Lecture 5

Elena Ivanova

Application of the continuum model with microstructure (continua with inner rotational degrees of freedom) to description on the macro-level of electro-magnetic processes: simplest theory leading to Maxwell's equations.

1 The balance equations in Euler's mechanics

Let us consider a collection of body-points $\mathcal{A}_i$, which we call a body $\mathcal{A}$ (see Fig. 1). All remaining body-points are called the environment of body $\mathcal{A}$ and denoted by a symbol $\mathcal{A}^e$.

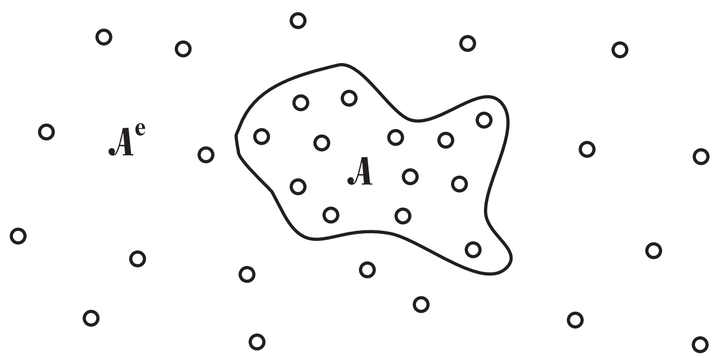

Figure 1: Body $\mathcal{A}$ and its environment $\mathcal{A}^e$

To model the action of the environment $\mathcal{A}^e$ on the body $\mathcal{A}$ we should assign a pair of vectors: a force vector and a moment vector. The force and moment vectors are additive on both the bodies compound the body $\mathcal{A}$ and the bodies compound its environment $\mathcal{A}^e$.

The equation of momentum balance. The rate in the momentum change of body $\mathcal{A}$ is equal to the force acting on the body $\mathcal{A}$ from its environment plus the rate of the momentum supply in body $\mathcal{A}$, namely:

$$\frac{d\mathbf{K}_1(\mathcal{A})}{dt} = \mathbf{F}(\mathcal{A}, \mathcal{A}^e) + \mathbf{k}_1(\mathcal{A}). \tag{1}$$

Here $\mathbf{F}(\mathcal{A}, \mathcal{A}^e)$ is the force acting on the body $\mathcal{A}$ from its environment $\mathcal{A}^e$, and $\mathbf{k}_1(\mathcal{A})$ is the rate of the momentum supply in body $\mathcal{A}$.

The equation of angular momentum balance. The rate in the angular momentum change of body $\mathcal{A}$, calculated with respect to a reference point Q, is equal to the moment acting on body $\mathcal{A}$ from its environment, calculated with respect to the same reference point Q, plus the rate of the angular momentum supply in a body $\mathcal{A}$, namely:

$$\frac{d\mathbf{K}_2^Q(\mathcal{A})}{dt} = \mathbf{M}^Q(\mathcal{A}, \mathcal{A}^e) + \mathbf{k}_2^Q(\mathcal{A}). \tag{2}$$

Here $\mathbf{M}^Q(\mathcal{A}, \mathcal{A}^e)$ is the moment acting on the body $\mathcal{A}$ from its environment $\mathcal{A}^e$, and $\mathbf{k}_2^Q(\mathcal{A})$ is the rate of the angular momentum supply in body $\mathcal{A}$.

The energy balance equation. The rate in the total energy change of body $\mathcal{A}$ is equal to the external force and moment power $N(\mathcal{A}, \mathcal{A}^e)$ plus the rate of supply of the energy of non-mechanical nature $\varepsilon(\mathcal{A})$:

$$\frac{dE(\mathcal{A})}{dt} = N(\mathcal{A}, \mathcal{A}^e) + \varepsilon(\mathcal{A}). \tag{3}$$

Here the total energy of a body $E(\mathcal{A})$ is a sum of the kinetic energy $K(\mathcal{A})$ and the internal energy $U(\mathcal{A})$. The power of external actions on body $\mathcal{A}$ consisting of body-points $\mathcal{A}_i$ is the bilinear form of velocities and actions:

$$N(\mathcal{A}, \mathcal{A}^e) = \sum_i \Big[\mathbf{F}(\mathcal{A}_i, \mathcal{A}^e) \cdot \mathbf{v}_i + \mathbf{L}(\mathcal{A}_i, \mathcal{A}^e) \cdot \boldsymbol{\omega}_i\Big]. \tag{4}$$

It is important to notice that the power of external actions depends on the proper moments $\mathbf{L}(\mathcal{A}_i, \mathcal{A}^e)$ rather than the full moments $\mathbf{M}^Q(\mathcal{A}_i, \mathcal{A}^e)$.

2 Continuum consisting of multi-rotor gyrostats

2.1 Kinematics of the continuum

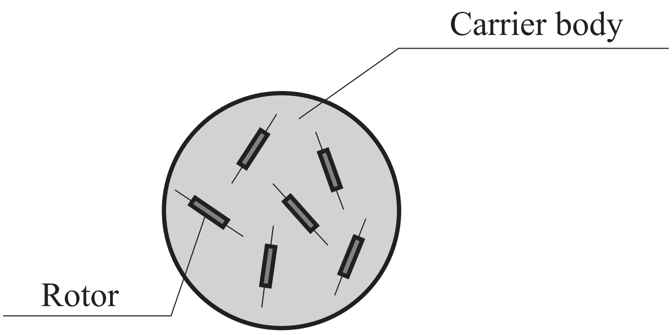

Figure 2: Multi-rotor gyrostat

The multi-rotor gyrostat (see Fig. 2) is a complex object which consists of the carrier body and N rotors. The carrier body and the rotors of the gyrostat are considered to be the infinitesimal rigid bodies. The carrier body has an arbitrary tensor of inertia. The rotors are the axisymmetric rigid bodies. The axes of symmetry of the rotors are fixed with respect to the carrier body. The rotors can not translate relative to the carrier body, and they can rotate independently of rotation of the carrier body only about their axes of symmetry.

Now we consider the material medium (see Fig. 3) consisting of multi-rotor gyrostats. Let vector $\mathbf{r}$ determine the position of some point of space. We introduce following notations: $\mathbf{v}(\mathbf{r},t)$ is the velocity field; $\mathbf{u}(\mathbf{r},t)$ is the displacement field; $\mathbf{P}_0(\mathbf{r},t)$, $\boldsymbol{\omega}_0(\mathbf{r},t)$ are the fields of the rotation tensors and the angular velocity vectors of the carrier bodies; $\mathbf{P}_i(\mathbf{r},t)$, $\boldsymbol{\omega}_i(\mathbf{r},t)$ are fields of the rotation tensor and the angular velocity vector of the rotor number i, where $i = 1, 2, ..., N$.

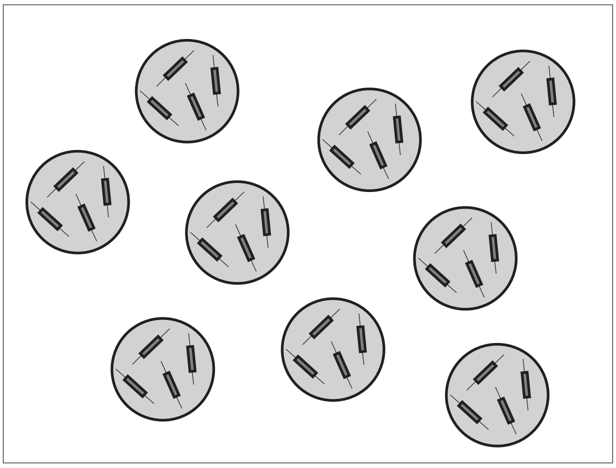

Figure 3: Elementary volume of continuum consisting of multi-rotor gyrostats

In view of the fact that the rotor number i can rotate independently of rotation of the carrier body only about its axis of symmetry the rotation tensor of the rotor is represented in the form

$$\mathbf{P}_i(\mathbf{r},t) = \mathbf{P}_0(\mathbf{r},t) \cdot \mathbf{P}(\beta_i \mathbf{n}_i), \tag{5}$$

where $\mathbf{n}_i$ is the unit vector which determines the direction of the axis of symmetry of the rotor at the reference position, $\beta_i(\mathbf{r},t)$ is the angle of rotation of the rotor with respect to the carrier body. Hence the unit vector $\mathbf{n}'_i(\mathbf{r},t)$ which determines the direction of the axis of symmetry of the rotor number i at the actual position takes the form

$$\mathbf{n}'_i(\mathbf{r},t) = \mathbf{P}_0(\mathbf{r},t) \cdot \mathbf{n}_i. \tag{6}$$

In the spatial description the angular velocity vector of the carrier body and angular velocity vectors of the rotors are calculated by the formulas:

$$\boldsymbol{\omega}_0 = -\frac{1}{2}\left(\frac{\delta \mathbf{P}_0}{\delta t} \cdot \mathbf{P}_0^T\right)_\times, \qquad \boldsymbol{\omega}_i = \boldsymbol{\omega}_0 + \frac{\delta \beta_i}{\delta t}\mathbf{n}'_i, \qquad i = 1, 2, ..., N. \tag{7}$$

2.2 The equations of motion

The multi-rotor gyrostat has $N+6$ degrees of freedom which are determined by the following functions:

$$\mathbf{v}(\mathbf{r},t), \quad \mathbf{P}_0(\mathbf{r},t), \quad \beta_i(\mathbf{r},t), \quad i = 1, 2, ..., N. \tag{8}$$

In order to find these unknown functions we need to formulate two vector and N scalar equations of motion.

The momentum balance equation for the gyrostats has the form

$$\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f} = \rho \frac{\delta \mathbf{v}}{\delta t}. \tag{9}$$

Here $\boldsymbol{\tau}$ is the stress tensor, $\mathbf{f}$ is the mass density of external forces, ρ is the mass density which satisfies the mass balance equation

$$\frac{\delta \rho}{\delta t} + \rho \nabla \cdot \mathbf{v} = 0. \tag{10}$$

The angular momentum balance equation for the gyrostats is

$$\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_{\times} + \rho \mathbf{L} = \rho \frac{\delta}{\delta t} \boldsymbol{\mathfrak{L}}(\mathbf{r}, t), \tag{11}$$

where $\boldsymbol{\mu}$ is the moment stress tensor, $\boldsymbol{\tau}_{\times}$ is the vector invariant of the stress tensor, $\mathbf{L}$ is the mass density of external moments, $\boldsymbol{\mathfrak{L}}$ is the mass density of the proper angular momentum of the gyrostat

$$\boldsymbol{\mathfrak{L}} = \mathbf{P}_0(\mathbf{r},t) \cdot \mathbf{C} \cdot \mathbf{P}_0^T(\mathbf{r},t) \cdot \boldsymbol{\omega}_0(\mathbf{r},t) + \sum_{i=1}^{N} \lambda_i \frac{\delta \beta_i(\mathbf{r},t)}{\delta t} \mathbf{n}_i'(\mathbf{r},t). \tag{12}$$

The first term on the right-hand side of Eq. (12) is the mass density of the proper angular momentum of the gyrostats when the gyrostats move as the rigid bodies, i.e. all rotors are fixed with respect to the carrier body of the gyrostat. The rest terms on the right-hand side of Eq. (12) characterize the influence of the independent rotation of rotors on the proper angular momentum of the gyrostat. Tensor $\mathbf{C}$ is the inertia tensor of the gyrostat at the reference configuration, λ_i is the axial moment of inertia if the rotor number i.

We should add the system of equations (9), (11) to the angular momentum balance equations for the rotors. The moment stress tensors characterizing the interaction between rotors are supposed to be zero. Then the projections of the angular momentum balance equations on the axes of the rotors take the form

$$\lambda_i \frac{\delta}{\delta t} \left(\frac{\delta \beta_i(\mathbf{r},t)}{\delta t} + \boldsymbol{\omega}_0(\mathbf{r},t) \cdot \mathbf{n}_i'(\mathbf{r},t) \right) = L_i, \qquad i = 1, 2, ..., N. \tag{13}$$

Here L_i is the mass density of external moments acting on the rotor number i. Let L_i take the form

$$L_i = -\nu_i \left(\frac{\delta \beta_i}{\delta t} - \Omega_i \right), \qquad \nu_i > 0, \tag{14}$$

where $\Omega_i = \text{const}$ and $\nu_i = \text{const}$ are the parameters of the particle. In relation to engineering problems the moment of the form (14) is usually called the limited power motor moment. The motor power is characterized by the parameter ν_i. The parameter Ω_i determines the nominal angular velocity of the rotor which is attained under the action of the limited power motor moment.

Now we have two vector and N scalar equations of motion. The system of equations is not closed. It is necessary to add these equations to the constitutive equations. In order to obtain the constitutive equations we should consider the equation of energy balance.

2.3 Equation of energy balance

Now we formulate the equation of energy balance (3) for the material medium in control volume V:

$$\frac{d}{dt}\int\limits_{(V)} \rho(K+U)dV = \int\limits_{(V)} \rho\big(\mathbf{f}\cdot\mathbf{v}+\mathbf{L}\cdot\boldsymbol{\omega}_0+Q\big)dV + \\ + \int\limits_{(S)} \big(\boldsymbol{\tau}_n\cdot\mathbf{v}+\boldsymbol{\mu}_n\cdot\boldsymbol{\omega}_0+H_{(n)}\big)\,dS - \int\limits_{(S)} \rho\mathbf{n}\cdot\mathbf{v}(K+U)dS. \quad (15)$$

Here U is the internal energy density per unit mass; $\boldsymbol{\tau}_n = \mathbf{n}\cdot\boldsymbol{\tau}$, $\boldsymbol{\mu}_n = \mathbf{n}\cdot\boldsymbol{\mu}$; quantities Q and H_n are the rate of the energy supply in volume and through surface S respectively. The rate of the energy supply through the surface can be expressed in term of energy-flux vector $\mathbf{H}$ by the formula $H_n = \mathbf{n}\cdot\mathbf{H}$. The mass density of kinetic energy for gyrostat has the form

$$K = \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \frac{1}{2}\boldsymbol{\omega}_0\cdot\mathbf{P}_0\cdot\mathbf{C}\cdot\mathbf{P}_0^T\cdot\boldsymbol{\omega}_0 + \frac{1}{2}\sum_{i=1}^{N}\lambda_i\left[\left(\frac{\delta\beta_i}{\delta t}\right)^2 + 2\,\frac{\delta\beta_i}{\delta t}\,\boldsymbol{\omega}_0\cdot\mathbf{n}_i'\right]. \quad (16)$$

By standard reasoning, taking into account the equation of mass balance (10) we transform the equation of energy balance (15) to the local form

$$\rho\frac{\delta}{\delta t}(K+U) = \rho\mathbf{f}\cdot\mathbf{v} + \rho\mathbf{L}\cdot\boldsymbol{\omega}_0 + (\nabla\cdot\boldsymbol{\tau})\cdot\mathbf{v} + (\nabla\cdot\boldsymbol{\mu})\cdot\boldsymbol{\omega}_0 + \\ + \boldsymbol{\tau}^T\cdot\cdot\nabla\mathbf{v} + \boldsymbol{\mu}^T\cdot\cdot\nabla\boldsymbol{\omega}_0 + \nabla\cdot\mathbf{H} + \rho Q. \quad (17)$$

Using expression (16) for the kinetic energy density and Eqs. (9)–(13) we transform the energy balance equation (17) to the form

$$\rho\frac{\delta U}{\delta t} = \boldsymbol{\tau}^T\cdot\cdot(\nabla\mathbf{v}+\mathbf{E}\times\boldsymbol{\omega}_0) + \boldsymbol{\mu}^T\cdot\cdot\nabla\boldsymbol{\omega}_0 + \nabla\cdot\mathbf{H} + \rho Q - \rho\sum_{i=1}^{N} L_i\,\frac{\delta\beta_i}{\delta t}. \quad (18)$$

Notice that the last term on the right-hand side of Eq. (18) plays role analogous to the term ρQ. It has the sense of the rate of energy supply. In the case of the elastic continuum the subsequent procedure of derivation of the Cauchy–Green relations is standard for continuum mechanics.

3 Maxwell's classical electrodynamics

In what follows we consider the model of electromagnetic field proposed by P. A. Zhilin. Now we consider the simplest variant of this model whose mathematical description is reduced to the Maxwell equations.

Let us consider the model discussed above accepting some simplifying assumption. We suppose the processes to be isothermal (or adiabatic) only. Moreover, we assume the following:

$$\mathbf{v} = \mathbf{0}, \qquad \boldsymbol{\tau} = \mathbf{0} \qquad \Rightarrow \qquad \rho = \text{const}. \quad (19)$$

In this case the material derivative coincides with the total time derivative. The moment stress tensor $\boldsymbol{\mu}$ is supposed to be antisymmetric one

$$\boldsymbol{\mu} = \varpi^{-1}\boldsymbol{\mathfrak{B}} \times \mathbf{E}, \qquad \varpi = \text{const} \qquad \Rightarrow \qquad \nabla \cdot \boldsymbol{\mu} = \varpi^{-1}\nabla \times \boldsymbol{\mathfrak{B}}, \tag{20}$$

where vector $\boldsymbol{\mathfrak{B}}$ is called the magnetic induction vector; ϖ is a dimensional constant. When accepted above restrictions the equation of momentum balance (9) becomes an identity, and the angular momentum balance equation (11) takes the form

$$\varpi^{-1}\nabla \times \boldsymbol{\mathfrak{B}} + \rho\mathbf{L} = \rho\frac{d\boldsymbol{\mathfrak{L}}}{dt}. \tag{21}$$

The equation of energy balance takes the very simple form

$$\rho\frac{d\mathfrak{U}}{dt} = -\varpi^{-1}\boldsymbol{\mathfrak{B}} \cdot \nabla \times \boldsymbol{\omega}_0. \tag{22}$$

We assume the angular momentum to have the simplest form

$$\boldsymbol{\mathfrak{L}} = \lambda\boldsymbol{\omega}_0, \qquad \boldsymbol{\mathfrak{E}} = \varpi c^2\rho\boldsymbol{\mathfrak{L}}, \qquad c = \text{const}. \tag{23}$$

Vector $\boldsymbol{\mathfrak{E}}$ introduced instead of the angular momentum vector is called the electric field vector. The constant c is the velocity of light in free space.

The rotations are assumed to be small. In this case the angular velocity vector can be expressed in terms of the vector of small rotation $\boldsymbol{\theta}$ by the simplest formula

$$\boldsymbol{\omega}_0 = \frac{d\boldsymbol{\theta}}{dt} \qquad \Rightarrow \qquad \boldsymbol{\mathfrak{E}} = \varpi c^2\rho\lambda\frac{d\boldsymbol{\theta}}{dt}. \tag{24}$$

Substituting Eq. (23) into Eq. (21) we obtain

$$\nabla \times \boldsymbol{\mathfrak{B}} + \varpi\rho\mathbf{L} = \frac{1}{c^2}\frac{d\boldsymbol{\mathfrak{E}}}{dt}. \tag{25}$$

This equation was first derived by J. Maxwell, and the electric current density was playing the role of the external moment $\varpi\rho\mathbf{L}$:

$$\varpi\rho\mathbf{L} = -\mu_0\mathbf{j} = -\frac{1}{\varepsilon_0 c^2}\,\mathbf{j}, \tag{26}$$

where $\mathbf{j}$ is the electric current density; μ_0 is the the magnetic constant; ε_0 is the the dielectric constant.

According to generally accepted ideas the electric current density is the rate of charge flow per unit area. Hence, the law of charge conservation takes place:

$$\nabla \cdot \mathbf{j} = -\frac{d\rho}{dt}. \tag{27}$$

By taking the divergence of both sides of Eq. (25) in view of Eq. (26) we obtain

$$\frac{d(\nabla \cdot \boldsymbol{\mathfrak{E}})}{dt} = -\frac{1}{\varepsilon_0}\,\nabla \cdot \mathbf{j}. \tag{28}$$

Integrating Eq. (28) with respect to time and taking into account the law of charge conservation (27) we obtain the Gauss law

$$\nabla \cdot \boldsymbol{\mathfrak{E}} = \frac{\rho}{\varepsilon_0}. \tag{29}$$

Now we consider the energy balance equation (22). In view of Eq. (24) it takes the form

$$\rho \frac{d\mathfrak{U}}{dt} = -\varpi^{-1} \boldsymbol{\mathfrak{B}} \cdot \frac{d}{dt} \nabla \times \boldsymbol{\theta}. \tag{30}$$

We assume the internal energy to have the simplest form

$$\mathfrak{U} = \frac{1}{2} \kappa \, |\nabla \times \boldsymbol{\theta}|^2, \qquad \kappa = \text{const} > 0. \tag{31}$$

Then according to Eq. (30) we obtain the following expression for the magnetic induction vector $\boldsymbol{\mathfrak{B}}$:

$$\boldsymbol{\mathfrak{B}} = -\varpi \rho \kappa \, \nabla \times \boldsymbol{\theta}. \tag{32}$$

It is easy to see that the Gauss law for the magnetic field

$$\nabla \cdot \boldsymbol{\mathfrak{B}} = 0 \tag{33}$$

follows from the constitutive equation (32).

Substituting Eqs. (24), (32) into Eq. (25) we can obtain the differential equation for the rotation vector $\boldsymbol{\theta}$

$$\kappa \, (\Delta \, \boldsymbol{\theta} - \nabla \nabla \cdot \boldsymbol{\theta}) + \mathbf{L} = \lambda \frac{d^2 \boldsymbol{\theta}}{dt^2}. \tag{34}$$

However, we can choose another way. Eliminating the rotation vector from the expression for the electric field vector $\boldsymbol{\mathfrak{E}}$ and the magnetic induction vector $\boldsymbol{\mathfrak{B}}$ we obtain the following continuity equation:

$$\nabla \times \boldsymbol{\mathfrak{E}} = -\frac{d\boldsymbol{\mathfrak{B}}}{dt}, \qquad \kappa = \lambda c^2. \tag{35}$$

This equation is known in physics as the Faraday law of induction. Maxwell's electrodynamics started from this equation. Maxwell himself has proposed Eq. (25).

Now we write down Eqs. (25), (29), (33), (35) as a system:

$$\begin{aligned} \nabla \times \boldsymbol{\mathfrak{E}} &= -\frac{d\boldsymbol{\mathfrak{B}}}{dt}, \qquad & \nabla \cdot \boldsymbol{\mathfrak{E}} &= \frac{\rho}{\varepsilon_0}, \\ \nabla \times \boldsymbol{\mathfrak{B}} + \varpi \rho \mathbf{L} &= \frac{1}{c^2} \frac{d\boldsymbol{\mathfrak{E}}}{dt}, \qquad & \nabla \cdot \boldsymbol{\mathfrak{B}} &= 0. \end{aligned} \tag{36}$$

Thus, we obtain Eqs. (36) which coincide with the classical Maxwell equations with the only difference that there are partial derivatives with respect to time on the right-hand side of the Maxwell equations whereas Eqs. (36) contains the total derivatives with respect to time. Notice that the operators of partial differentiation with respect to time are not objective. Therefore, from the viewpoint of classical mechanics these operators can not be present in the equations describing some physical process.

4 Concluding remarks

- Above we consider the continuum of a special type. By using the line of reasonings standard for continuum mechanics we have derived the basic relations describing this continuum and as a result we have obtained the close set of equations (36) which coincide with the Maxwell equations.

- The equations (36) have a clear mechanical interpretation. This is important because of the following reasons. Suppose that the classical Maxwell equations do not satisfy us for some reason and they should be changed in some way. This is not hypothetical assumption since it is known that the classical equations do not allow us to construct construct a consistent theory of the atom. Therefore, changes are needed, but what exactly needs to be changed? The classical equations do not answer this question. The presence of a mechanical interpretation gives direction refinements.

- The mechanical sense of the equations (36) answers the question: "How the Earth can move through an elastic medium (the luminiferous ether)?" (Lord Kelvin, 1900). Above we consider the elastic medium which does not act on bodies by forces since this medium can interact with bodies only by moments.