Chapter 12
High-Frequency Free Vibrations of Plates in the Reissner's Type Theory

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Abstract The classic plate theory by Kirchhoff allows to accurately describe the processes slowly varying by time. To solve the problems of the plate vibrations in the case of the external loads quickly varying by time the Reissner's type plate theory should be used. The Reissner's type plate theory includes three eigenfrequency spectra: one low-frequency spectrum whose asymptotic order is $O(h)$, and two high-frequency spectra whose asymptotic order is $O(h^{-1})$. Solving the problems of plate vibrations under the action of the quickly varying by time loads it is necessary to take into account vibrations with eigenfrequencies from the high-frequency spectra. That is why the problem of plate free vibrations with eigenfrequencies whose asymptotic order is $O(h^{-1})$ is interesting and practically important. In this paper asymptotic analysis of the equations of the Reissner's type plate theory for high-frequency free vibrations is carried out and the approximate equations of plate vibrations with eigenfrequencies of the asymptotic order $O(h^{-1})$ are proposed. Asymptotic analysis of the equations of plate free vibrations shows that behavior of the functions defining the stress-strain state of the plate for high-frequency free vibrations differs from it for low-frequency free vibrations. For high-frequency free vibrations the solution includes the functions which quickly vary along the space coordinates but which are not the boundary layer type functions. Because of that using of the exact equations of the Reissner's type theory in numerical procedures is difficult. Approximate equations of high-frequency free vibrations of plates independent of quickly varying along the space coordinates functions are formulated in this paper. These equations describe vibrations with eigenfrequencies from the high-frequency spectra only, like the classic plate theory describes vibrations with eigenfrequencies from the low-frequency spectrum only.

Keywords Reissner plate · High-frequency vibration

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12.1 Introduction

It is known that the transverse shear deformations [1–4] and the inertia of rotation [5,6] must be taken into account in some problems of forced vibrations of plates, in particular, for the plate vibrations under impact and other rapidly time-varying loads, for the thick plates and for the composite laminates. [7–10]. Therefore, the solution of the problem of eigenfrequencies and modes of plate vibrations for the Reissner theory [11, 12] is of great importance. The problem of law-frequency free vibrations of the Reissner plate is studied in detail. It is well known that the solution of the problem includes the slowly varying functions of the space coordinates and the rapidly varying functions of boundary-layer type [13]. In contrast to law-frequency free vibrations, the problem of high-frequency free vibrations of plate has been studied insufficiently. As far as we know, the problem was studied only in the paper [14] where the equations of free plate vibrations are obtained on the basis of three-dimensional elasticity by a variational-asymptotic method for various frequency spectra.

In contrast to [14], we obtain the approximate equations of high-frequency free plate vibrations starting from the asymptotic analysis of the equations of the Reissner plate theory. We show that the functions that describe the stress-strain state of the plate for the law-frequency and high-frequency vibrations, have quite a different character of varying with respect to the space coordinates. For high-frequency free vibrations the solutions include rapidly varying functions as well as for low-frequency ones, but for the high-frequency case these functions are not the boundary-layer type functions. They deeply penetrate into the plate domain. In contrast to [14], our attempt revises the problem of high-frequency vibrations so as to make it more convenient for numerical implementation and to cover all possible types of boundary conditions. We also note that the given equations for high-frequency vibrations differ from the equations in [14].

12.2 Summary of the Basic Equations of Free Vibrations of Reissner's Plate

Let us consider the problem of vibrations of a plate with taking account of the inertia of rotation, and of the transverse shear deformation. The deflection w, the vector Ψ of rotation angles, the vector N of shear forces, and the moment tensor M are related to the displacements and stresses in three-dimensional elasticity as follows [15, 16]:

$$
\begin{gathered}
hw = \int_{-h/2}^{h/2} \mathrm{u}\cdot\mathrm{n}\,dz, \qquad h^3\Psi = \int_{-h/2}^{h/2} \mathrm{u}\,z\,dz \\
\mathrm{N} = \int_{-h/2}^{h/2} \mathrm{a}\cdot\tau\cdot\mathrm{n}\,dz, \qquad \mathrm{M} = \int_{-h/2}^{h/2} \mathrm{a}\cdot\tau\cdot\mathrm{a}\,z\,dz, \qquad \mathrm{a} = \mathrm{E}-\mathrm{nn}.
\end{gathered}
\tag{12.1}
$$

Here u and τ are are the displacement vector and the stress tensor in the three-dimensional theory, h is the plate thickness, n is the vector of the unit normal to the plate plane, E is the unit tensor.

The theory of plate with taking account of the inertia of rotation, and of the transverse shear deformation includes the following equations.

The equations of the motion are

$$\nabla \cdot \mathrm{N} + \rho h P = \rho h \ddot{w}, \qquad \nabla \cdot \mathrm{M} - \mathrm{N} = \frac{1}{12}\rho h^3 \ddot{\Psi}, \tag{12.2}$$

where $P(x,y,t)$ is the external load, ρ is the mass density.

Constitutive equations take the form

$$\mathrm{N} = Gh\Gamma\gamma, \qquad \mathrm{M} = D[(1-\mu)æ + \mu\,\mathrm{tr}\,æ\,\mathrm{a}]. \tag{12.3}$$

Here γ is the transverse shear deformation vector, æ is the bending–twisting tensor, $D = Eh^3/[12(1-\mu^2)]$ is the bending stiffness, E is the Young modulus, μ is Poisson's ratio, $Gh\Gamma$ is the shear stiffness, Γ is the coefficient of transverse shear, $G = E/[2(1+\mu)]$ is the shear modulus.

Geometric relations are

$$\gamma = \nabla w + \Psi, \qquad æ = \frac{1}{2}\left(\nabla\Psi + \nabla\Psi^T\right). \tag{12.4}$$

The kinematic boundary conditions acquire the form

$$w|_c = w^*, \qquad \nu \cdot \Psi|_c = \Psi_\nu^*, \qquad \tau \cdot \Psi|_c = \Psi_\tau^*. \tag{12.5}$$

The force boundary conditions can be written as follows

$$\nu \cdot \mathrm{N}|_c = N_\nu^*, \qquad \nu \cdot \mathrm{M} \cdot \nu|_c = M_\nu^*, \qquad \nu \cdot \mathrm{M} \cdot \tau|_c = M_\tau^*. \tag{12.6}$$

Here ν and τ the unit outward normal vector and the unit tangent vector to the plate contour, respectively; the vectors ν, τ, and n, are assumed to form a right-handed system; Ψ_ν^* and Ψ_τ^* are angles of rotation about the tangent vector and the normal vector to the plate contour, respectively; N_ν^* is the lateral force, M_ν^* is bending moment, M_τ^* is the torque.

Introducing potentials Φ and F we reduce the equations of the plate theory to the more convenient form [17]:

$$D\Delta\Delta\Phi + \rho h\ddot{\Phi} - \frac{\rho h^3}{12}\left(1 + \frac{2}{\Gamma(1-\mu)}\right)\Delta\ddot{\Phi} + \frac{\rho^2 h^3}{12G\Gamma}\ddddot{\Phi} + \rho h P = 0, \tag{12.7}$$

$$\Delta F - \frac{12\Gamma}{h^2}F - \frac{\rho}{G}\ddot{F} = 0. \tag{12.8}$$

The quantities characterizing the stress-strain state of a plate are expressed in terms of the potentials Φ and F by the formulas

$$w = -\Phi + \frac{h^2}{6\Gamma(1-\mu)}\Delta\Phi - \frac{\rho h^2}{12G\Gamma}\ddot{\Phi}, \qquad \Psi = \nabla\Phi + \nabla F \times \mathrm{n},$$
$$\mathrm{N} = D\nabla\Delta\Phi - \frac{\rho h^3}{12}\nabla\ddot{\Phi} + Gh\Gamma\nabla F \times \mathrm{n}, \tag{12.9}$$
$$\mathrm{M} = D\Big[(1-\mu)\nabla\nabla\Phi + \mu\Delta\Phi\,\mathrm{a} + \frac{1-\mu}{2}(\nabla\nabla F \times \mathrm{n} - \mathrm{n} \times \nabla\nabla F)\Big].$$

It is known [18] that in the Reissner's type theory of plates there are three spectra of eigenfrequencies, which satisfy the following asymptotic estimates:

$$\omega_i^{(1)} = h\omega_{1i}^{(1)} + h^2\omega_{2i}^{(1)} + \dots$$
$$\omega_i^{(2)} = \sqrt{\frac{12G\Gamma}{\rho h^2}} + \omega_{0i}^{(2)} + \dots \qquad \omega_i^{(3)} = \sqrt{\frac{12G\Gamma}{\rho h^2}} + \omega_{0i}^{(3)} + \dots \tag{12.10}$$

where the first spectrum in Eqs (12.10) describes the low frequency bending vibrations, whereas the second and the third spectra in (12.10) characterize the high frequency shear and bending vibrations.

12.3 Asymptotic Analysis of the Equations of Reissner's Plate Theory

For the free high-frequency vibrations, the functions Φ and F substantially differ from those for the low-frequency vibrations or static bending. The function F varies slowly with respect to the spatial coordinates and is not of boundary-layer type. This is due to the fact that the leading terms in the first and in the second components in Eq. (12.8) cancel each other. For the approximate statement of the problem, Eq. (12.8) remains the same. A very important feature of the high-frequency vibrations is that the asymptotic orders of the functions F and Φ are the same: $F \sim \Phi$. The penetrating potential Φ for the high-frequency vibrations has quite a different structure than in the low-frequency case; namely, along with functions, slowly varying with respect to the spatial coordinates, it includes a rapidly varying function, which was lacking in the preceding cases. Let us denote this rapidly varying function by φ, and retain the notation Φ for the slowly varying component as well as for the penetrating potential itself. This seems convenient, since there is an asymptotic relation $\varphi \sim h^2\Phi$. The function φ seemingly need not be taken into account since it is relatively small. However, this is not the case, and the function φ may exert influence on the leading terms of some characteristics of the stress-strain state, which can depend not only on the penetrating potential, but also on its derivatives of order ≤ 3. Let us proceed from Eq. (12.7) to the approximate equations for the components Φ and φ. We suppose that for the functions Φ, and φ the following asymptotic estimates hold

$$\ddot{F} = \Big[-\frac{12G\varGamma}{\rho h^2} + O(1)\Big]F, \quad \ddot{\varPhi} = \Big[-\frac{12G\varGamma}{\rho h^2} + O(1)\Big]\varPhi, \quad \ddot{\varphi} = \Big[-\frac{12G\varGamma}{\rho h^2} + O(1)\Big]\varphi,$$

$$\frac{\partial F}{\partial x} \sim \frac{\partial F}{\partial y} \sim F, \quad \frac{\partial \varPhi}{\partial x} \sim \frac{\partial \varPhi}{\partial y} \sim \varPhi, \quad \frac{\partial \varphi}{\partial x} \sim \frac{\partial \varphi}{\partial y} \sim \frac{1}{h}\varphi, \quad \varphi \sim h^2\varPhi \sim h^2 F.$$

12.4 Approximate Formulation of the Problem of High-Frequency Free Vibrations

On substituting the expression for the penetrating potential $\varPhi + \varphi$ into Eq. (12.7) and by retaining only the leading terms, we obtain

$$\begin{gathered} GhA(\varPhi) + DB(\varphi) = 0, \\ A(\varPhi) = \Big(\varGamma + \frac{2}{1-\mu}\Big)\varDelta\varPhi - \frac{12\varGamma}{h^2}\varPhi - \frac{\rho}{G}\ddot{\varPhi}, \\ B(\varphi) = \varDelta\Big[\varDelta\varphi + \frac{12}{h^2}\Big(1 + \frac{\varGamma(1-\mu)}{2}\Big)\varphi\Big]. \end{gathered} \tag{12.11}$$

Note that the first equation in (12.11) contains an obvious contradiction. On the one hand, $A(\varPhi)$ is a slowly varying function, since it depends on $\varPhi$, and $B(\varphi)$ is a rapidly varying function, since it depends on φ. On the other hand, according to the first equation in (12.11), the functions $A(\varPhi)$ and $B(\varphi)$ are proportional. Thus, $A(\varPhi)$ and $B(\varphi)$ are slowly varying and rapidly varying simultaneously, that is, they are both zero. Hence, the first equation in (12.11) represents the two equations

$$A(\varPhi) = 0, \qquad B(\varphi) = 0. \tag{12.12}$$

The first equation in (12.12) has the form

$$\Big(\varGamma + \frac{2}{1-\mu}\Big)\varDelta\varPhi - \frac{12\varGamma}{h^2}\varPhi - \frac{\rho}{G}\ddot{\varPhi} = 0. \tag{12.13}$$

Equation (12.13) permits us to find the leading term of the slowly varying part of the penetrating potential. It should be noted that Eq. (12.13) for $\varPhi$ coincides with Eq. (12.8) for F to within a constant coefficient of the first summand, and the behavior of $\varPhi$ is similar to that of F in the case of high-frequency vibrations.

The second equation in (12.12) is

$$\varDelta z(\varphi) = 0, \qquad z(\varphi) = \varDelta\varphi + \frac{12}{h^2}\Big(1 + \frac{\varGamma(1-\mu)}{2}\Big)\varphi. \tag{12.14}$$

The solution of the equation $\varDelta z = 0$ is a slowly varying function. Since $z(\varphi)$ varies rapidly, all solution except for zero are excluded: $z \equiv 0$. Then the equation for φ acquires the form

$$\Delta\varphi + \frac{12}{h^2}\Big(1 + \frac{\Gamma(1-\mu)}{2}\Big)\varphi = 0. \tag{12.15}$$

Equation (12.15) allows one to find the leading term of the rapidly varying part of the penetrating potential. Note that in contrast to the low-frequency vibrations, the rapidly varying function for the high-frequency vibrations is not of boundary-layer type, but penetrates into the entire plate domain.

For the high-frequency vibrations, the characteristics of stress-strain state of the plate have the following asymptotic representations:

$$\begin{gathered} w = -\frac{h^2}{12}\Delta\Phi - \Big(1 + \frac{2}{\Gamma(1-\mu)}\Big)\varphi, \\ \Psi = \nabla\Phi + \nabla F\times \mathrm{n}, \quad \mathrm{N} = Gh\Gamma(\nabla\Phi + \nabla F\times\mathrm{n}), \\ \mathrm{M} = D\Big[(1-\mu)\nabla\nabla(\Phi+\varphi) + \mu\Delta(\Phi+\varphi)\,\mathrm{a} + \frac{1-\mu}{2}(\nabla\nabla F\times\mathrm{n} - \mathrm{n}\times\nabla\nabla F)\Big]. \end{gathered} \tag{12.16}$$

Before proceeding to the statement of the boundary conditions, let us focus our attention on the following important property of φ. Consider a boundary condition depending on φ, for example, $w|_c = 0$. Obviously, φ cannot vary rapidly on the boundary, since the other components in the boundary condition vary slowly. A similar conclusion holds for any boundary condition that depends on φ. In addition, φ is generally nonzero on the plate contour, since otherwise it would obviously be zero in the entire plate domain. The most natural conclusion follows: on the plate boundary, the function φ loses the property of being rapidly varying and becomes a slowly varying function along the contour. We point out that φ varies slowly only on the plate boundary; it rapidly varies in every direction in the interior of the domain arbitrarily close to the boundary. Thus, the tangent derivative of φ on the plate contour has the same asymptotic order as φ: $\partial\varphi/\partial\tau|_c \sim \varphi$.

The kinematic boundary conditions acquire the form

$$\Big[-\frac{h^2}{12}\Delta\Phi - \Big(1 + \frac{2}{\Gamma(1-\mu)}\Big)\varphi\Big]\Big|_c = 0, \qquad \Big(\frac{\partial\Phi}{\partial\nu} + \frac{\partial F}{\partial\tau}\Big)\Big|_c = 0, \qquad \Big(\frac{\partial\Phi}{\partial\tau} - \frac{\partial F}{\partial\nu}\Big)\Big|_c = 0. \tag{12.17}$$

The force boundary conditions are written as follows:

$$\begin{gathered} Gh\Gamma\Big(\frac{\partial\Phi}{\partial\nu} + \frac{\partial F}{\partial\tau}\Big)\Big|_c = 0, \\ \Big[D(1-\mu)\Big[\frac{\partial^2 F}{\partial\nu\partial\tau} - \Big(\frac{1}{R}\frac{\partial\Phi}{\partial\nu} + \frac{\partial^2\Phi}{\partial\tau^2}\Big)\Big] + D\Big[\Delta\Phi - \frac{12}{h^2}\Big(1 + \frac{\Gamma(1-\mu)}{2}\Big)\varphi\Big]\Big]\Big|_c = 0, \\ \Big[D(1-\mu)\Big[\frac{\partial^2\Phi}{\partial\nu\partial\tau} + \Big(\frac{1}{R}\frac{\partial F}{\partial\nu} + \frac{\partial^2 F}{\partial\tau^2}\Big)\Big] - \Big(\frac{\rho h^3}{12}\ddot{F} + Gh\Gamma F\Big)\Big]\Big|_c = 0. \end{gathered} \tag{12.18}$$

The physical meaning of the boundary conditions (12.17) and (12.18) is the same as that of conditions (12.5) and (12.6), respectively. Equations (12.8), (12.13), and (12.15) supplemented with the boundary conditions (12.17) and (12.18) form an approximate statement of the problem of free high-frequency vibrations, which permits us to find the leading terms of the characteristics of the stress-strain state and the eigenfrequencies with a relative error $O(h^4)$. Recall that the main terms of the eigenfrequencies are equivalent to $\sqrt{12G\Gamma/(\rho h^2)}$.

12.5 Formulation of the Problem of High-Frequency Free Vibrations of the Reissner's Plate without Taking Account of Rapidly Varying Function

As was noted in the preceding, φ is a rapidly varying function and penetrates into the entire plate domain. Hence, the cited statement for the high-frequency vibrations is practically invalid in numerical implementations. Is it possible to revise this problem without taking account of φ? Before answering this question, let us note the following: the vector Ψ of rotation angles and the vector N of shearing forces, which do not depend on φ (see Eqs (12.16)), are two orders of magnitude larger than the deflection w and the moment tensor M respectively. Thus allows one to claim, that the stress-strain state is mainly characterized by the vector of rotation angles and the vector of shear forces, whereas the deflection and the moment tensor play a less important role. Therefore, the statement of this problem without the function φ is consistent in principle.

Thus, let us eliminate Eq. (12.15) from the system of equations for the high-frequency vibrations. Then the order of the system with respect to the spatia] derivatives is reduced from six to four. Hence, the three boundary conditions in the original statement must be replaced by two boundary conditions independent of φ. One of the original three boundary conditions is the third condition in (12.17) or in (12.18). Since these conditions do not depend on φ, they are retained. Two other conditions are replaced by one condition according to the following rule:

- the second condition in (12.17) and the first condition in (12.18) are equivalent, and if they are given simultaneously, there is no need to choose one of them (note that the function φ is identically zero for these boundary conditions);
- if the first and the second conditions in (12.17) are given, the second condition must be retained, since it is independent of φ (whereas the first condition depends on φ);
- if the first and the second conditions in (12.18) are given, the first condition must be retained, since it is independent of φ (whereas the second condition depends on φ);
- since the first condition in (12.17) and the second condition in (12.13) depend on φ they can be replaced by the combination

$$\left[D(1-\mu)\Big[\frac{\partial^2 F}{\partial\nu\partial\tau}-\Big(\frac{1}{R}\frac{\partial\Phi}{\partial\nu}+\frac{\partial^2\Phi}{\partial\tau^2}\Big)\Big]+\Big(\frac{\rho h^3}{12}\ddot{\Phi}+Gh\Gamma\Phi\Big)\right]\Bigg|_c=0. \qquad (12.19)$$

The physical meaning of the boundary conditions without the function φ is the following. The third condition in (12.17) means that the angle Ψ_τ of rotation about the normal to the plate contour is zero. The second condition in (12.17) (and the equivalent first condition in (12.18)) indicates that the angle Ψ_ν of rotation about the tangent to the plate contour is zero. The third condition in (12.18) means that the torque M_τ is zero. Condition (12.19) indicates that the reduced bending moment $M_\nu - Gh\Gamma w$ is zero.

12.6 Asymptotic and Numerical Analysis of High-Frequency Free Vibrations of Rectangular Plates

Free vibrations of rectangular plates with frequencies belonging to high-frequency spectra are studied. The results predicted by the exact Reissner's type theory are compared with those predicted by an approximate theory of high-frequency free vibrations which takes into account only functions slowly varying with respect to the spatial coordinates. It is well known that in solving some dynamical problems of plates, in particular, problems on forced vibrations under the action of impact loadings, one cannot ignore high-frequency vibrations which are associated with the inertia of rotation and the transverse shear deformation. As pointed out above, for high-frequency vibrations, the solution contains functions rapidly varying with respect to spatial coordinates and penetrating into the entire domain of the plate. The presence of such functions makes the exact equations of the Reissner theory practically unsuitable for the numerical analysis of the problems. Above an approximate statement of the problem on high-frequency free vibrations of a plate was suggested; only functions that vary slowly with respect to the spatial coordinates are taken into account. The asymptotic accuracy of this statement is $O(h)$ compared with unity in determining eigenfrequencies and $O(h^4)$ in determining eigenfrequencies. This difference in the accuracy is accounted for by the fact that the leading terms of asymptotic expansions for all eigenfrequencies coincide and are known, whereas the approximate theory defines the first correcting term in the asymptotic expansions for the eigenfrequencies.

Of course, the asymptotic accuracy of a theory is an important characteristic. However, to assess an asymptotic theory from the viewpoint of its practical significance, the actual accuracy of the theory is of importance rather than its asymptotic accuracy. (By the actual accuracy we mean the relative difference of the value of a quantity predicted by the approximate theory and the value of that quantity predicted by the exact theory for the given value of the small parameter.) In what follows we deal with the analysis of the actual accuracy of the approximate theory of high-frequency vibrations which was suggested above. The purpose on this research is to determine the area of applicability of the theory.

The investigation is exemplified by problems having exact analytical solution, which allows us to rule out practically any errors of calculations. Now we consider rectangular plates two opposite sides of which are hinged. Let us consider a plate occupying a domain $-a \le x \le a, -b \le y \le b$. The constrained hinged support conditions are assumed to be satisfied at the sides $y = \pm b$:

$$w|_c = 0, \qquad M_\nu|_c = 0, \qquad \Psi_\tau|_c = 0. \tag{12.20}$$

The boundary conditions at the sides $x = \pm a$ can be arbitrary. We study vibrations symmetric with respect to the axes $x = 0$ and $y = 0$. The eigenforms satisfying the differential equations (12.7) and (12.8) and the boundary conditions (12.20) at $y = \pm b$ have the form

$$\begin{aligned}
&\Phi_n(x,y) = [C_{1n}\cos(\lambda_{1n}x) + C_{2n}\cos(\lambda_{2n}x)]\cos(\mu_n y),\\
&F_n(x,y) = C_{3n}\sin(\delta_n x)\sin(\mu_n y),\\
&\mu_n = (2n-1)\pi/(2b), \qquad \lambda_{1n} = \sqrt{A_n - B_n}, \qquad \lambda_{2n} = \sqrt{A_n + B_n},\\
&\delta_n = \sqrt{\rho\omega_n^2/G - 12\Gamma/h^2 - \mu_n^2}, \qquad A_n = [1 + \Gamma(1-\mu)/2]\rho\omega_n^2/(2G\Gamma) - \mu_n^2,\\
&B_n = \sqrt{\rho h/D + ([1 - \Gamma(1-\mu)/2]\rho\omega_n^2/(2G\Gamma))^2}.
\end{aligned} \tag{12.21}$$

By satisfying the boundary conditions at $x = \pm a$ one reduces the problem to solving a system of homogeneous algebraic equations for the coefficients C_{1n}, C_{2n}, C_{3n}. By equating the determinant of this system to zero, one obtains an equation for determining eigenfrequencies. We considered all types of boundary conditions possible in the Reissner's type theory and obtained the frequency equations for each of them.

Let us discuss the solution of the problem according to the approximate theory of high-frequency free vibrations. It can be readily shown that the eigenforms satisfying the differential equations (12.8) and (12.13) and the constrained hinged support conditions (12.20) at $y = \pm b$ have the form

$$\begin{aligned}
&\Phi_n(x,y) = C_{1n}\cos(\lambda_{1n}x)\cos(\mu_n y), \qquad F_n(x,y) = C_{3n}\sin(\delta_n x)\sin(\mu_n y),\\
&\lambda_{1n} = \sqrt{\frac{\rho}{G}\frac{\omega_{0n}}{\Gamma + 2/(1-\mu)} - \mu_n^2}, \quad \delta_n = \sqrt{\frac{\rho}{G}\omega_{0n} - \mu_n^2}, \quad \omega_{0n} = \omega_n^2 - \frac{12G\Gamma}{\rho h^2}.
\end{aligned} \tag{12.22}$$

The asymptotic analysis shows that the frequency equations and the eigenforms predicted by tho asymptotic theory in question follow from the exact frequency equations and the exact eigenforms within an $O(h)$ asymptotic error for all types of boundary conditions (by exact frequency equations and exact eigenforms we mean those obtained by the Reissner's type theory).

The numerical analysis was carried out for the problem discussed above. Computations were performed for plates of dimensions $a = b = 1$ m and thicknesses $h = 0{,}1$ m and $h = 0{,}04$ m with the elastic constants $E = 2{,}1 \cdot 10^{11}$ Pa, $\mu = 0{,}25$, $\Gamma = 5/6$, and $\rho = 7{,}951 \cdot 10^3$ kg/m^3.

The key results can be summarized as follows. The first 10 eigenfrequencies of high-frequency spectra are found. The calculations were performed by the exact theory and by the approximate theory for all types of boundary conditions possible in the Reissner's type theory. The actual errors of the approximate theory are found. The calculations were performed for all types of boundary conditions and the plate thicknesses $0,1$ m and $0,04$ m. For the case in which the free edge conditions were imposed at the sides $x = \pm a$, more detailed investigation was carried out. The first 10 eigenfrequencies for plates of thickness $0,2$ m, $0,3$ m, $0,4$ m, and $0,5$ m and the corresponding actual errors are calculated. The eigenforms corresponding to the first 10 eigenfrequencies are determined. The calculations were performed by the exact and approximate theories for all types of boundary conditions and the plate thickness $0,1$ m. In the case of the free edge conditions, the eigenforms are found for the plate of thickness $0,04$ m as well. For the eigenforms in calculation of which the approximate theory leads to the largest errors, the graphs of the rotation angles Ψ_x and Ψ_y versus the coordinate x are constructed.

Let us point out some general features characteristic of high-frequency spectra and the approximation of these spectra by the approximate theory.

The frequencies of the bending spectrum are higher than those of the shear spectrum. (Calculation show that for all types of boundary conditions at $x = \pm a$, only two frequencies of the first ten belong to the bending spectrum.) The accuracy for the frequencies belonging to the bending spectrum is, as a rule, less than that for the frequencies belonging to the shear spectrum. This is quite natural, since in the approximate theory, the equation responsible for the shear vibrations is exact, whereas the equation responsible for the bending vibrations is approximate. While the approximate theories when applied to low-frequency vibrations yield higher values for the eigenfrequencies compared with the exact ones, this is not the case for high-frequency vibrations. For high-frequency vibrations, no monotonic increase in the relative error with the mode number is observed either. Of course, lower frequencies are, on the average, predicted more accurately than higher frequencies. However, for high frequency vibrations, the situation in which a frequency with a larger mode number is predicted more accurately than many frequencies with smaller mode numbers is usual.

Let us briefly dwell on the results of calculation of eigenforms. We carried out the investigation of the accuracy provided by the approximate theory for eigenforms as follows. For all types of boundary conditions, analytical expressions for the potentials $F(x,y)$ and $\Phi(x,y)$ were obtained according to the exact and approximate theories. Then the eigenforms represented by $F(x,y)$ were compared with those represented by $\Phi(x,y)$. As a result, it was established that most of eigenforms predicted by the approximate theory virtually coincide with those predicted by the exact Reissner's type theory. The errors turned out to be noticeable only in the case of the free edge conditions at $x = \pm a$. Therefore, the subsequent discussion pertains just to this type of boundary condition ($h = 0,1$ m). It has been established that the eigenforms predicted by the approximate theory and by the exact theory are in quite good agreement, which allows us to conclude that the approximate theory provides high accuracy in predicting eigenforms as well. This statement is valid for the overwhelming majority of eigenforms. Nevertheless, in exceptional cases, the difference in eigenforms predicted by the exact and approximate theories can be large. It should be emphasized that such cases are encountered rather rarely: among eigenforms found in the course of the present investigation (10 eigenforms were calculated for each of the eight types of boundary conditions), the exact and the approximate theories were established to disagree only in one case.

12.7 Discussion of the Physical Meaning of Obtained Results

From the physical viewpoint, it seems obvious that the high-frequency vibrations are produced by the shear phenomena. The presented asymptotic estimates confirm this assertion. Indeed, for the low-frequency vibrations the vector $\Psi = -\nabla w + \gamma$ of rotation angles is actually determined by the deflection w, and the vector γ of transverse shear deformation represents the unimportant refinements: $\gamma \sim h\Psi$ in the vicinity of the boundary and $\gamma \sim h^2\Psi$ inside the domain. For the high-frequency vibrations the situation is quite opposite: the vector Ψ of rotation angles practically coincides with the vector of transverse shear deformation γ, and the deflection w adds unimportant refinements:$\Psi \sim \gamma$, $\nabla w \sim h\Psi$. Since the nature of the high-frequency shear and high-frequency bending vibrations is the same (in particular, this is confirmed by the coincidence of leading eigenfrequency terms in the shear and bending spectra), the equations for both vibrations naturally seem to be similar. This is just the case in the suggested statement of problem for the high-frequency vibrations: Eq. (12.13) for the bending vibrations practically coincides with Eq. (12.8) for the shear vibrations.

Thus, if we assume similarity of the bending and shear vibrations, then the order of the system with respect to the space coordinates is reduced. This means that yet another function is not included. Since we already have the equations for the shear and bending vibrations, the equation for this function is likely to be the static equation. In the presented statement, Eq. (12.15) for φ just does not contain the time derivatives. From the physical considerations, it is obvious that this function characterizes the bending phenomena and has no importance in the problem of high-frequency vibrations. This assertion is supported by the fact that this function rapidly varies with respect to the space coordinates, as we can see from Eq. (12.15), and penetrates into the entire plate domain. Would this function be of primary importance for high-frequency vibrations, we should adopt that the plate theory does not apply to this problem. Thus, the possibility of stating the problem without rapidly varying functions seems to be natural from the physical viewpoint. The boundary conditions in the proposed statement are also worth saying a few words. First, it must be noted that there are four possible boundary conditions, and two of them are kinematic: $\Psi_\nu|_c = 0$ (the angle of rotation about the tangent) and $\Psi_\tau|_c = 0$ (the angle of rotation about the normal); the other two are the force conditions: $M^*_\nu\big|_c = 0$ (reduced bending moment) and $M_\tau|_c = 0$ (torque).

Note that the meaning of the kinematic and the force conditions slightly change in proceeding from the original statement to the approximate statement in the Reissner's type theory. For the approximate statement, the conditions $\Psi_\nu|_c = 0$ and

$N_\nu|_c = 0$ are equivalent, since $\mathrm{N} = Gh\Gamma\Psi$, and the force condition $N_\nu|_c = 0$ becomes kinematic. The kinematic condition $w|_c = 0$ disappears in the approximate statement, and the deflection appears in the condition $M_\nu^*\big|_c = 0$, since $M_\nu^* = M_\nu - Gh\Gamma w$, that is, it becomes part of the force boundary condition. A certain symmetry for the problem of high-frequency vibrations is not restricted by the similarity of Eqs (12.8) and (12.13). The similarity also occurs for the boundary conditions. In fact, the condition $\Psi_\nu|_c = 0$ yields $\Psi_\tau|_c = 0$, and the condition $M_\nu^*\big|_c = 0$ yields $M_\tau|_c = 0$ by the substitution $\Phi \to F$, $F \to -\Phi$. Such symmetry also supports the cited statement (rather aesthetically than physically).

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