given in handbooks. However, these values correspond to the low-frequency range in which another relation ($\gamma/\omega^2=\mathrm{const}$) should be satisfied. According to the Landau–Rumer theory, the relation $\gamma/\omega=\mathrm{const}$ is satisfied in the hypersonic frequency range. The experimental data corresponding to this frequency range are absent in handbooks. That is why we cannot use Eq. (162) to determine the numerical values of the isentropic volume viscosity.

15 Viscoelastic shear vibrations

Now we consider the problem (146) of viscoelastic shear vibrations. Representing the solution of problem as

$$\nabla \times \mathbf{u} = \mathbf{A}_u \, e^{i\omega t-(\gamma+i\delta)s}, \qquad \nabla \times \boldsymbol{\theta}_a = \mathbf{A}_\theta \, e^{i\omega t-(\gamma+i\delta)s} \tag{164}$$

we obtain the dispersion relations the asymptotic analysis of which can be found in Appendix C. It is proved that for low frequencies, the dispersion relations can be written in the approximate form

$$\left(1-\frac{\lambda(K_{ad}-K_{iz})}{\beta c_v \eta_v^2}\right)\left(\frac{\rho_* c^2}{G_{ad}}-1\right) = \frac{\lambda(K_{ad}-K_{iz})}{c_v\eta_v^2}\left(\frac{1}{\beta}-\frac{\eta_s}{G_{ad}}\right),$$
$$\frac{2\,\gamma}{\omega^2}\left(1-\frac{\lambda\eta_s(K_{ad}-K_{iz})}{G_{ad}\,c_v\eta_v^2}\right) = \frac{1}{c^3\rho_*}\left(\eta_q-2\eta_s-\frac{2\lambda\eta_s(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\right)\left(\frac{\rho_* c^2}{G_{ad}}-1\right). \tag{165}$$

If the well-known appoximate formula $c^2 = G_{ad}/\rho_*$ for the velocity of transverse wave propagation is used, then from the first equation in Eq. (165), it follows that $\beta^{-1} = \eta_s/G_{ad}$, and from the second equation in Eq. (165), we obtain that $\gamma = 0$. This solution is unacceptable.

By using the experimental values of the velocity of transverse wave propagation c and the attenuation characteristic γ/ω^2 from Eq. (165), we can find the values of the shear viscosities η_s, η_q, and after that, we can calculate all remaining parameters concerned with the shear vibrations. However, as well as in the case of volume vibrations, this method of determination of parameters of the proposed model does not give satisfactory results.

Assuming that the first equation in the system (165) is satisfied approximately, we exclude it from consideration. To determine η_s and η_q, we use the relation $\eta_q - \eta_s \approx \dfrac{G_{ad}}{\beta}$ substantiation of which is given in Appendix C, the second equation in the system (165) and the experimental values of c^2 and γ/ω^2. As shown by calculations, the sufficiently high accuracy of determination of the shear viscosities is provided by the following approximate form of the second equation in (165)

$$\eta_q - 2\eta_s = \rho_* c^3 \left(\frac{2\,\gamma}{\omega^2}\right) \Big/ \left(\frac{\rho_* c^2}{G_{ad}}-1\right). \tag{166}$$

The results of calculation of the shear viscosities and the third elastic modulus A for some solids are given in Table 3. For substance marked by asterisk, we failed to find the value of transverse wave attenuation factor. Instead of it, we used the value of longitudinal wave attenuation factor. It is easy to see that the shear viscosity η_s can be both positive and negative, but the difference $\eta_q - \eta_s$ is always positive. The third elastic modulus A is negligibly small compared to the shear modulus. The results of calculation of the parameters B_a, J_a, D_a, and $\varGamma_a$ for some solids are given in Table 4. It is easy to see that the values of $1 - B_a^2/J_a$ are positive that guarantees the positive definiteness of kinetic energy. An examination of the values of parameters A, D_a and $\varGamma_a$ shows that if the stability for shear disturbances is not ensured by the shear modulus, then the vortex motions would occur in the matter.

An asymptotic analysis of the dispersion relations that is based on the assumptions valid for hypersonic frequencies can be found in Appendix C.

Table 3 Shear viscosities for solids calculated by using the velocity of transverse wave propagation and the transverse wave attenuation factor

Substance	$\eta_s\left(\frac{\mathrm{kg}}{\mathrm{m\cdot s}}\right)$	$\eta_q-\eta_s\left(\frac{\mathrm{kg}}{\mathrm{m\cdot s}}\right)$	$A\left(\frac{\mathrm{N}}{\mathrm{m}^2}\right)$	$\frac{A}{G_{ad}}$
Glass* (s)	$-1,87\cdot 10^{2}$	$3,29\cdot 10^{-4}$	$7,59\cdot 10^{-7}$	$3,16\cdot 10^{-17}$
Copper (s)	$7,84\cdot 10^{2}$	$3,63\cdot 10^{-1}$	$12,2$	$2,56\cdot 10^{-10}$
Aluminium (s)	$-1,08\cdot 10^{2}$	$6,62\cdot 10^{-2}$	$4,12$	$1,67\cdot 10^{-10}$
Lead (s)	$4,09\cdot 10^{3}$	$4,50\cdot 10^{-2}$	$8,39\cdot 10^{-2}$	$1,31\cdot 10^{-11}$

Table 4 Values of the parameters of proposed model for solids calculated by using the velocity of transverse wave propagation and the transverse wave attenuation factor

Substance	$B_a\ \left(\frac{\mathrm{K}\cdot\mathrm{m}^2}{\mathrm{N}}\right)$	$J_a\ \left(\frac{\mathrm{K}^2\cdot\mathrm{m}^4}{\mathrm{N}^2}\right)$	$D_a\ (K)$	$\Gamma_a\ \left(\frac{\mathrm{K}^2\cdot\mathrm{m}^2}{\mathrm{N}}\right)$
Glass* (s)	$2,62\cdot10^{-16}$	$2,17\cdot10^{-15}$	$-3,58$	$5,21\cdot10^{-5}$
Copper (s)	$3,39\cdot10^{-13}$	$4,49\cdot10^{-16}$	$48,2$	$2,15\cdot10^{-5}$
Aluminium (s)	$4,54\cdot10^{-13}$	$1,23\cdot10^{-15}$	$-18,2$	$3,03\cdot10^{-5}$
Lead (s)	$2,60\cdot10^{-13}$	$5,20\cdot10^{-15}$	$1,52\cdot10^{2}$	$3,34\cdot10^{-5}$

16 Conclusion

The new theory of thermoviscoelasticity is proposed. Neither the standard approaches based on the hypothesis of fading memory nor the method of rheological models are used to construct this theory. The proposed approach is based on the idea of using the mechanical model of a one-rotor gyrostat continuum in order to describe behavior of the ordinary material medium possessing not only mechanical properties but also the thermal ones. The analysis of mathematical formulation of the proposed theory shows the following.

1. In special cases, the mathematical description of the mechanical model of a one-rotor gyrostat continuum can be reduced to the equations of the coupled problem of thermoelasticity, the self-diffusion equation, and the equation describing the flow of viscous incompressible fluid.
2. Proceeding from some theoretical considerations based on the concept of the "thermal ether" and the analysis of some model problems the original treatment of physical nature of the mechanism of thermal conduction and internal damping is proposed. The volume and shear viscosities introduced in the context of the proposed model are proved to be different from the analogous quantities used in the known theories.
3. In the framework of proposed theory, the generalized equation of the vortex motion of a viscous fluid is obtained. Emphasize that the foregoing equation is obtained in the framework of a linear theory. In the case of slow process, the asymptotically principal term of this equation coincides with the equation of the vortex motion of an incompressible Newtonian viscous fluid. In the case of fast process, the asymptotically principal term of this equation contains increasing in time solutions.
4. The proposed theory (constructed in frame of the classical mechanics) describes the acoustic wave absorption law being in quantitative agreement with the law obtained on the basis of phonon theory. Namely, in the range of relatively low frequencies, the attenuation factor is proportional to the squared frequency (the Akhiezer mechanism of absorption), and in the area of hypersonic frequencies, the attenuation factor is proportional to the frequency (the Landau–Rumer mechanism of absorption).
5. The formula for the heat flow relaxation timescale obtained on the basis of the asymptotic analysis of the dispersion relations in the area of hypersonic frequencies coincides with the formula obtained by means of the quantum-mechanical approach.
6. The proposed theory of thermoviscoelasticity is the hyperbolic type theory. The terms ensuring the hyperbolicity of the heat conduction equation are not important to describe the process of heat transfer in macroscopic objects. However, leaving out these terms we cannot describe the quantum-mechanical effects.

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A Analysis of dispersion relations in the case of thermoviscoelastic volume vibrations at low frequencies

Representing the solution of problem (145) of thermoviscoelastic volume vibrations in the form of (149), we obtain the dispersion relation

$$\Big(K_{iz}+\frac{4}{3}G\Big)(\gamma+i\delta)^4+\rho_*\left(1-\frac{2(K_{ad}-K_{iz})}{\beta\eta_v}+\frac{c_v}{\beta\lambda}\Big(K_{ad}+\frac{4}{3}G\Big)\right)(\gamma+i\delta)^2\omega^2+ \\ +\frac{\rho_*^2c_v}{\beta\lambda}\left(1-\frac{\lambda(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\right)\omega^4-i\,\frac{\rho_*c_v}{\lambda}\left(K_{ad}+\frac{4}{3}G-\frac{\lambda(K_{ad}-K_{iz})}{c_v\eta_v}\right)(\gamma+i\delta)^2\omega- \\ -i\,\frac{\rho_*^2c_v}{\lambda}\left(1-\frac{\lambda(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\right)\omega^3=0. \quad (167)$$

Now we rewrite Eq. (167) by using the notations (151) and separate this equation into its imaginary part

$$\frac{c}{\delta}\,\frac{\rho_*c_v}{\lambda}\left[\rho_*\Big(1-\frac{\lambda(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\Big)c^2-\Big(K_{ad}+\frac{4}{3}G-\frac{\lambda(K_{ad}-K_{iz})}{c_v\eta_v}\Big)\Big(1-\tilde\gamma^2\Big)\right]= \\ =2\,\tilde\gamma\left[\rho_*\Big(1-\frac{2(K_{ad}-K_{iz})}{\beta\eta_v}+\frac{c_v}{\beta\lambda}\Big(K_{ad}+\frac{4}{3}G\Big)\Big)c^2-2\Big(K_{iz}+\frac{4}{3}G\Big)\Big(1-\tilde\gamma^2\Big)\right] \quad (168)$$

and its real part

$$2\,\tilde\gamma\,\frac{c}{\delta}\,\frac{\rho_*c_v}{\lambda}\Big(K_{ad}+\frac{4}{3}G-\frac{\lambda(K_{ad}-K_{iz})}{c_v\eta_v}\Big)+\frac{\rho_*^2c_v}{\beta\lambda}\Big(1-\frac{\lambda(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\Big)c^4-\Big(1-\tilde\gamma^2\Big)\times \\ \times\rho_*\Big(1-\frac{2(K_{ad}-K_{iz})}{\beta\eta_v}+\frac{c_v}{\beta\lambda}\Big(K_{ad}+\frac{4}{3}G\Big)\Big)c^2+\Big(1-6\tilde\gamma^2+\tilde\gamma^4\Big)\Big(K_{iz}+\frac{4}{3}G\Big)=0. \quad (169)$$

Notice that the diagram of dependence of sound attenuation factor on frequency that corresponds to the dispersion relations (168) and (169) has the same form as the diagram corresponding to the quantum-mechanical concepts — see Fig. 7.

As a result of the analysis of Eqs. (168) and (169) analogous to that carried out in the case of classical theory of thermoelasticity, we obtain the following approximate form of Eqs. (168) and (169)

$$\rho_*\left(1-\frac{\lambda(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\right)c^2-\left(K_{ad}+\frac{4}{3}G-\frac{\lambda(K_{ad}-K_{iz})}{c_v\eta_v}\right)=0, \\ 2\,\tilde\gamma\,\frac{c}{\delta}\,\frac{\rho_*c_v}{\lambda}\left(K_{ad}+\frac{4}{3}G-\frac{\lambda(K_{ad}-K_{iz})}{c_v\eta_v}\right)+\frac{\rho_*^2c_v}{\beta\lambda}\left(1-\frac{\lambda(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\right)c^4- \\ -\rho_*\left(1-\frac{2(K_{ad}-K_{iz})}{\beta\eta_v}+\frac{c_v}{\beta\lambda}\Big(K_{ad}+\frac{4}{3}G\Big)\right)c^2+K_{iz}+\frac{4}{3}G=0. \quad (170)$$

Transforming Eq. (170) by using the notations (151), we get the more convenient form of these equations

$$\left(1-\frac{\lambda(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\right)\left(\frac{\rho_*c^2}{K_{ad}+4G/3}-1\right)=\frac{\lambda(K_{ad}-K_{iz})}{c_v\eta_v^2}\left(\frac{1}{\beta}-\frac{\eta_v}{K_{ad}+4G/3}\right), \\ \frac{2\,\gamma}{\omega^2}\left(1-\frac{\lambda(K_{ad}-K_{iz})}{c_v\eta_v\,(K_{ad}+4G/3)}\right)=\frac{1}{c^3\rho_*}\left(\frac{\lambda}{c_v}-\eta_v\right)\left(\frac{\rho_*c^2}{K_{ad}+4G/3}-1\right). \quad (171)$$

B Analysis of dispersion relations in the case of thermoviscoelastic volume vibrations at hypersonic frequencies

Now we consider Eqs. (168) and (169). Assuming that in the case of acoustical vibrations $\tilde\gamma$ is small we neglect the second-order and higher-order terms of $\tilde\gamma$ in the foregoing equations. Next eliminating $\tilde\gamma$ from the obtained set of equations, we get the equation for phase velocity

$$\frac{c^2}{\delta^2}\left(\frac{\rho_*c_v}{\lambda}\right)^2\left(K_{ad}+\frac{4}{3}G-\frac{\lambda(K_{ad}-K_{iz})}{c_v\eta_v}\right)\left[\rho_*\left(1-\frac{\lambda(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\right)c^2-\left(K_{ad}+\frac{4}{3}G-\frac{\lambda(K_{ad}-K_{iz})}{c_v\eta_v}\right)\right]= \\ =-\left[\rho_*\left(1-\frac{2(K_{ad}-K_{iz})}{\beta c_v\eta_v}+\frac{c_v}{\beta\lambda}\Big(K_{ad}+\frac{4}{3}G\Big)\right)c^2-2\Big(K_{iz}+\frac{4}{3}G\Big)\right]\times \\ \times\left[\frac{\rho_*^2c_v}{\beta\lambda}\Big(1-\frac{\lambda(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\Big)c^4-\rho_*\Big(1-\frac{2(K_{ad}-K_{iz})}{\beta\eta_v}+\frac{c_v}{\beta\lambda}\Big(K_{ad}+\frac{4}{3}G\Big)\Big)c^2+\Big(K_{iz}+\frac{4}{3}G\Big)\right]. \quad (172)$$

The velocity of acoustic signal propagation is known to weakly depend on frequency. Therefore, we suppose that the phase velocity does not depend on frequency (or, equivalently, the wave number) with asymptotic error $O(\tilde{\gamma}^2)$, i. e., with the accuracy of Eq. (172). It is obvious that c^2 does not depend on δ^2 if and only if a root of the polynomial on the left-hand side of Eq. (172) coincides with one of the roots of the polynomial of third degree with respect to c^2 on the right-hand side of this equation. There are two possibilities. The root of the polynomial on the left-hand side of Eq. (172) coincides either with the root of the polynomial of second degree or with the root of the polynomial of first degree on the right-hand side of this equation. The latter case leads to meaningless results. Therefore, we consider the case when the root of the polynomial on the left-hand side of Eq. (172)

$$c^2 = \left(K_{ad} + \frac{4}{3}G - \frac{\lambda(K_{ad} - K_{iz})}{c_v\eta_v}\right)\Big/\left[\rho_*\left(1 - \frac{\lambda(K_{ad} - K_{iz})}{\beta c_v\eta_v^2}\right)\right] \tag{173}$$

coincides with the root of the polynomial of second degree, i. e., satisfies the equation

$$\frac{\rho_*^2 c_v}{\beta\lambda}\left(1 - \frac{\lambda(K_{ad} - K_{iz})}{\beta c_v\eta_v^2}\right)c^4 - \rho_*\left(1 - \frac{2(K_{ad} - K_{iz})}{\beta\eta_v} + \frac{c_v}{\beta\lambda}\left(K_{ad} + \frac{4}{3}G\right)\right)c^2 + \left(K_{iz} + \frac{4}{3}G\right) = 0. \tag{174}$$

It is easy to show that Eq. (173) satisfies Eq. (174) only when the following relation between the parameters takes place:

$$\beta^{-1} \approx \frac{\eta_v}{K_{ad} + 4G/3}\,. \tag{175}$$

We emphasize that Eq. (175) is approximate since we obtained it proceeding from the idea of behavior of the asymptotically principal term of phase velocity. It is easy to show that the exact satisfaction of Eq. (175) leads to the acoustic oscillations turn out undamped.

Substituting Eq. (175) into Eq. (173) after simple transformations, we obtain the well-known expression for the longitudinal acoustic wave propagation velocity

$$c^2 \approx \frac{K_{ad} + 4G/3}{\rho_*}\,. \tag{176}$$

Now we return to Eqs. (168) and (169). Eliminating the ratio c/δ from this set of equations, we transform the obtained equation in view of Eqs. (175) and (176) retaining only the asymptotically principal term $\tilde{\gamma}$ in it. In view of $\tilde{\gamma} = \gamma/\delta$ and $c = \omega/\delta$, we get

$$1 - \frac{c_v\eta_v}{\lambda} = \frac{K_{iz} + 4G/3}{\rho_*}\left(\frac{\gamma}{\omega}\right)^2 + \frac{K_{iz} + 4G/3}{K_{ad} + 4G/3}\left(\frac{\rho_* c^2}{K_{ad} + 4G/3} - 1\right). \tag{177}$$

C Analysis of dispersion relations in the case of viscoelastic shear vibrations

Representing the solution of problem (146) of viscoelastic shear vibrations in the form of (164), we obtain the dispersion relation

$$\begin{aligned}
&\left((\eta_q - \eta_s)\,G_{ad} - \frac{\lambda\,\eta_s^2(K_{ad} - K_{iz})}{c_v\eta_v^2}\right)(\gamma + i\delta)^4 + \frac{\rho_*^2}{\beta}\left(1 - \frac{\lambda(K_{ad} - K_{iz})}{\beta c_v\eta_v^2}\right)\omega^4 + \\
&\quad + \rho_*\left[\eta_q - \eta_s + \frac{1}{\beta}\left(G_{ad} - \frac{2\lambda\,\eta_s(K_{ad} - K_{iz})}{c_v\eta_v^2}\right)\right](\gamma + i\delta)^2\omega^2 - \\
&\quad - i\,\rho_*\left(G_{ad} - \frac{\lambda\,\eta_s(K_{ad} - K_{iz})}{c_v\eta_v^2}\right)(\gamma + i\delta)^2\omega - i\,\rho_*^2\left(1 - \frac{\lambda(K_{ad} - K_{iz})}{\beta c_v\eta_v^2}\right)\omega^3 = 0.
\end{aligned} \tag{178}$$

We rewrite Eq. (178) by using the notations (151) and separate this equation into its imaginary part

$$\begin{aligned}
&\frac{c}{\delta}\left[\rho_*\left(1 - \frac{\lambda(K_{ad} - K_{iz})}{\beta c_v\eta_v^2}\right)c^2 - \left(G_{ad} - \frac{\lambda\,\eta_s(K_{ad} - K_{iz})}{c_v\eta_v^2}\right)\left(1 - \tilde{\gamma}^2\right)\right] = \\
&= 2\,\tilde{\gamma}\left[\left(\eta_q - \eta_s + \frac{1}{\beta}\left(G_{ad} - \frac{2\lambda\,\eta_s(K_{ad} - K_{iz})}{c_v\eta_v^2}\right)\right)c^2 - \frac{2}{\rho_*}\left((\eta_q - \eta_s)\,G_{ad} - \frac{\lambda\,\eta_s^2(K_{ad} - K_{iz})}{c_v\eta_v^2}\right)\left(1 - \tilde{\gamma}^2\right)\right]
\end{aligned} \tag{179}$$

and its real part

$$2\,\tilde\gamma\,\frac{c}{\delta}\left(G_{ad}-\frac{\lambda\,\eta_s(K_{ad}-K_{iz})}{c_v\eta_v^2}\right)+\frac{\rho_*}{\beta}\left(1-\frac{\lambda(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\right)c^4- \\ -\left(1-\tilde\gamma^2\right)\left[\eta_q-\eta_s+\frac{G_{ad}}{\beta}\left(1-\frac{2\lambda\,\eta_s(K_{ad}-K_{iz})}{G_{ad}\,c_v\eta_v^2}\right)\right]c^2+ \\ +\left(1-6\tilde\gamma^2+\tilde\gamma^4\right)\frac{G_{ad}}{\rho_*}\left(\eta_q-\eta_s-\frac{\lambda\,\eta_s^2(K_{ad}-K_{iz})}{G_{ad}\,c_v\eta_v^2}\right)=0. \quad (180)$$

Notice that the structure of the dispersion relations (179) and (180) is the same as the structure of the dispersion relations (168) and (169) corresponding to the problem of volume vibrations. Let us carry out the analysis of Eqs. (179) and (180) analogous to what was carried out in Appendix A for the case of volume vibrations. The approximate form of Eqs. (179) and (180) is

$$\left(1-\frac{\lambda(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\right)c^2-\frac{G_{ad}}{\rho_*}\left(1-\frac{\lambda\,\eta_s(K_{ad}-K_{iz})}{G_{ad}\,c_v\eta_v^2}\right)=0, \\ 2\,\tilde\gamma\,\frac{c}{\delta}\left(G_{ad}-\frac{\lambda\,\eta_s(K_{ad}-K_{iz})}{c_v\eta_v^2}\right)+\frac{\rho_*}{\beta}\left(1-\frac{\lambda(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\right)c^4- \\ -\left[\eta_q-\eta_s+\frac{G_{ad}}{\beta}\left(1-\frac{2\lambda\,\eta_s(K_{ad}-K_{iz})}{G_{ad}\,c_v\eta_v^2}\right)\right]c^2+\frac{G_{ad}}{\rho_*}\left(\eta_q-\eta_s-\frac{\lambda\,\eta_s^2(K_{ad}-K_{iz})}{G_{ad}\,c_v\eta_v^2}\right)=0. \quad (181)$$

Transforming Eq. (181) by using the notations (151), we obtain the more convenient form of these equations:

$$\left(1-\frac{\lambda(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\right)\left(\frac{\rho_*c^2}{G_{ad}}-1\right)=\frac{\lambda(K_{ad}-K_{iz})}{c_v\eta_v^2}\left(\frac{1}{\beta}-\frac{\eta_s}{G_{ad}}\right), \\ \frac{2\,\gamma}{\omega^2}\left(1-\frac{\lambda\eta_s(K_{ad}-K_{iz})}{G_{ad}\,c_v\eta_v^2}\right)=\frac{1}{c^3\rho_*}\left(\eta_q-2\eta_s-\frac{2\lambda\eta_s(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\right)\left(\frac{\rho_*c^2}{G_{ad}}-1\right). \quad (182)$$

Now we carry out the analysis of Eqs. (179) and (180) analogous to what was carried out in Appendix B for the case of volume vibrations. In view of the fact that in the case of mechanical vibrations $\tilde\gamma$ is small we neglect the second-order and higher-order terms of $\tilde\gamma$ in Eqs. (179) and (180). Eliminating $\tilde\gamma$ from the obtained set of equations, we obtain the equation for phase velocity

$$\frac{c^2}{\delta^2}\left[G_{ad}-\frac{\lambda\,\eta_s(K_{ad}-K_{iz})}{c_v\eta_v^2}\right]\left[\rho_*\left(1-\frac{\lambda(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\right)c^2-G_{ad}+\frac{\lambda\,\eta_s(K_{ad}-K_{iz})}{c_v\eta_v^2}\right]= \\ =-\left[\frac{\rho_*}{\beta}\left(1-\frac{\lambda(K_{ad}-K_{iz})}{\beta c_v\eta_v^2}\right)c^4-\left(\eta_q-\eta_s+\frac{G_{ad}}{\beta}\left(1-\frac{2\lambda\,\eta_s(K_{ad}-K_{iz})}{G_{ad}\,c_v\eta_v^2}\right)\right)c^2+\right. \\ \left.+\frac{G_{ad}}{\rho_*}\left(\eta_q-\eta_s-\frac{\lambda\,\eta_s^2(K_{ad}-K_{iz})}{G_{ad}\,c_v\eta_v^2}\right)\right]\left[\left(\eta_q-\eta_s+\frac{G_{ad}}{\beta}\left(1-\frac{2\lambda\,\eta_s(K_{ad}-K_{iz})}{G_{ad}\,c_v\eta_v^2}\right)\right)c^2-\right. \\ \left.-\frac{2\,G_{ad}}{\rho_*}\left(\eta_q-\eta_s-\frac{\lambda\,\eta_s^2(K_{ad}-K_{iz})}{G_{ad}\,c_v\eta_v^2}\right)\right]. \quad (183)$$

It is easy to show that c^2 does not depend on δ^2 (i. e., c^2 is the root of the polynomial on the left-hand side of Eq. (183) and the root of the polynomial of second degree on the right-hand side of this equation) if the following conditions are valid

$$\eta_s\approx\frac{G_{ad}}{\beta}\,,\qquad c^2\approx\frac{G_{ad}}{\rho_*}\,. \quad (184)$$

The relations (184) are approximate since we obtained them proceeding from the idea of behavior of the asymptotically principal term of phase velocity. It can be shown that the exact satisfaction of the first equation in (184) leads to the shear vibrations turn out undamped.

We return to Eqs. (179) and (180). Eliminating the ratio c/δ from this set of equations, we transform the obtained equation in view of Eq. (184) retaining only the asymptotically principal term $\tilde{\gamma}$ in it. As a result, we get

$$\eta_q - 2\eta_s = \beta^{-1} G_{ad} \left(1 - \frac{\lambda(K_{ad} - K_{iz})}{\beta c_v \eta_v^2}\right) \left[\frac{G_{ad}}{\rho_*}\left(\frac{\gamma}{\omega}\right)^2 + \frac{\rho_* c^2}{G_{ad}} - 1\right]. \tag{185}$$

Suppose that the velocity of transverse wave propagation is accurately approximated by the expression $c^2 \approx \dfrac{G_{ad}}{\rho_*}$, i. e.,

$$\left|\frac{\rho_* c^2}{G_{ad}} - 1\right| \ll c^2 \left(\frac{\gamma}{\omega}\right)^2. \tag{186}$$

In this case, Eq. (185) can be simplified by retaining only the first term in the expression in square brackets:

$$\eta_q - 2\eta_s = \frac{G_{ad}^2}{\beta\rho_*} \left(1 - \frac{\lambda(K_{ad} - K_{iz})}{\beta c_v \eta_v^2}\right) \left(\frac{\gamma}{\omega}\right)^2. \tag{187}$$

Assuming the right-hand side of Eq. (187) is small and using the approximate expression for shear viscosity $\eta_s \approx \dfrac{G_{ad}}{\beta}$ we obtain

$$\eta_q - 2\eta_s \approx 0 \qquad \Rightarrow \qquad \eta_q - \eta_s \approx \frac{G_{ad}}{\beta}. \tag{188}$$

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