On effective stiffness of a three-layered plate with a core filled by a capillary fluid

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ABSTRACT: We discuss the effective (apparent) stiffness of a nano- or microsized thin-walled structure consisting of two elastic flexible faces and a capillary fluid between them. We take into account the capillary forces acting on a free surface of the fluid. These capillary effects play a significant role at the micro- and nanoscales and influence mechanical properties of such structures. We introduce the effective stiffness parameters of considered three-layered plate as functions depending on the elastic properties of faces as well as the surface tension.

1 INTRODUCTION

Capillarity phenomena coupled with deformations of elastic flexible structures play a significant role in design of MEMS, medicine, and biology. At the nano- and microscales such phenomena can be observed in various flexible structures interacting with fluids (Madasu & Cairncross 2003, Hazel & Heil 2003, Grotberg & Jensen 2004, Hazel & Heil 2005, Squires & Quake 2005, de Langre, Baroud, & Reverdy 2010, Liu & Feng 2012). In the theory of capillarity there are two approaches. The first one relates to seminal works (Young 1805) and (Laplace 1805, Laplace 1806) where they introduced a sharp interface with the surface energy while the second approach was developed in (Korteweg 1901) and (van der Waals 1893) who suggested to use unified bulk energy density depending on the second gradient of the material density (Rowlinson & Widom 2003, Finn 1986, dell'Isola & Seppecher 1995, dell'Isola & Seppecher 1997, de Gennes, Brochard-Wyart, & Quéré 2003, Fried & Gurtin 2006, Rosi, Giorgio, & Eremeyev 2013). Both models lead to complex boundary-value problems with a priori unknown sharp interface or an interfacial layer.

In this paper we consider the bending of two elastic plates with a capillary fluid between them. Capillarity influences the plates interaction. From the mechanical point of view the capillary forces acting on the free surface can be considered as a surface or/and contour reinforcement. This means that the effective (apparent) stiffness of the structure consisting of two elastic plates and a capillary fluid depends on the elastic stiffness parameters of the elastic plates as well as on the fluid-solid interaction and the surface tension.

For derivation of effective stiffness parameters we use the variational approach replacing the total energy of the system by the effective one using the first-order shear deformable theory of plates.

We propose a simple rheological model replacing the capillary forces by spring-dashpot structures. The model gives the formulas for the effective stiffness parameters. We show that the effective bending stiffness determined almost by the stiffness of the elastic plates while the transverse shear stiffness depends significantly on the boundary reinforcement. Within the proposed framework we analyze the difference between hydrophobic and hydrophillic fluids.

2 GOVERNING EQUATIONS OF CAPILLARY FLUID AND ELASTIC PLATE

Let us consider two identical elastic plates with a fluid layer between them, see Fig. 1. In what follows we

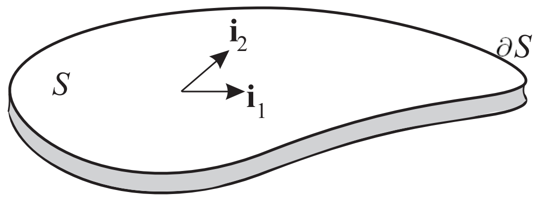

Figure 1: Two plates with fluid layer

consider the infinitesimal bending and shear of this structure and the first-order shear deformable theory. The bending and shear of the plate are described by the deflection w and the independent rotation vector $\boldsymbol{\vartheta}$. For the derivation of the governing equations we use the variational approach. The functional of the total energy is given by

$$\mathcal{E} = \mathcal{E}_{\text{pl}} + \mathcal{E}_{\text{fl}}, \tag{1}$$

where $\mathcal{E}_{\text{pl}}$ is the total energy of the plates and $\mathcal{E}_{\text{fl}}$ is the energy functional of the fluid. For $\mathcal{E}_{\text{pl}}$ we use the following formula

$$\mathcal{E}_{\text{pl}} = \frac{1}{2}\int_S (\Gamma\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} + \boldsymbol{\kappa}\cdot\cdot\,\mathbf{D}\cdot\cdot\,\boldsymbol{\kappa})\,\mathrm{d}S, \tag{2}$$

$$\boldsymbol{\kappa} = \frac{1}{2}\left(\nabla\boldsymbol{\vartheta}\cdot\mathbf{A} + \mathbf{A}\cdot(\nabla\boldsymbol{\vartheta})^T\right), \quad \boldsymbol{\gamma} = \nabla w - \boldsymbol{\vartheta}.$$

Here $\boldsymbol{\kappa}$ and $\boldsymbol{\gamma}$ are the second-order tensor of bending strain and the vector of the transverse shear strain, respectively, $\mathbf{A} = \mathbf{i}_1\otimes\mathbf{i}_1 + \mathbf{i}_2\otimes\mathbf{i}_2$ is the 2D unit tensor. In Eq. (2) Γ is the transverse shear stiffness, $\mathbf{D}$ is fourth-order tensor of bending stiffness having the following form (Altenbach & Eremeyev 2008b)

$$\mathbf{D} = D_{22}(\mathbf{a}_2\otimes\mathbf{a}_2 + \mathbf{a}_4\otimes\mathbf{a}_4) + D_{33}\mathbf{a}_3\otimes\mathbf{a}_3,$$

$$\mathbf{a}_2 = \mathbf{i}_1\otimes\mathbf{i}_1 - \mathbf{i}_2\otimes\mathbf{i}_2, \quad \mathbf{a}_3 = \mathbf{i}_1\otimes\mathbf{i}_2 - \mathbf{i}_2\otimes\mathbf{i}_1,$$

$$\mathbf{a}_4 = \mathbf{i}_1\otimes\mathbf{i}_2 + \mathbf{i}_2\otimes\mathbf{i}_1.$$

$\mathbf{i}_1$, $\mathbf{i}_2$ are the unit base vectors, D_{22} and D_{33} are the stiffness parameters given by

$$D_{22} = \frac{1}{24}\frac{E_{\text{f}}(h^3 - h_{\text{c}}^3)}{1+\nu_{\text{f}}}, \quad D_{33} = \frac{1}{24}\frac{E_{\text{f}}(h^3 - h_{\text{c}}^3)}{1-\nu_{\text{f}}},$$

where E_{f} and ν_{f} are the Young modulus and the Poisson ratio of the plate material, h and h_{c} are the thickness of the two plates and the fluid layer, respectively. The classical bending stiffness can be computed as $D = D_{22} + D_{33}$.

The surface energy of the fluid is given by (Finn 1986)

$$\mathcal{E}_{\text{fl}} = \int_A \sigma\sqrt{1 + |\nabla_{\text{s}} Z|^2}\,\mathrm{d}A, \tag{3}$$

where σ is the surface tension, ∇_{s} is the surface nabla-operator, Z is the function describing the capillary meniscus form.

The governing equations with respect to the unknown functions $w, \boldsymbol{\vartheta}$ and Z can be obtained using the variational equation $\delta\mathcal{E} = 0$ supplemented by proper boundary equations. The corresponding boundary-value problem is very complicated and cannot be solved analytically, in general. Instead of the solution of this problem we assume that $\mathcal{E}_{\text{fl}}$ can be approximated as the boundary elastic reinforcement. We assume that $\mathcal{E}_{\text{fl}}$ takes the form

$$\mathcal{E}_{\text{fl}} = \frac{1}{2}\int_{\partial S} (\Gamma_{\text{S}}\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} + \boldsymbol{\kappa}\cdot\cdot\,\mathbf{D}_{\text{S}}\cdot\cdot\,\boldsymbol{\kappa})\,\mathrm{d}s, \tag{4}$$

where Γ_{S} and $\mathbf{D}_{\text{S}}$ are the shear stiffness and the tensor of bending and torsion. In other words, we replace the functional (3) defined on the 2D field Z by the functional (4) which is similar to the elastic energy of the Timoshenko-type beam.

3 EFFECTIVE PROPERTIES

To calculate the effective properties of the considered structure we introduce the homogeneous plate which total energy is given by

$$\mathcal{E}_* = \frac{1}{2}\int_S (\Gamma_*\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} + \boldsymbol{\kappa}\cdot\cdot\,\mathbf{D}_*\cdot\cdot\,\boldsymbol{\kappa})\,\mathrm{d}S,$$

where Γ_* and $\mathbf{D}_*$ are the effective stiffness parameters. We assume that $\mathbf{D}_*$ has the same structure as $\mathbf{D}$. Equating $\mathcal{E}_*$ to $\mathcal{E}$ in the case of homogeneous strains we obtain

$$\begin{aligned}&(\Gamma S + \Gamma_{\text{S}} L)\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} + \boldsymbol{\kappa}\cdot\cdot\,(S\mathbf{D} + L\mathbf{D}_{\text{S}})\cdot\cdot\,\boldsymbol{\kappa}\\ &= (\Gamma_*\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} + \boldsymbol{\kappa}\cdot\cdot\,\mathbf{D}_*\cdot\cdot\,\boldsymbol{\kappa})S.\end{aligned} \tag{5}$$

Here S and L are the area and the contour length of the plate, respectively. From (5) it follows

$$\Gamma_* = \Gamma + \frac{L}{S}\Gamma_{\text{S}}, \quad \mathbf{D}_* = \mathbf{D} + \frac{L}{S}\mathbf{D}_{\text{S}}. \tag{6}$$

Let us note that in (6) the influence of the surface tension on the stiffness parameters is different. Since $\mathbf{D} \neq \mathbf{0}$ it is almost negligible in the case of the bending stiffness. For the transverse shear stiffness the influence of the surface tension is more essential. Indeed, for two plates connected by a fluid layer without surface tension the effective transverse shear stiffness is close to zero. Thus the surface tension is essential when $\Gamma \approx 0$.

We re-write $(6)_1$ as follows

$$\Gamma_* = \Gamma + \alpha h \quad \text{with} \quad \alpha = \frac{Lh}{S}\Gamma_{\text{S}} \tag{7}$$

α is the coefficient which depends on the surface tension σ. Taking $\Gamma_{\text{S}} = 2\sigma h_c$ we find that

$$\alpha = \frac{2\sigma L h_c}{Sh}.$$

The latter formula relates to the rule proposed by (Wang, Duan, Huang, & Karihaloo 2006) for the size-effect in nanomaterials. Indeed, we have the formula

$$\frac{\Gamma_*}{\Gamma} = 1 + \frac{2\sigma L h_\mathrm{c}}{kSh\mu_\mathrm{c}},$$

which coincides with (Wang, Duan, Huang, & Karihaloo 2006). Besides, the second term in (7) depends significantly on the surface tension for plates with curved contour. Equation (7) describes the averaged influence of the surface tension.

4 THREE-LAYERED PLATE

Following (Altenbach, Eremeyev, Ivanova, & Morozov 2012) in this section we briefly present another way of determination of the effective stiffness of the structure under consideration. Here we use the technique presented in (Altenbach & Eremeyev 2008a, Altenbach & Eremeyev 2009) for the case of viscoelastic non-homogeneous plates. We assume the fluid as a standard incompressible viscoelastic solid with the shear relaxation function

$$\mu_\mathrm{c} = \mu_\infty + (\mu_0 - \mu_\infty) \exp\left(-\frac{t}{\eta}\right),$$

where μ_0 and μ_∞ are the viscoelastic moduli, η is the relaxation time. For the Maxwell fluid $\mu_\infty = 0$. Thus, the case of a classical fluid can be obtained as a limit case if $t \to \infty$.

For the viscoelastic sandwich plate we have the formulas (Altenbach & Eremeyev 2008a, Altenbach & Eremeyev 2009)

$$D_{22} = \frac{1}{24}\left[\frac{E_\mathrm{f}(h^3 - h_\mathrm{c}^3)}{1+\nu_\mathrm{f}} + \frac{E_\mathrm{c} h_\mathrm{c}^3}{1+\nu_\mathrm{c}}\right],$$

$$D_{33} = \frac{1}{24}\left[\frac{E_\mathrm{f}(h^3 - h_\mathrm{c}^3)}{1-\nu_\mathrm{f}} + \frac{E_\mathrm{c} h_\mathrm{c}^3}{1-\nu_\mathrm{c}}\right],$$

where due to the incompressibility of the fluid $E_\mathrm{c}(t) = 3\mu_\mathrm{c}(t)$ and $\nu_\mathrm{c} = 1/2$. The bending stiffness is given by

$$D = D_{22} + D_{33} = \frac{1}{12}\left[\frac{E_\mathrm{f}(h^3 - h_\mathrm{c}^3)}{1-\nu_\mathrm{f}^2} + \frac{E_\mathrm{c} h_\mathrm{c}^3}{1-\nu_\mathrm{c}^2}\right]. \quad (8)$$

Since $E_\mathrm{c} \ll E_\mathrm{f}$ instead of (8) one can use

$$D = \frac{1}{12}\frac{E_\mathrm{f}(h^3 - h_\mathrm{c}^3)}{1-\nu_\mathrm{f}^2}. \quad (9)$$

For the transverse shear stiffness using certain assumptions we obtain the following approximated formula (Altenbach, Eremeyev, Ivanova, & Morozov 2012):

$$\Gamma = \frac{26}{9}\mu_\mathrm{c} h. \quad (10)$$

From (10) it follows that as in the previous model Γ is almost determined by the fluid properties.

5 RHEOLOGICAL MODEL

For capillary fluids the type the fluid-structure interaction is essential. It depends on the hydrophillic (wetting) or hydrophobic (dewetting) properties of the fluid. These properties influences the shape of the meniscus as well as the adhesion of the fluid and plates, see Fig 2. Here F stands for the fluid area while M denotes the meniscus.

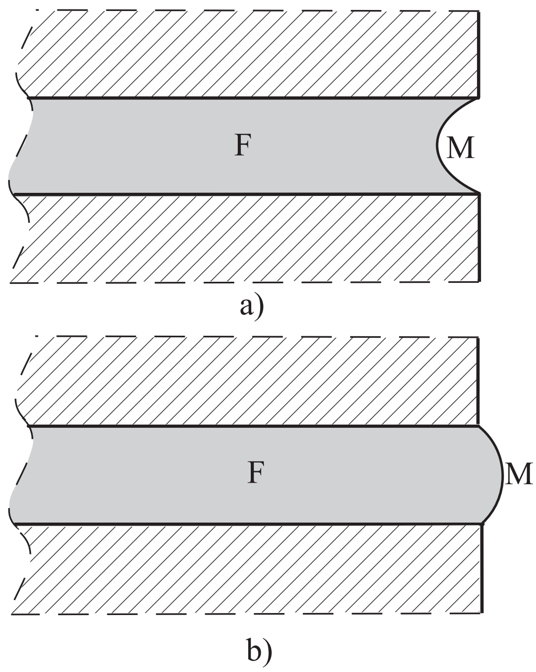

Figure 2: Meniscus shape for hydrophillic (a) and hydrophobic (b) fluids

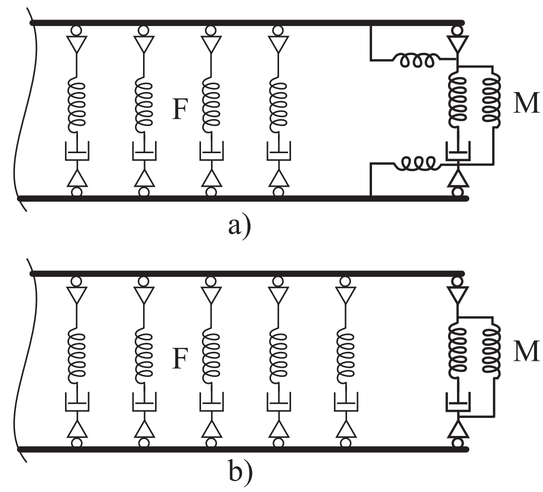

Figure 3: Rheological models for hydrophillic (a) and hydrophobic (b) fluids

For the sake of simplicity we consider the following rheological model which demonstrate the discussed above difference between hydrophobic and hydrophillic fluids, see Fig. 3. Here we introduce the Maxwell spring-dashpot rheological element for the fluid and the standard viscoelastic solid for the capillary surface. We also assume that there is no friction between fluid and plates. In addition, we introduce two horizontal springs modeling the adhesion properties of the wetting fluid, see Fig. 3 a). Thus, we replace the fluid area F and the meniscus M by the corresponding spring-dashpots. In other words instead of consideration of complex problem for the capillary fluid with free surface we use simple one-dimensional model. Within the model the meniscus is replaced by simple contour reinforcement which describes qualitatively deformations of a capillary surface. The presented rheological model corresponds to both presented in Sects 2–4 approaches since the model takes into account both viscoelastic properties of the fluid and the surface tension.

As a result, within the framework of the rheological model the difference between hydrophobic and hydrophillic fluids relates to the difference in Γ_*. For the hydrophillic fluid Γ_* is bigger than for hydrophobic one.

6 CONCLUSIONS

We discussed the effective stiffness of two elastic plates connected by capillary fluid and analyzed the influence of the surface tension on the stiffness parameters. We present few models describing the action of surface tension. As a result we have shown that the surface tension influences significantly the effective transverse shear stiffness of the whole structure at the micro- or nanoscale.

ACKNOWLEDGEMENTS

The first author acknowledges support by the German Research Foundation (AL341/33-1) and by the Russian Foundation of Basic Research (12-01-00038).

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