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# STEADY-STATE MOTIONS OF A NONSYMMETRIC TOP

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Steady-state motions of a nonsymmetric top with a fixed point under the action of elastic and driving torques are considered. The asymmetry of the top is provided by an unbalanced point mass. Three independent steady-state motion modes are singled out. One of the motions is shown to be a convenient model for the investigation of general properties of forced nonlinear vibrations. The analysis of this motion is given to a first nonlinear approximation.

The stability of the basic steady-state motion is investigated. For special cases (the motion driven by a motor of infinitely large power and the motion without drive), simple conditions of stability are obtained and the location of stability regions on the amplitude-frequency diagram of the system is considered. For motion without drive, the concept of modified frequency of rotation is introduced which provides a correspondence between the arrangement of the stability regions and the shape of the amplitude-frequency characteristic.

## 1. INTRODUCTION

Problems of dynamics of rigid bodies and problems of dynamics of rigid rotors are considered in numerous publications. However, there is a gap at the boundary between these areas. First of all, this is the allowance for nonlinearities in the equations of rotor dynamics due to inertial terms. In the present paper, we investigate the influence of the inertial terms on the system dynamics for a relatively simple model. Particular interest in steady-state motions is accounted for by the fact that this sort of motion is a natural extension of the motion described by linearized equations to the nonlinear area. Nevertheless, the values of parameters for which the problem becomes related to the dynamics of rigid bodies (finite amplitudes, negative stiffness, etc.) are also considered.

For a two-dimensional analog of the problem treated in the present paper, nonlinear steady-state motions and their stability were investigated in [1–3], where the nonlinearity was associated with the properties of elastic parts of the system. From the viewpoint of dynamics of rigid bodies, the motion considered in the present paper is close to Staude's permanent rotations of a heavy rigid body [4, 5]; the difference is in the form of the external torque. The present work was prompted by the problems of dynamics of centrifuges considered in [6, 7].

A special form of equations of motion of a symmetric rigid body was derived in [8] in nondegenerating variables that can be clearly interpreted. In the present paper, similar equations are derived for some cases of nonsymmetric body.

We use the tensor notation [9, 10]; the tensor product of vectors **a** and **b** is denoted by **ab** and the identity tensor by **E**. The application of the tensor of rotation to dynamics of rigid body is described in [11]. The operator method for description of rotations of a rigid body, which is close to the tensor method, is considered in [12].

## 2. BASIC NOTATION AND RELATIONS

**2.1. Mass-geometric characteristics.** We consider an axisymmetric rigid body with a fixed point (a symmetric top); a point mass (unbalanced mass), which is not generally assumed to be small, is rigidly attached to the body (see Fig. 1).

Let **k**, **n**, and $\sigma$ be the unit vectors of the vertical axis, the symmetry axis of the top, and the direction to the unbalanced mass, respectively. Denote by $\gamma$, $|\vartheta|$, and $|\chi|$ the angles between **n** and $\sigma$, **k** and **n**, and **k** and $\sigma$, respectively.

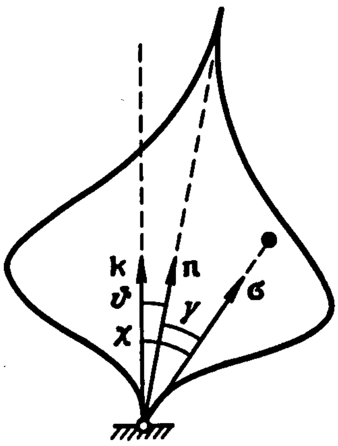


Fig. 1

The quantity $\gamma$ is positive, the signs of $\vartheta$ and $\chi$ will be defined later. The angle $\gamma$ defining the top geometry is constant, and the angles $\vartheta$ and $\chi$ are variable. Denote $\eta \stackrel{\text{def}}{=} \mathbf{k} \cdot \mathbf{n} = \cos\vartheta$.

The tensor of inertia of a symmetric top about the fixed point can be expressed as $\boldsymbol{\theta} = \theta_{12}(\mathbf{E} - \mathbf{nn}) + \theta_3\mathbf{nn}$, where $\theta_{12}$ is the equatorial moment of inertia and $\theta_3$ is the axial moment of inertia. Note that the inequality $\theta_3 < 2\theta_{12}$ is valid. Denote by $\theta \stackrel{\text{def}}{=} \theta_{12} - \theta_3$ the gyroscopic moment of inertia. We will distinguish between the prolate top ($\theta > 0$, $\theta_3 < \theta_{12}$) and the oblate top ($\theta < 0$, $\theta_3 > \theta_{12}$).

The inertial properties of the unbalanced mass are determined by its moment of inertia $\theta_* \stackrel{\text{def}}{=} m_*\rho^2$, where $m_*$ is the magnitude of the unbalanced mass and $\rho$ is the distance from the fixed point to the unbalanced mass.

The mass-geometric characteristics of the system can be described by four parameters, for example, $\theta_{12}$, $\theta_3$, $\theta_*$, and $\gamma$. One can reduce the number of parameters to three by introducing dimensionless variables.

**2.2. Angular momentum.** Let us derive vector relations for the angular momentum of the system. Since the unit vector $\mathbf{n}$ is rigidly connected with the top, we have $\dot{\mathbf{n}} = \boldsymbol{\omega} \times \mathbf{n}$, where $\boldsymbol{\omega}$ is the angular velocity of the top. By premultiplying these relations vectorially by $\mathbf{n}$, we arrive at the formula

$$\boldsymbol{\omega} = \mathbf{n} \times \dot{\mathbf{n}} + \Omega\mathbf{n}, \qquad \Omega \stackrel{\text{def}}{=} \mathbf{n} \cdot \boldsymbol{\omega}, \tag{2.1}$$

for the angular velocity. The quantity $\Omega$ will be referred to as *the angular velocity of proper rotation*.

Using (2.1), one can obtain the simple formula

$$\mathbf{K}_0 = \boldsymbol{\theta} \cdot \boldsymbol{\omega} = [\theta_{12}(\mathbf{E} - \mathbf{nn}) + \theta_3\mathbf{nn}] \cdot [(\mathbf{n} \times \dot{\mathbf{n}}) + \Omega\mathbf{n}] = \theta_{12}\mathbf{n} \times \dot{\mathbf{n}} + \theta_3\Omega\mathbf{n} \tag{2.2}$$

for the angular momentum $\mathbf{K}_0$ of a symmetric top. This formula, in a different notation, was obtained in [8]. The angular momentum of the unbalanced mass is given by

$$\mathbf{K}_* = \rho\boldsymbol{\sigma} \times \frac{d}{dt}(m_*\rho\boldsymbol{\sigma}) = \theta_*\boldsymbol{\sigma} \times \dot{\boldsymbol{\sigma}}. \tag{2.3}$$

By adding expressions (2.2) and (2.3), we obtain the formula

$$\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_* = \theta_{12}\mathbf{n} \times \dot{\mathbf{n}} + \theta_*\boldsymbol{\sigma} \times \dot{\boldsymbol{\sigma}} + \theta_3\Omega\mathbf{n} \tag{2.4}$$

for the total angular momentum $\mathbf{K}$ of the nonsymmetric top with respect to the fixed point.

**2.3. External disturbances.** We assume that the top is acted upon by an external elastic torque $\mathbf{M}_e = C\mathbf{n} \times \mathbf{k}$. For simplicity, we consider the case where $C$ is constant (independent of $\vartheta$). The results can be readily extended to the case of arbitrary dependence $C(\vartheta)$. Both positive $C$ (restoring torque) and negative $C$ (destabilizing torque) are considered. The signs of $C$ and $\theta$ classify tops into four types with essentially different behavior.

We also assume that a driving torque $\mathbf{M}_d$ is exerted on the top. From now on, by $\mathbf{M}_d$ we understand either the "dead" torque $\mathbf{M}_d = M_d\mathbf{k}$ or the torque $\mathbf{M}_d = M_d\mathbf{n}$ pointing along the symmetry axis of the top. For convenience, we will use the unified notation $\mathbf{M}_d = M_1\mathbf{k} + M_2\mathbf{n}$, where $M_1$ and $M_2$ are equal either to $M_d$ or to zero, depending on

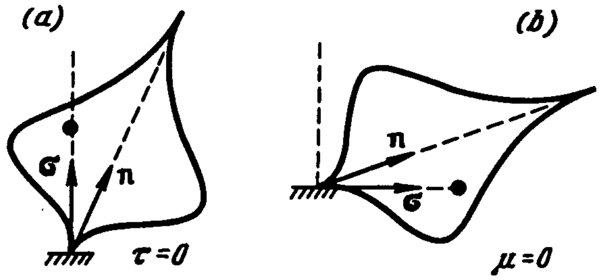


Fig. 2

which of the moments is considered. The quantity $M_d$ is a function of the speed of rotation of the top; this function will be specified later.

Thus, the total external torque applied to the system has the form

$$\mathbf{M} = \mathbf{M}_e + \mathbf{M}_d = C\mathbf{n}\times\mathbf{k} + M_1\mathbf{k} + M_2\mathbf{n}. \tag{2.5}$$

**2.4. Equations of dynamics.** The principle of angular momentum implies the equation

$$\dot{\mathbf{K}} = \mathbf{M}. \tag{2.6}$$

By substituting (2.4) and (2.5) into (2.6), we obtain the vector differential equation

$$\theta_{12}\mathbf{n}\times\ddot{\mathbf{n}} + \theta_*\boldsymbol{\sigma}\times\ddot{\boldsymbol{\sigma}} + \theta_3\frac{d}{dt}(\Omega\mathbf{n}) + C\mathbf{k}\times\mathbf{n} - M_1\mathbf{k} - M_2\mathbf{n} = 0 \tag{2.7}$$

governing the system motion. To complete the system of equations, one must add kinematic equations relating the parameters $\mathbf{n}$, $\boldsymbol{\sigma}$, $\Omega$, $M_1$, and $M_2$. However, this is practically unnecessary when considering steady-state motions.

## 3. STEADY-STATE MOTIONS

**3.1. Three types of steady-state motions.** We will seek the steady-state motions in the form of permanent rotations

$$\boldsymbol{\omega} \overset{\text{def}}{=} \omega\mathbf{k}, \qquad \dot{\omega} = 0, \tag{3.1}$$

where $\omega$ can be interpreted as the precession angular velocity. From (3.1), we immediately obtain

$$\dot{\mathbf{n}} = \boldsymbol{\omega}\times\mathbf{n} = \omega\mathbf{k}\times\mathbf{n}, \quad \dot{\boldsymbol{\sigma}} = \boldsymbol{\omega}\times\boldsymbol{\sigma} = \omega\mathbf{k}\times\boldsymbol{\sigma}, \quad \dot{\eta} = \mathbf{k}\cdot\dot{\mathbf{n}} = 0, \quad \Omega = \dot{\mathbf{n}}\cdot\boldsymbol{\omega} = \omega\eta. \tag{3.2}$$

Develop the vectors $\mathbf{n}$ and $\boldsymbol{\sigma}$ into the horizontal and vertical components:

$$\mathbf{n} = \boldsymbol{\epsilon} + \eta\mathbf{k}, \quad \boldsymbol{\sigma} = \boldsymbol{\tau} + \mu\mathbf{k}, \qquad \mathbf{k}\cdot\boldsymbol{\epsilon} = \mathbf{k}\cdot\boldsymbol{\tau} = 0.$$

Substitute (3.2) into (2.7). Then the vertical component of the resulting equation yields $M_1 = 0$ and $M_2 = 0$, and the horizontal component leads to the basic equation of steady-state motion:

$$(\theta\eta - C/\omega_2)\boldsymbol{\epsilon} + \theta_*\mu\boldsymbol{\tau} = \mathbf{0}. \tag{3.3}$$

All scalar quantities in Eq. (3.3) are constant and the vectors $\boldsymbol{\epsilon}$ and $\boldsymbol{\tau}$ are uniformly rotating.

Let us analyze Eq. (3.3). This equation has three different solutions characterized, respectively, by the relations

$$\boldsymbol{\tau} = \mathbf{0}, \quad \mu = 0, \quad \boldsymbol{\epsilon}\times\boldsymbol{\tau} = \mathbf{0}. \tag{3.4}$$

These solutions define three modes of steady-state motion. The first two modes are degenerate; in these modes, the top moves so that the inertial forces due to the unbalanced mass do not create a torque with respect to the fixed point (Fig. 2). From (3.3), we obtain

$$\theta\eta = \frac{C}{\omega^2} \quad \Longleftrightarrow \quad \omega^2 = \frac{C}{\theta\cos\vartheta} \tag{3.5}$$

for these steady-state motion modes.

For the second mode, relation (3.5) is the amplitude-frequency characteristic; the first mode can occur only for fixed values of the nutation angle, namely, either $|\vartheta| = |\gamma|$ or $|\vartheta| = \pi - |\gamma|$. For small $\vartheta$, these two steady-state motion modes can occur only for special values of $\gamma$. For this reason, they are not investigated in detail in the present paper.

Consider now the third, basic mode, in which the motion of the top is similar to forced oscillations. According to (3.4), the vectors $\boldsymbol{\epsilon}$ and $\boldsymbol{\tau}$ are collinear, and hence the vectors **k**, **n**, and $\boldsymbol{\sigma}$ are coplanar. This makes it possible to introduce the signs for the angles $\vartheta$ and $\chi$. Assume that $|\vartheta| < \gamma < \frac{1}{2}\pi$ and $|\chi| < \frac{1}{2}\pi$. This leads to little loss of generality but significantly simplifies the consideration.

Let us agree that $\vartheta > 0$ if the vectors **n** and $\boldsymbol{\sigma}$ lie on the same side of the vector **k**, and $\vartheta < 0$ if the vectors **n** and $\boldsymbol{\sigma}$ lie on different sides of **k**. As was mentioned above, when in the third mode, the top performs forced oscillations, with the inertial force due to the unbalanced mass being the exciting force. The sign of $\vartheta$ indicates whether these oscillations are in-phase or anti-phase. We consider the angles $\chi$ and $\gamma$ to be positive. In this case, the relation $\chi = \gamma + \vartheta$ is valid.

Introduce the parameters

$$\varepsilon = \sin\vartheta, \quad \tau = \sin\chi, \quad \alpha = \sin\gamma, \quad \eta = \cos\vartheta, \quad \mu = \cos\chi, \quad \beta = \cos\gamma. \tag{3.6}$$

Parameters $\eta$ and $\mu$ have already been introduced. Using the formula $\chi = \gamma + \vartheta$, one can obtain different formulas relating the parameters of (3.6); for example, $\tau = \beta\varepsilon + \alpha\eta$ and $\mu = \beta\eta - \alpha\varepsilon$. Since the vectors $\boldsymbol{\epsilon}$ and $\boldsymbol{\tau}$ are parallel, we can obtain the scalar analog of the basic equation of steady-state motion (3.3)

$$\theta\varepsilon\eta + \theta_*\tau\mu = C\varepsilon/\omega^2, \tag{3.7}$$

where $\varepsilon$, $\eta$, $\tau$, and $\mu$ are defined in (3.6).

By expressing $\varepsilon$, $\eta$, $\tau$, and $\mu$ in terms of $\vartheta$, we arrive at the equation describing the amplitude-frequency characteristic in an implicit form. It is rather complicated to obtain an explicit form of the function $\vartheta(\omega)$. In practice, it is more convenient to use the inverse function, i.e., the frequency versus the amplitude:

$$\omega^2 = \frac{C\varepsilon}{\theta\varepsilon\eta + \theta_*\tau\mu} = \frac{2C\sin\vartheta}{\theta\sin 2\vartheta + \theta_*\sin[2(\vartheta+\gamma)]}. \tag{3.8}$$

By setting $\theta_* = 0$ in (3.7) or (3.8), we obtain the equation of the skeleton curve. Thus, we arrive at the following result: the amplitude-frequency characteristic for the first mode of steady-state motion is a point (for $\vartheta = \gamma$) on the amplitude-frequency characteristic of the second mode, and the latter characteristic coincides with the skeleton curve for the third (basic) mode. Apparently, such an interaction of different steady-state motions plays an important role in the formation of finite (nonsmall) amplitudes.

Note that so far all results have been exact; the assumption of smallness of the angles has nowhere been used. The only constraint on the angles, imposed above, can be readily eliminated by introducing additional definitions, with formula (3.8) remaining unchanged. Analysis of the amplitude-frequency characteristics for arbitrary amplitudes is an extremely wide area for investigation. In the next subsection, we will consider the case of small (but nonlinear) oscillations.

**3.2. Resonance relation.** Assume that the unbalance is small,* i.e., $\theta_* \ll \theta$. Assume, in addition, that the quantities $\gamma$, $\frac{1}{2}\pi - \gamma$, $\theta_3/\theta$, and $\theta_{12}/\theta$ are not small. In this case, the system contains two independent small parameters: the inertial parameter $\theta_*/\theta$ and the kinematic parameter $\varepsilon$. We will consider two cases: the case of small amplitudes ($\varepsilon \sim \theta_*/\theta$) and the case of large amplitudes ($\varepsilon \gg \theta_*/\theta$). In both the cases, we assume that $\varepsilon \ll 1$.

The case of small amplitudes leads to the conventional linear problem. For example, Eq. (3.7) in the first approximation gives the classical relation

$$\varepsilon = \frac{\theta_*\alpha\beta\omega^2}{\theta(C/\theta - \omega^2)}.$$

For $C\theta > 0$, this is the resonance relation for the linear system with natural frequency $\omega_{\rm res} \stackrel{\rm def}{=} \sqrt{C/\theta}$. For $C\theta < 0$, resonance does not take place in the system.

In what follows, we will consider the case of large amplitudes. This case is possible only in a near-resonance zone, where the inequality $C\theta > 0$ holds, which corresponds either to a prolate top with restoring torque or to an oblate

* From now on the notations $x \ll y$ means $|x| \ll |y|$.

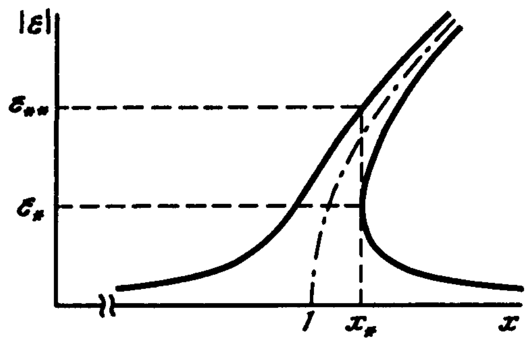


Fig. 3

top with destabilizing torque. Expand expression (3.8) into a series in powers of small parameters. Taking into account the relation $\theta\varepsilon \gg \theta_*$, we retain the terms of order $\leq 2$. Thus we obtain

$$\omega^2 = \frac{C\varepsilon}{\theta\varepsilon\eta + \theta_*\tau\mu} \approx \frac{C\varepsilon}{\theta\varepsilon\left(1-\frac{1}{2}\varepsilon^2\right)+\theta_*\alpha\beta} = \frac{C}{\theta}\frac{1}{1-\frac{1}{2}\varepsilon^2+\theta_*\alpha\beta/(\theta\varepsilon)} \approx \frac{C}{\theta}\left(1+\frac{1}{2}\varepsilon^2-\frac{\theta_*\alpha\beta}{\theta\varepsilon}\right),$$

or, equivalently,

$$x = 1+\frac{1}{2}\varepsilon^2-\frac{p}{\varepsilon}; \qquad x \overset{\text{def}}{=} \frac{\omega^2}{\omega_{\text{res}}^2} = \frac{\theta}{C}\omega^2, \quad p \overset{\text{def}}{=} \frac{\theta_*\alpha\beta}{\theta}. \tag{3.9}$$

This equation is a special case (for $k=\frac{1}{2}$) of the universal amplitude-frequency characteristic

$$x = 1 + k\varepsilon^2 - p/\varepsilon \tag{3.10}$$

which occurs in the vicinity of resonance for oscillatory systems with small cubic nonlinearity. In particular, the amplitude-frequency characteristic for Duffing's oscillator [13] has the form of Eq. (3.10). Note, however, that to obtain relation (3.10) for the examined system, one need not use, unlike a single-degree-of-freedom system, approximate methods of averaging. In this sense, gyroscopic systems are more convenient to investigate forced nonlinear oscillations.

Consider relation (3.10) in more detail. We will refer to the coefficient $k$ as the nonlinearity coefficient, and to $p$ as the intensity coefficient. Recall that Eq. (3.10) is valid for $p \ll \varepsilon \ll 1$. The amplitude-frequency characteristic of the system (the amplitude magnitude, $|\varepsilon|$, versus the square of the dimensionless frequency, $x$), corresponding to relation (3.9), is shown in Fig. 3. The skeleton curve $x = 1 + k\varepsilon^2$ is shown by the dot-and-dash line. The amplitude-frequency characteristic has two branches, the right-hand and left-hand ones. These branches are characterized by different signs of $\varepsilon$, lie on different sides of the skeleton curve, and asymptotically approach this curve as $\varepsilon$ increases. In the case in question, the inequality $k > 0$ holds, and the amplitude-frequency characteristic is hard.

A specific feature of the amplitude-frequency characteristic of a nonlinear system is the fact that the function $\varepsilon(x)$ has a bifurcation point at which the derivative $d\varepsilon/dx$ becomes infinite. By differentiating (3.10), we can find the coordinates $(\varepsilon_1, x_*)$ of the bifurcation point:

$$\varepsilon_1 = -\left(\tfrac{1}{2}p/k\right)^{1/3}, \quad \varepsilon_* \overset{\text{def}}{=} |\varepsilon_1|, \quad x_* = 1+\tfrac{3}{2}\left(2kp^2\right)^{1/3}.$$

For $k=\frac{1}{2}$, the relations are simplified: $\varepsilon_1 = -p^{1/3}$, $x_* = 1+\frac{3}{2}p^{2/3}$. At the bifurcation point, the system can jump from the lower branch to the upper branch of the amplitude-frequency characteristic, with the amplitude abruptly increasing to $\varepsilon_2$ (the notation $\varepsilon_{**} \overset{\text{def}}{=} |\varepsilon_2|$ is used in Fig. 3). According to (3.10), to find $\varepsilon_2$ one must solve the cubic equation $k\varepsilon_2^3 + (1-x_*)\varepsilon_2 - p = 0$. By making use of the relation $x_* = x(\varepsilon_1)$, we can reduce this equation to the form $\varepsilon_2^3 - 3\varepsilon_1^2\varepsilon_2 + 2\varepsilon_1^3 = 0$, which has three roots: the double root $\varepsilon_2 = \varepsilon_1$ and the root $\varepsilon_2 = -2\varepsilon_1$ which is the desired value of $\varepsilon_2$. Thus, we have arrived at the remarkable result that the jump to the upper branch of the amplitude-frequency characteristic leads to an increase in the amplitude exactly by a factor of 2, irrespective of the values of $k$ and $p$. Note that such an amplitude doubling occurs for all systems whose amplitude-frequency characteristics have the form (3.10), for example, for the Duffing equation. Although this fact is simple and universal, the author of the present paper failed to find a citation of it in the literature.

Turn to Fig. 3. We will refer to the part of the lower branch of the amplitude-frequency characteristic above the bifurcation point as the back slope part. All that was said above about the jumps is valid only if the back slope part is the only unstable part of the amplitude-frequency characteristic. It is however known [1] that, in the general case, there is no direct correspondence between the bifurcation points and the points of change of stability. It seems quite natural that such a correspondence occurs if the system is equipped with a drive maintaining a prescribed rotation frequency $\omega$ (and thereby $x$). If a drive is absent or its power is bounded, the issue becomes more than problematic. A hypothetical stability condition, stating that the only unstable part on the amplitude-frequency characteristic of the form (3.9) is the back slope part, can be expressed as

$$\varepsilon/\varepsilon_1 < 1 \iff \theta\varepsilon^3 + \theta_* \alpha\beta > 0. \tag{3.11}$$

In the subsequent sections, we will define stability criteria for steady-state motions and compare them with condition (3.11).

## 4. STABILITY ANALYSIS

**4.1. Kinematic relations.** To investigate the stability of the steady-state motion, a complete system of equations of motion is needed. As was mentioned above, Eq. (2.7) contains the redundant set of variables $\mathbf{n}$, $\sigma$, $\Omega$, $M_1$, and $M_2$. Therefore, additional kinematic relations are required to close this system. It is convenient to introduce a nonredundant set of three generalized coordinates in terms of which the other coordinates could be expressed. Take two components of the unit vector $\mathbf{n}$ of the axis of the ideal top and the angle $\varphi$ of rotation of the top about this axis as the basic generalized coordinates.

As the reference position of the top, we choose a position for which the axis of the top is vertical, i.e., $\mathbf{n} = \mathbf{k}$. To specify a current position, we use the rotation tensor $\mathbf{P}$, describing the rotation of the top with respect to the reference position. The rotation tensor $\mathbf{P}$ can be represented as a composition of two rotations: the inclination $\mathbf{P}(\mathbf{n}, \mathbf{k})$ (the rotation from $\mathbf{k}$ to $\mathbf{n}$ about a horizontal axis) and the proper rotation $\mathbf{P}(\varphi\mathbf{n})$ through the angle $\varphi$ about the axis $\mathbf{n}$. As a result, we have

$$\mathbf{P} = \mathbf{P}(\varphi\mathbf{n}) \cdot \mathbf{P}(\mathbf{n}, \mathbf{k}).$$

The inclination tensor has the form

$$\mathbf{P}(\mathbf{n}, \mathbf{k}) = \mathbf{E} - \frac{1}{1+\eta}(\mathbf{n}+\mathbf{k})(\mathbf{n}+\mathbf{k}) + 2\mathbf{nk} \tag{4.1}$$

and is an algebraic function of $\mathbf{n}$. Note that the top can be brought to the same position by rotation $\mathbf{P}(\varphi\mathbf{k})$ through the angle $\varphi$ about the vertical axis, followed by the inclination $\mathbf{P}(\mathbf{n}, \mathbf{k})$, so that

$$\mathbf{P} = \mathbf{P}(\mathbf{n}, \mathbf{k}) \cdot \mathbf{P}(\varphi\mathbf{k}). \tag{4.2}$$

The angular velocity $\boldsymbol{\omega}$ corresponding to the rotation tensor $\mathbf{P}$ can be found by solving Poisson's equation $\dot{\mathbf{P}} = \boldsymbol{\omega} \times \mathbf{P}$. With allowance for (4.2), we obtain

$$\boldsymbol{\omega} = \frac{1}{1+\eta}(\mathbf{n}+\mathbf{k}) \times \dot{\mathbf{n}} + \dot{\varphi}\mathbf{n}. \tag{4.3}$$

Using (4.3), we obtain the expression

$$\Omega = \mathbf{n} \cdot \boldsymbol{\omega} = -\frac{1}{1+\eta}\mathbf{k}(\mathbf{n} \times \dot{\mathbf{n}}) + \dot{\varphi} \tag{4.4}$$

for the angular velocity of proper rotation. The manipulations with the rotation tensor are described in more detail in [11].

Note that by angle of proper rotation, one of Euler's angles is commonly understood. Euler's angles are related to the angles introduced above by the formulas $\varphi = \varphi_e + \psi_e$ and $|\vartheta| = \vartheta_e$; the subscript "e" indicates Euler's angles. By angular velocity of proper rotation, the quantity $\dot{\varphi}_e$ is usually understood. The quantities $\Omega$ and $\dot{\varphi}_e$ are related by $\Omega = \dot{\varphi}_e + \eta\dot{\psi}_e$. The quantity $\Omega$ is an invariant characteristic of motion, which favorably distinguishes it from $\dot{\varphi}_e$, which depends on the direction of $\mathbf{k}$. Moreover, it can be demonstrated that $\Omega$ better corresponds to an intuitive idea of the proper rotation.

**4.2. Variation of equations of motion.** To investigate the stability, consider small oscillations of the system in a neighborhood of the basic steady-state motion. To this end, we vary the nonlinear dynamic equation (2.7). In the general form, the variational equation can be represented as

$$\mathbf{A}_2 \cdot \delta\ddot{\mathbf{n}} + \omega \mathbf{A}_1 \cdot \delta\dot{\mathbf{n}} + \omega^2 \mathbf{A}_0 \cdot \delta\mathbf{n} + \mathbf{B}_2 \cdot \delta\ddot{\boldsymbol{\sigma}} + \omega^2 \mathbf{B}_0 \cdot \delta\boldsymbol{\sigma} + \omega \mathbf{C}_1\, \delta\dot{\Omega} + \omega^2 \mathbf{C}_0\, \delta\Omega - \delta\mathbf{M}_d = \mathbf{0}, \tag{4.5}$$

where $\delta$ is the symbol of variation and $\mathbf{A}_s$, $\mathbf{B}_s$, and $\mathbf{C}_s$ are tensor coefficients of the variations. These coefficients are calculated for the steady-state motion, which renders them substantially simplified. These coefficients are time-varying: the vectors on which these coefficients depend rotate at the angular velocity $\omega\mathbf{k}$. To obtain the equations with constant coefficients, we introduce the dimensional relative derivative $(\cdot)'$ according to the formulas

$$\dot{\mathbf{a}} = \omega(\mathbf{a}' + \mathbf{k} \times \mathbf{a}), \quad \dot{\mathbf{b}} = \omega \mathbf{b}'.$$

The variations $\delta\boldsymbol{\sigma}$, $\delta\Omega$, and $\delta\mathbf{M}_d$ are expressed in terms of the basic generalized coordinates $\delta\mathbf{n}$ and $\delta\varphi$. The expression for $\delta\Omega$ can be found by varying relation (4.4). When finding the variation $\delta\mathbf{M}_d$, we assume that the driving torque is a function of $\dot{\varphi}$, i.e., $M_d = M(\dot{\varphi})$. Note that the relation $\dot{\varphi} = \omega$ is valid for the steady-state motion. To find $\delta\boldsymbol{\sigma}$, it is convenient to use an analogy between variation and differentiation. For example, the variation of the unit vector $\boldsymbol{\sigma}$ can be obtained in the form $\delta\boldsymbol{\sigma} = \delta\boldsymbol{\xi} \times \boldsymbol{\sigma}$, where the vector $\delta\boldsymbol{\xi}$ serves in varying as an analog of the angular velocity $\boldsymbol{\omega}$: $\delta\mathbf{P} = \delta\boldsymbol{\xi} \times \mathbf{P}$. This fact allows one to express $\delta\boldsymbol{\xi}$ by the formula

$$\delta\boldsymbol{\xi} = \frac{1}{1+\eta}(\mathbf{n} + \mathbf{k}) \times \delta\mathbf{n} + \mathbf{n}\,\delta\varphi,$$

which is similar to (4.3). After the transformations and substitutions indicated above are made, Eq. (4.5) becomes

$$\mathbf{a}_2 \cdot \delta\mathbf{n}'' + \mathbf{a}_1 \cdot \delta\mathbf{n}' + \mathbf{a}_0\, \delta\mathbf{n} + \mathbf{c}_2\, \delta\varphi'' + \mathbf{c}_1\, \delta\varphi' + \mathbf{c}_0\, \delta\varphi = \mathbf{0}, \tag{4.6}$$

where the coefficients $\mathbf{a}_s$, $\mathbf{c}_s$ and $\mathbf{A}_s$, $\mathbf{B}_s$, $\mathbf{C}_s$ are related linearly. To obtain the system of differential equations of perturbed motion in a standard form, we collect all unknown variables to form a single three-dimensional vector $\delta\boldsymbol{\zeta} \overset{\text{def}}{=} \delta\boldsymbol{\epsilon} + \mathbf{k}\,\delta\varphi$, where $\boldsymbol{\epsilon}$ is the horizontal component of $\mathbf{n}$. The variations $\delta\mathbf{n}$ and $\delta\varphi$ can be expressed via $\delta\boldsymbol{\zeta}$ according to the formulas $\delta\mathbf{n} = (\mathbf{E} - \mathbf{k}\mathbf{n}/\eta)\,\delta\boldsymbol{\zeta}$ and $\delta\varphi = \mathbf{k} \cdot \delta\boldsymbol{\zeta}$. By substituting these expressions into (4.6), we obtain

$$\mathbf{p}_2 \cdot \delta\boldsymbol{\zeta}'' + \mathbf{p}_1 \cdot \delta\boldsymbol{\zeta}' + \mathbf{p}_0 \cdot \delta\boldsymbol{\zeta} = \mathbf{0}, \tag{4.7}$$

where the $\mathbf{p}_s$ are linearly related to $\mathbf{a}_s$ and $\mathbf{c}_s$.

Relation (4.7) is the desired system of differential equations with constant (for the relative derivative) coefficients. By substituting the solution of the form $\boldsymbol{\zeta} = \boldsymbol{\zeta}_0 e^{\lambda t}$ (where $t$ is the dimensionless time) into (4.7), we obtain the characteristic equation

$$\Delta(\lambda) = 0, \qquad \Delta(\lambda) \overset{\text{def}}{=} \det(\mathbf{p}_2\lambda^2 + \mathbf{p}_1\lambda + \mathbf{p}_0).$$

Following the above procedure, one can obtain a chain of formulas to express $\Delta(\lambda)$ in terms of the coefficients of $\mathbf{A}_s$, $\mathbf{B}_s$, and $\mathbf{C}_s$. The implementation of this algorithm is rather complicated, because the coefficients are cumbersome. However, the recursive character of the formulas makes them convenient for computer implementation. When solving the problem, we used the computer algebra system Reduce [14]. A library of special Reduce procedures was created which allowed us to perform symbolic manipulations with vectors and tensors. Apart from a constant coefficient, the determinant $\Delta(\lambda)$ can finally be represented in the form

$$\begin{vmatrix} (\theta+\theta_*+\theta_3)\lambda^2+\theta\varepsilon^2+\theta_*\left(\dfrac{\alpha\mu}{\varepsilon}+\tau^2\right) & -\left(2\theta\eta+2\theta_*\mu\dfrac{\tau-\alpha}{\varepsilon}+\theta_3\right)\lambda & (-2\theta_*\alpha\mu+\theta_3\varepsilon)\lambda \\ (2\theta\eta^2+2\theta_*\mu^2+\theta_3)\lambda & \left(\theta\eta+\theta_*\mu\dfrac{\tau-\alpha}{\varepsilon}+\theta_3\right)\lambda^2+\theta_*\dfrac{\alpha\mu}{\varepsilon} & (\theta_*\alpha\mu-\theta_3\varepsilon)\lambda^2+\dfrac{M_2'}{\omega}\varepsilon\lambda-\theta_*\alpha\mu \\ 2(\theta\varepsilon\eta+\theta_*\tau\mu)\lambda & \left(\theta\varepsilon+\theta_*\tau\dfrac{\tau-\alpha}{\varepsilon}+\theta_3\dfrac{1-\eta}{\varepsilon}\right)\lambda^2 & (\theta_*\alpha\tau+\theta_3\eta)\lambda^2-\left(\dfrac{M_1'}{\omega}+\dfrac{M_2'}{\omega}\eta\right)\lambda \end{vmatrix},$$

where the prime denotes differentiation with respect to $\omega$. Note that this expression corresponds to the most general case; it is valid for any (not necessarily small) amplitudes and unbalanced masses and any power of the drive.

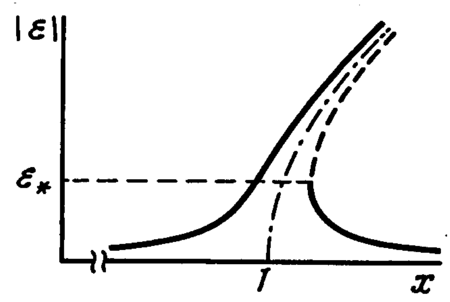


Fig. 4

However, analyzing the characteristic determinant is a much more difficult problem than obtaining the expression for it. It is hardly possible to obtain closed-form results in the general setting. However, the obtained expression allows investigating numerous interesting special cases. Let us now consider one of these cases.

**4.3. Stability conditions.** Consider the characteristic determinant for two limit cases, namely, for the system with a drive of infinitely large power and for the system without a drive.

These systems are conservative and therefore the results obtained on the basis of the analysis of the perturbed equations in the first approximation provide, generally speaking, only necessary conditions for Lyapunov stability. To obtain sufficient conditions, two ways are possible. First, one can try to obtain the sufficient conditions in the framework of the conservative model by considering more precise equations of the perturbed motion. However, the results obtained in such a way will be structurally unstable: they can be destroyed by adding small nonconservative perturbation. Second, one can introduce a small dissipation into the system, which allows confining oneself to the equations of the first approximation. However, any dissipation brings specific effects to the system, which can hinder singling out properties characteristic of the basic (conservative) system. For this reason, it is the analysis of the conservative system on the basis of the equations of the first approximation that allows investigating major properties intrinsic to the conservative system.

Introduce the variable $L \stackrel{\text{def}}{=} -M_{\text{d}}'(\omega)$ characterizing the power of the drive. We will consider limit values of $L$, namely, $L = \infty$ (a drive of infinitely large power) or $L = 0$ (no drive). In these cases, to determine the characteristic exponent $\lambda$, one has to solve a biquadratic equation. We will treat the relation $\operatorname{Re}\lambda = 0$ as the stability condition. Consider the small parameters introduced in Section 3.2. For small amplitudes ($\varepsilon \sim \theta_*/\theta$), the stability conditions are trivial; for large amplitudes ($\varepsilon \gg \theta_*/\theta$), they lead to the inequalities

$$\theta\varepsilon^3 + \theta_*\alpha\beta > 0 \qquad \text{for} \quad L = \infty,$$
$$\theta(4\theta + \theta_3)\varepsilon^3 + \theta_3\theta_*\alpha\beta > 0 \qquad \text{for} \quad L = 0.$$

Note that these conditions are valid both for dead and follower driving torques.

Compare these stability conditions with the hypothetical conditions (3.11) based on the shape of the amplitude-frequency characteristic. For $L = \infty$ both the conditions coincide and the critical value of the magnitude is equal to $\varepsilon_*$. Thus, in this case, the amplitude-frequency characteristic (Fig. 4) describes the motion perfectly well; the dashed line shows the instability region. However, for $L = 0$, the conditions differ and the critical value of $|\varepsilon|$ is given by

$$\varepsilon_0 = \left(\frac{\theta_3}{|4\theta_{12} - 3\theta_3|}\right)^{1/3} \varepsilon_*.$$

Consider separately three ranges of the values of $\theta_3$ and $\theta_{12}$

(1) A prolate top with a restoring torque: $\theta_3 < \theta_{12}$ ($C > 0$). In this case, we have $\varepsilon_0 < \varepsilon_*$; the corresponding stability region is shown in Fig. 5*a*. The result obtained is close to that predicted by the analysis of the amplitude-frequency characteristic, but instability arises earlier.
(2) A moderately oblate top with a destabilizing torque: $\theta_{12} < \theta_3 < \frac{4}{3}\theta_{12}$ ($C < 0$). In this case, the inequality $\varepsilon_0 > \varepsilon_*$ holds and, unlike the previous case, instability arises later (Fig. 5*b*).
(3) A strongly oblate top with a destabilizing torque: $\frac{4}{3}\theta_{12} < \theta_3 < 2\theta_{12}$ ($C < 0$). In this case, we have $\varepsilon_0 > \varepsilon_*$, but the instability arises in the subresonance rather than superresonance zone (Fig. 5*c*).

Thus, these results do not agree with those obtained from the analysis of the amplitude-frequency characteristics even approximately.

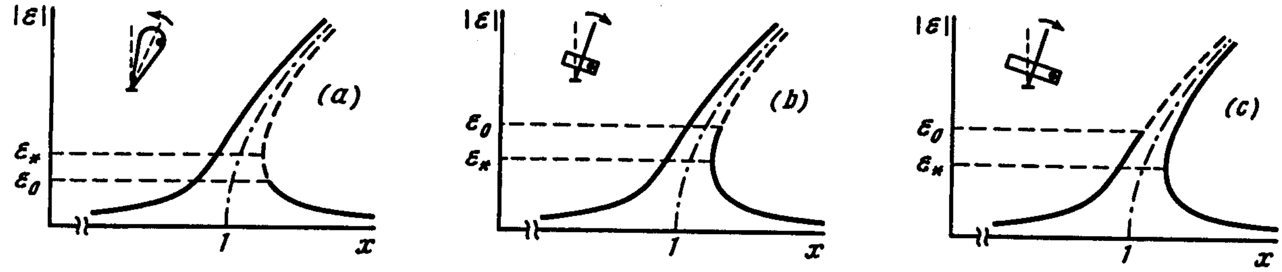


Fig. 5

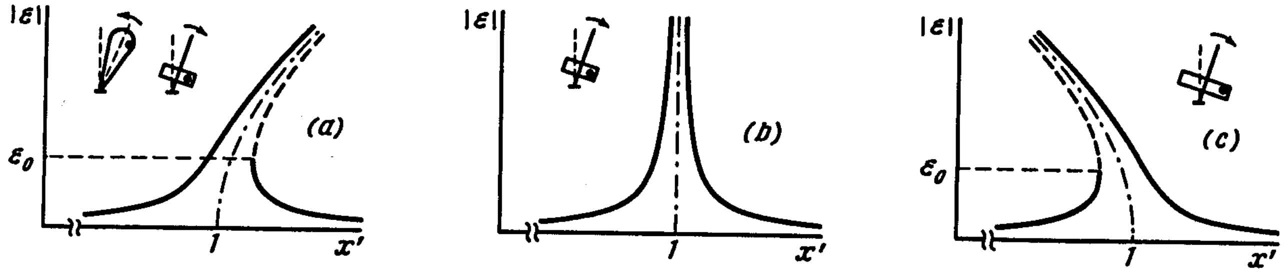


Fig. 6

Let us summarize the results. If the power of the drive is infinitely large, then the stability region obtained from the analysis of the characteristic determinant coincides with that obtained from the analysis of the amplitude-frequency characteristics. For the system without a drive, the results are different. The difference is especially distinct for $\theta_3 > \frac{4}{3}\theta_{12}$; in this case, the results contradict each other. Similar effects have been revealed earlier for the case of plane-parallel motion of a rigid body. It is shown [1] that for the system without a drive, the instability region begins below the bifurcation point, which corresponds to Fig. 5*a*. The cases corresponding to Fig. 5*b* and Fig. 5*c* can occur only for spatial motion. Since a destabilizing torque acts in these cases, the gyroscopic stabilization is necessary to maintain the permanent rotation.

Consider the case $L = 0$ in more detail. The discrepancy between the results obtained on the basis of the characteristic determinant and on the basis of the amplitude-frequency characteristic can be accounted for as follows. Unless the frequency $\omega$ is rigidly prescribed, there is no reason to prefer this quantity to be used as the argument of the amplitude-frequency characteristic. For example, we can well use the proper rotation frequency $\Omega$ as the argument. However, in this case, the form of the amplitude-frequency characteristic will dramatically change. It turns out that one can find a modified frequency, close to $\omega$, so that the corresponding amplitude-frequency characteristic agrees with the stability condition perfectly well. Introduce the modified frequency

$$\omega' \overset{\text{def}}{=} \left(1 + \frac{\theta_{12} - \theta_3}{\theta_3}\varepsilon^2\right)\omega.$$

The amplitude-frequency characteristic corresponding to $\omega'$ differs from the original amplitude-frequency characteristic in the nonlinearity coefficient $k' = (4\theta_{12} - 3\theta_3)/(2\theta_3)$.

The modified amplitude-frequency characteristics are shown in Figs. 6*a*, 6*b*, and 6*c* for the cases $\theta_3 < \frac{4}{3}\theta_{12}$, $\theta_3 = \frac{4}{3}\theta_{12}$, and $\theta_3 > \frac{4}{3}\theta_{12}$, respectively. The instability region completely corresponds to the form of the amplitude-frequency characteristic. One can see from these figures that the amplitude-frequency characteristic can be either hard or soft. Thus, the introduction of the concepts of hardness and softness make sense only after analyzing stability. The case $\theta_3 = \frac{4}{3}\theta_{12}$ is of particular interest. In this case, nonlinearity in the amplitude-frequency characteristic disappears, and the system becomes stable for all frequencies (Fig. 6*b*). Let us give the values of the modified parameters for some particular kinds of tops. For the infinitely prolate top ($\theta_3 \to 0$), we have $k' \to +\infty$. For the top with spherical tensor of inertia ($\theta_3 = \theta_{12}$), the relations $k' = \frac{1}{2}$ and $\omega' = \omega$ are valid. For the infinitely oblate top ($\theta_3 = 2\theta_{12}$), we have $k' = -\frac{1}{2}$ and $\omega' = \Omega + o(\varepsilon^2)$. In the case $\theta_3 \to 0$, the system becomes essentially nonlinear.

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