Theory of Shells and Theory of Curvilinear Rods: A Comparative Analysis

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Preliminary remarks

- Now conferences on the theory of plates and shells are held regularly and many articles in periodicals are published. It is not the case with the theory of rods.
- The question arises: what is the reason for the lack of interest in the theory of rods?
- It would seem that the theory of rods and the theory of shells have much in common. They are both moment theories, and the basic equations are derived by the same methods.
- One can assume that a reason for the lack of active research in the area of rod theory is that the theory of rods is mathematically much simpler the theory of shells and all problems in the theory of rods have long been solved.
- This is not the case. Further we give examples.

Literature on the theory of curvilinear rods

- *P. A. Zhilin.* Applied mechanics. Theory of thin elastic rods. St. Petersburg. 2007. (In Russian)
- *V. A. Svetlitskii.* Mechanics of rods. Moscow. 1987. (In Russian).
- *V. A. Svetlitskii.* Mechanics of flexible filaments and rods. Moscow. 1978. (In Russian).
- *M. B. Rubin.* Cosserat theories: shells, rods and points. Kluwer, Dordrecht. 2000.
- *H. Altenbach, V. A. Eremeyev (Eds).* Generalized continua — from the theory to engineering applications. Springer. 2013.
- *V. A. Eremeyev, L. P. Lebedev, H. Altenbach.* Foundations of micropolar mechanics. Springer. 2013.
- *E. I. Grigilyuk, I. T. Selezov.* Nonclassical theories of rod, plate and shell vibrations. Moscow. 1973. (In Russian)

The basic equations of shell theory and rod theory

Theory of shells	Theory of rods

Equations of motion

$\nabla\cdot\mathbf{T}+\rho\mathbf{F}=\rho\,(\mathbf{v}+\boldsymbol{\Theta}_1\cdot\omega)^{\cdot}$	$\mathbf{T}'+\rho\mathbf{F}=\rho\,(\mathbf{v}+\boldsymbol{\Theta}_1\cdot\omega)^{\cdot}$
$\nabla\cdot\mathbf{M}+\mathbf{T}_{\times}+\rho\mathbf{L}=\rho\,(\mathbf{v}\cdot\boldsymbol{\Theta}_1+\boldsymbol{\Theta}_2\cdot\omega)^{\cdot}$	$\mathbf{M}'+\mathbf{R}'\times\mathbf{T}+\rho\mathbf{L}=\rho\,(\mathbf{v}\cdot\boldsymbol{\Theta}_1+\boldsymbol{\Theta}_2\cdot\omega)^{\cdot}$

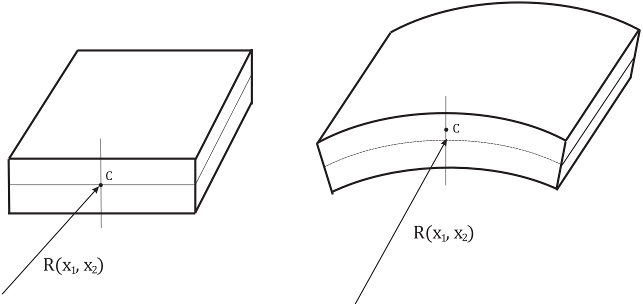
Deformations (in the case of linear theory)

$\boldsymbol{\varepsilon}=\nabla\mathbf{u}+\mathbf{a}\times\boldsymbol{\psi},\quad \boldsymbol{\phi}=\nabla\boldsymbol{\psi}$	$\boldsymbol{\varepsilon}=\mathbf{u}'+\mathbf{t}\times\boldsymbol{\psi},\quad \boldsymbol{\phi}=\boldsymbol{\psi}'$

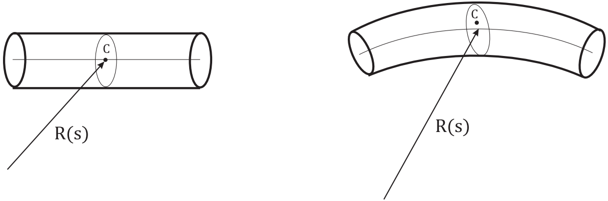
Constitutive equations (without thermal effects)

$$\mathbf{T}=\frac{\partial(\rho U)}{\partial\boldsymbol{\varepsilon}},\quad \mathbf{M}=\frac{\partial(\rho U)}{\partial\boldsymbol{\phi}},\quad \rho U=\rho U(\boldsymbol{\varepsilon},\boldsymbol{\phi})$$

Effect of curvature



In the case of curvilinear rods and shells the mass centers are offset towards the convexity.



Internal energy of shell (linear theory)

In the case of linear theory the internal energy of shell is

$$\rho_0 U = \frac{1}{2}\varepsilon\cdot\cdot\mathbf{A}\cdot\cdot\varepsilon + \varepsilon\cdot\cdot\mathbf{B}\cdot\cdot\phi + \frac{1}{2}\phi\cdot\cdot\mathbf{C}\cdot\cdot\phi$$

Stiffness tensors **A**, **C** are well known, **B** has the following structure:

$$\mathbf{B} = \frac{Eh^2}{12(1-\nu^2)}\Big(2hH\big[B_1\mathbf{a}_1\mathbf{a}_3 + B_2(\mathbf{a}_2\mathbf{a}_4 - \mathbf{a}_4\mathbf{a}_2)\big] + \\ + 2hH_1\big[B_3\mathbf{a}_1\mathbf{a}_4 + B_4\mathbf{a}_4\mathbf{a}_1 + B_5\mathbf{a}_2\mathbf{a}_3\big]\Big)$$

$$B_1 = -\frac{\nu(1+\nu)}{2(1-\nu)},\ \ B_2 = 0,\ \ B_3 = \frac{1+\nu}{2},\ \ B_4 = -\frac{1-\nu}{4},\ \ B_5 = -\frac{1}{2}$$

P. A. Zhilin. Applied mechanics. Foundations of the shells theory. St. Petersburg. 2006. (In Russian)

Internal energy of curvilinear rod

In the case of physically linear theory the internal energy of rod is

$$\rho_0 U = \frac{1}{2}\boldsymbol{\varepsilon}\cdot\mathbf{P}\cdot\mathbf{A}\cdot\mathbf{P}^T\cdot\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}\cdot\mathbf{P}\cdot\mathbf{B}\cdot\mathbf{P}^T\cdot\boldsymbol{\phi} + \frac{1}{2}\boldsymbol{\phi}\cdot\mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}^T\cdot\boldsymbol{\phi},$$

Stiffness tensors **A**, **C** are the same as those of rectilinear rod:

$$\mathbf{A} = A_1\mathbf{d}_1\mathbf{d}_1 + A_2\mathbf{d}_2\mathbf{d}_2 + A_3\mathbf{tt}, \quad \mathbf{C} = C_1\mathbf{d}_1\mathbf{d}_1 + C_2\mathbf{d}_2\mathbf{d}_2 + C_3\mathbf{tt}$$

Stiffness tensor **B** has the following structure:

$$\mathbf{B} = \frac{1}{R_t}\big(B_1\mathbf{d}_1\mathbf{d}_1 + B_2\mathbf{d}_2\mathbf{d}_2 + B_3\mathbf{tt}\big) + \\ + \frac{1}{R_c}\big[(B_{23}\mathbf{d}_2\mathbf{t} + B_{32}\mathbf{td}_2)\cos\alpha + (B_{13}\mathbf{d}_1\mathbf{t} + B_{31}\mathbf{td}_1)\sin\alpha\big]$$

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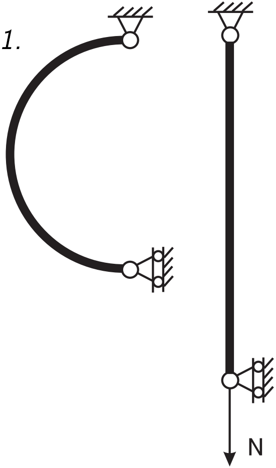
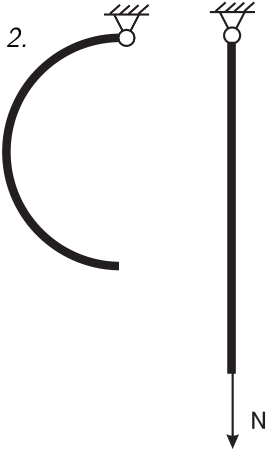
St. Petersburg. 2007. (In Russian)

First example: rod is a half circle in non-deformed state

Variant 1.



Variant 2.



Is it possible to bring the rods into linear state by applying force **N** as shown in the Figures?

Solution of the problem without tensor $\mathbf{B}$

The nonlinear strains of a rod are determined as

$$\boldsymbol{\varepsilon} = \mathbf{R}' - \mathbf{P}\cdot\mathbf{t}, \qquad \mathbf{P}' = \boldsymbol{\phi}\times\mathbf{P}$$

When passing the rod in the rectilinear position:

$$\mathbf{R}' = (1+\varepsilon)\mathbf{k}, \quad \mathbf{P}\cdot\mathbf{t} = \mathbf{k} \quad \Rightarrow \quad \boldsymbol{\varepsilon} = \varepsilon\mathbf{k}, \quad \boldsymbol{\phi} = \frac{1}{R}\,\mathbf{b}$$

In view of the boundary conditions the solution of static equations is

$$\mathbf{T}' = \mathbf{0} \quad \Rightarrow \quad \mathbf{T} = N\mathbf{k}; \qquad \mathbf{M}' + \mathbf{R}'\times\mathbf{T} = \mathbf{0} \quad \Rightarrow \quad \mathbf{M} = \mathbf{0}$$

The constitutive equations without tensor $\mathbf{B}$ take the form:

$$\mathbf{T} = A_3\varepsilon\mathbf{k}, \qquad \mathbf{M} = \frac{C_1}{R}\,\mathbf{b}$$

It is easy to see that we have a contradiction. Hence, the solution does not exist. It is very strange.

Solution of the problem taking into account tensor **B**

The solution of static equations and the expressions for strains are

$$\mathbf{T} = N\mathbf{k}, \quad \mathbf{M} = \mathbf{0}, \quad \boldsymbol{\varepsilon} = \varepsilon\mathbf{k}, \quad \boldsymbol{\phi} = \frac{1}{R}\,\mathbf{b}$$

The stiffness tensor at the actual configuration is

$$\mathbf{P}\cdot\mathbf{B}\cdot\mathbf{P}^T = \frac{1}{R_c}(B_{13}\mathbf{bk} + B_{31}\mathbf{kb}), \qquad R_c = -R$$

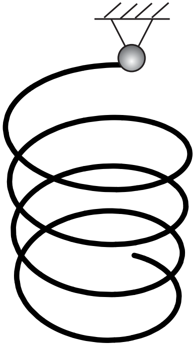
In view of the expressions for strains the constitutive equations are

$$\mathbf{T} = \left(A_3\varepsilon - \frac{B_{31}}{R^2}\right)\mathbf{k}, \quad \mathbf{M} = \left(-\frac{B_{31}\varepsilon}{R} + \frac{C_1}{R}\right)\mathbf{b}$$

In view of the solution of static equations we have

$$\varepsilon = \frac{C_1}{B_{31}}, \quad N = \frac{A_3 C_1}{B_{31}} - \frac{B_{31}}{R}$$

Second example: rod is a cylindrical helical spring



R is the radius of spring;
h is the pitch of spring;
β is the helix angle;

$$\tan\beta = \frac{h}{2\pi R}$$

Is it possible to straighten the rod by putting force **N** on the free end?

The solution of the problem without taking into account the tensor **B** compels us to conclude that it is impossible.

Solution of static equations:

$$\mathbf{T} = N\mathbf{k}, \quad \mathbf{M} = \mathbf{0}$$

Constitutive equation for **M**:

$$\mathbf{M} = \mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}^T\cdot\phi$$

Solution of the problem taking into account tensor **B**

The the expressions for strain vectors are

$$\boldsymbol{\varepsilon} = \varepsilon \mathbf{k}, \quad \boldsymbol{\phi} = -\frac{1}{R_t}\mathbf{k} - \frac{1}{R_c}\tilde{\mathbf{b}}$$

The stiffness tensor at the actual configuration is

$$\mathbf{P}\cdot\mathbf{A}\cdot\mathbf{P}^T = A_3\mathbf{kk} + A_1(\mathbf{E}-\mathbf{kk}), \quad \mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}^T = C_3\mathbf{kk} + C_1(\mathbf{E}-\mathbf{kk})$$

$$\mathbf{P}\cdot\mathbf{B}\cdot\mathbf{P}^T = \frac{1}{R_t}\left[B_3\mathbf{kk} + B_1(\mathbf{E}-\mathbf{kk})\right] + \frac{1}{R_c}(B_{13}\tilde{\mathbf{b}}\mathbf{k} + B_{31}\mathbf{k}\tilde{\mathbf{b}})$$

The constitutive equations are

$$\mathbf{T} = \mathbf{P}\cdot\mathbf{A}\cdot\mathbf{P}^T\cdot\boldsymbol{\varepsilon} + \mathbf{P}\cdot\mathbf{B}\cdot\mathbf{P}^T\cdot\boldsymbol{\phi}$$

$$\mathbf{M} = \boldsymbol{\varepsilon}\cdot\mathbf{P}\cdot\mathbf{B}\cdot\mathbf{P}^T + \mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}^T\cdot\boldsymbol{\phi}$$

Solution of the problem taking into account tensor **B**

From the equation $\mathbf{M} = \mathbf{0}$ it follows that $\phi = -\dfrac{B_3 \varepsilon}{R_t C_3}\,\mathbf{k} - \dfrac{B_{31}\varepsilon}{R_c C_1}\,\tilde{\mathbf{b}}$

Comparing the two obtained expressions for ϕ we have

$$\varepsilon = \frac{C_1}{B_{31}}, \qquad \frac{C_1}{B_{31}} = \frac{C_3}{B_3}$$

From the equation $\mathbf{T} = N\mathbf{k}$ it follows that

$$N = \left(A_3 - \frac{B_3^2}{C_3 R_t^2} - \frac{B_{31}^2}{C_1 R_c^2} \right) \varepsilon, \qquad \frac{B_3 B_{13}}{C_3} + \frac{B_1 B_{31}}{C_1} = 0$$

Thus, the solution exists if the parameters satisfy the conditions:

$$B_{31} = \frac{B_3 C_1}{C_3}, \qquad B_{13} = -B_1$$

Conclusion

- At present the theory of rods is not only applied engineering sciences. There are many unsolved theoretical problems in the rod theory.
- The determination of stiffness tensor $\mathbf{B}$ is one of unsolved theoretical problems in the rod theory.
- To solve this problem we can use the methods and approaches that are well developed in the shell theory.

Thanks for attention!