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DESCRIPTION OF MOTION OF AN AXISYMMETRIC RIGID BODY IN A LINEARLY VISCOUS MEDIUM IN TERMS OF QUASICOORDINATES

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Free rotation of a rigid body under the action of a linear viscous torque is considered. A simple mechanical system permitting one to model this interaction is presented. Mathematically, the problem is reduced to determining a unit vector the absolute values of whose first and second derivatives are prescribed functions of time. The trajectory described by the end of the unit vector of the body symmetry axis on the unit sphere is investigated. A closed-form solution of the problem is constructed using quasicoordinates.

1. INTRODUCTION

Consider free motion of an axisymmetric rigid body under the action of the linear viscous torque

$$\mathbf{M} = -\mathbf{B} \cdot \boldsymbol{\omega}, \qquad \mathbf{B} \overset{\text{def}}{=} B_{12}(\mathbf{E} - \mathbf{nn}) + B_3\mathbf{nn}, \tag{1.1}$$

where $\boldsymbol{\omega}$ is the angular velocity, $\mathbf{n}$ is the unit vector of the body axis of symmetry, B_{12} and B_3 are constant positive coefficients, and $\mathbf{nn}$ is the dyadic product of the unit vector $\mathbf{n}$.

The motion of this system is governed by the following equation:

$$\frac{d}{dt}(\boldsymbol{\theta} \cdot \boldsymbol{\omega}) = -\mathbf{B} \cdot \boldsymbol{\omega}, \qquad \boldsymbol{\theta} \overset{\text{def}}{=} \theta_{12}(\mathbf{E} - \mathbf{nn}) + \theta_3\mathbf{nn}, \tag{1.2}$$

where θ_{12} and θ_3 are the equatorial and polar moments of inertia of the body, respectively.

The problem of Eqs. (1.1) and (1.2) has been known for a long time. Routh addressed this problem in the case where the viscosity tensor $\mathbf{B}$ was proportional to the inertia tensor of the body, i.e., $\mathbf{B} = \alpha\boldsymbol{\theta}$ [1]. In this case Eq. (1.2) becomes a linear differential equation for the angular momentum $\mathbf{K} = \boldsymbol{\theta} \cdot \boldsymbol{\omega}$. This equation is easy to integrate, thus obtaining

$$\dot{\mathbf{K}} = -\alpha\mathbf{K} \quad \Longrightarrow \quad \mathbf{K} = \mathbf{K}_0 e^{-\alpha t}. \tag{1.3}$$

Hence, in this particular case, the angular momentum remains constant in direction, and the motion of the body is similar to the motion in vacuum but with exponentially decaying angular velocity. Note that this solution is valid for an arbitrary tensor of inertia.

Grammel [2] considered a spherical tensor of viscosity, i.e., $\mathbf{B} = \alpha\mathbf{E}$. This case is substantially more complicated than that of proportionality of the tensors $\mathbf{B}$ and $\boldsymbol{\theta}$ and differs insignificantly from the general case of Eq. (1.1) considered in [3]. In the works cited, the projections of the angular velocity onto the moving body-fixed axes are determined. Among more recent publications, it is worthwhile mentioning the book [4] in which the projections of the angular velocity onto the body-fixed axes are determined and also a more general case, where the viscosity tensor is not axially symmetric, is considered. However, in all these works, neither the position of the angular velocity vector in the fixed space nor the angles identifying the position of the body are determined. This task needs further integrating the equations of motion. An exact solution of this problem has been obtained in [5] in the form of a uniformly convergent series. In the present paper, we suggest a method allowing constructing a simple solution of this problem with the aid of quasicoordinates. This solution is not a solution in the classical sense. Nevertheless, it permits one to describe graphically the spatial motion of the body, without using functions more complicated than exponential ones.

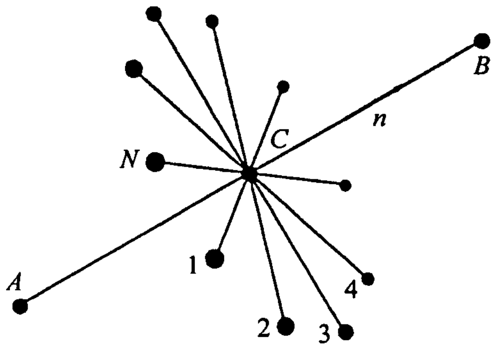

Fig. 1

2. PHYSICAL INTERPRETATION OF THE RESISTANCE TORQUE

Prior to proceeding to the solution of the problem, we consider a simple physical model for which the friction torque completely corresponds to Eq. (1.1). This is a rigid structure formed of rods and identical balls attached to these rods. See Fig. 1. Balls $1, 2, \ldots, N$ lie in the plane perpendicular to rod AB and are spaced uniformly along the circle of radius a centered at the midpoint C of rod AB. We assume that $N \geq 3$ and is not too large so that the distances between the balls substantially exceed their diameters. It is obvious that the tensor of inertia of such a system is transversally isotropic. Consider the motion of this system in a linearly viscous medium. We assume the thickness of the rods to be negligibly small as compared with the diameters of the balls. In this case, the resistance is defined by the force of viscous friction acting on the balls, $\mathbf{f} = -\beta \mathbf{v}$, where $\mathbf{v}$ is the velocity of the ball and β is the coefficient of viscosity. The total resistance torque with respect to the center of mass C is expressed as

$$\mathbf{M} = -\beta \sum_{k=1}^{N} \mathbf{r}_k \times \mathbf{v}_k - \beta \mathbf{r}_A \times \mathbf{v}_A - \beta \mathbf{r}_B \times \mathbf{v}_B, \tag{2.1}$$

where $\mathbf{r}_k$, $\mathbf{r}_A$, and $\mathbf{r}_B$ are the position vectors from the point C for the respective balls. The velocity of each ball can be represented in the form

$$\mathbf{v} = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{r} \quad \Longrightarrow \quad \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \mathbf{v}_C + \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = \mathbf{r} \times \mathbf{v}_C + (r^2 \mathbf{E} - \mathbf{rr}) \cdot \boldsymbol{\omega},$$

where $\boldsymbol{\omega}$ and $\mathbf{v}_C$ are the angular velocity and the velocity of the structure center of mass, respectively. Substitute this expression into that of (2.1) to obtain

$$\mathbf{M} = -\beta \sum_{k=1}^{N} \mathbf{r}_k \times \mathbf{v}_C - \beta \left[\sum_{k=1}^{N} (a^2 \mathbf{E} - \mathbf{r}_k \mathbf{r}_k) + 2(b^2 \mathbf{E} - \mathbf{r}_A \mathbf{r}_A) \right] \cdot \boldsymbol{\omega}. \tag{2.2}$$

In this expression we used the conditions $|\mathbf{r}_k| = a$, $|\mathbf{r}_A| = b$, and $\mathbf{r}_B = -\mathbf{r}_A$. Let $\mathbf{n}$ be the unit vector of the axis AB such that $\mathbf{r}_A = -b\mathbf{n}$. Consider relation (2.2) in more detail. Due to symmetry, the sum $\sum_{k=1}^{N} \mathbf{r}_k$ vanishes. The tensor $\sum_{k=1}^{N} \mathbf{r}_k \mathbf{r}_k$ possesses the rotational symmetry of order N about the axis $\mathbf{n}$ and, hence, is proportional to the transversally isotropic tensor $\mathbf{E} - \mathbf{nn}$. By calculating the trace of these tensors we find the coefficient of proportionality:

$$\sum_{k=1}^{N} \mathbf{r}_k \mathbf{r}_k = k(\mathbf{E} - \mathbf{nn}) \quad \Longrightarrow \quad \sum_{k=1}^{N} a^2 = k(3 - 1) \quad \Longrightarrow \quad k = \frac{N}{2} a^2.$$

The relation (2.2) becomes

$$\mathbf{M} = -\beta \left(\frac{N}{2} a^2 (\mathbf{E} + \mathbf{nn}) + 2b^2 (\mathbf{E} - \mathbf{nn}) \right) \cdot \boldsymbol{\omega},$$

and, hence,

$$\mathbf{M} = -\mathbf{B} \cdot \boldsymbol{\omega}, \qquad \mathbf{B} = B_{12}(\mathbf{E} - \mathbf{nn}) + B_3 \mathbf{nn}, \quad B_{12} = \beta \left(\frac{N}{2} a^2 + 2b^2 \right), \quad B_3 = \beta N a^2.$$

Thus, the resistance torque acting on the structure in question conforms to Eq. (1.1) exactly.

3. FIRST INTEGRALS

We proceed now to solving Eq. (2.2)

$$\frac{d}{dt}(\boldsymbol{\theta}\cdot\boldsymbol{\omega}) = -\mathbf{B}\cdot\boldsymbol{\omega}. \tag{3.1}$$

Following [6], we represent the angular momentum $\mathbf{K} = \boldsymbol{\theta}\cdot\boldsymbol{\omega}$ as $\mathbf{K} = \theta_{12}\mathbf{n}\times\dot{\mathbf{n}} + \theta_3\Omega\mathbf{n}$, where Ω is the angular velocity of proper rotation defined as the projection of the angular velocity vector onto $\mathbf{n}$. Then Eq. (3.1) becomes

$$\theta_{12}\mathbf{n}\times\ddot{\mathbf{n}} + \theta_3(\dot{\Omega}\mathbf{n} + \Omega\dot{\mathbf{n}}) = -B_{12}\mathbf{n}\times\dot{\mathbf{n}} - B_3\Omega\mathbf{n}.$$

Multiplying this relation scalarly by $\mathbf{n}$, $\dot{\mathbf{n}}$, and $\mathbf{n}\times\dot{\mathbf{n}}$, after some transformations we obtain the following three scalar equations

$$\theta_3\dot{\Omega} = -B_3\Omega, \quad \theta_{12}\mathbf{n}\cdot(\dot{\mathbf{n}}\times\ddot{\mathbf{n}}) - \theta_3\Omega\dot{\mathbf{n}}\cdot\dot{\mathbf{n}} = 0, \quad \theta_{12}\mathbf{n}\cdot\ddot{\mathbf{n}} = -B_{12}\dot{\mathbf{n}}\cdot\dot{\mathbf{n}}. \tag{3.2}$$

The first and third equations of (3.2) are easy to integrate to determine Ω and $\dot{\mathbf{n}}\cdot\dot{\mathbf{n}}$ as functions of time:

$$\Omega = \Omega_0\exp\left(-\frac{B_3t}{\theta_3}\right), \quad \dot{\mathbf{n}}\cdot\dot{\mathbf{n}} = \nu_0^2\exp\left(-\frac{2B_{12}t}{\theta_{12}}\right), \tag{3.3}$$

where Ω_0 and ν_0 are integration constants.

To determine the motion of the body, it suffices to define the time histories of the unit vector $\mathbf{n}$, identifying the direction of the body axis of symmetry, and the angular velocity of rotation about this axis, Ω. The quantity Ω is known from Eq. (3.3). The unit vector $\mathbf{n}$ is a two-parameter quantity and, hence, it can be defined by two scalar equations. We will take relation (3.3) to be one of these equations, and the second equation can be obtained from the second relation of (3.2). Introduce two exponential functions of time

$$\Omega(t) \overset{\text{def}}{=} \Omega_0\exp\left(-\frac{B_3t}{\theta_3}\right), \quad \nu(t) \overset{\text{def}}{=} \nu_0\exp\left(-\frac{B_{12}t}{\theta_{12}}\right). \tag{3.4}$$

Then we obtain the following equations for Ω and $\mathbf{n}$:

$$\Omega = \Omega(t), \quad \dot{\mathbf{n}}\cdot\dot{\mathbf{n}} = \nu^2(t), \quad \mathbf{n}\cdot(\dot{\mathbf{n}}\times\ddot{\mathbf{n}}) = \frac{\theta_3}{\theta_{12}}\Omega(t)\nu^2(t) \overset{\text{def}}{=} f^3(t). \tag{3.5}$$

If the constant ν_0 in (3.4) is chosen positive, then the quantity $\nu(t)$ is the absolute value of the velocity of the end of the vector $\mathbf{n}$. Thus the problem is reduced to that of restoration of the unit vector $\mathbf{n}$ by the two scalar relations

$$\dot{\mathbf{n}}\cdot\dot{\mathbf{n}} = \nu^2(t), \quad \mathbf{n}\cdot(\dot{\mathbf{n}}\times\ddot{\mathbf{n}}) = f^3(t), \tag{3.6}$$

where $\nu(t)$ and $f(t)$ are known function of time having the dimension of frequency. The second equation of (3.6) can be replaced by an equivalent one. Using the relation [7]

$$[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]^2 = a^2b^2c^2 - a^2(\mathbf{b}\cdot\mathbf{c})^2 - b^2(\mathbf{c}\cdot\mathbf{a})^2 - c^2(\mathbf{a}\cdot\mathbf{b})^2 + 2(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c})(\mathbf{c}\cdot\mathbf{a}),$$

we have

$$[\mathbf{n}\cdot(\dot{\mathbf{n}}\times\ddot{\mathbf{n}})]^2 = \dot{\mathbf{n}}\cdot\dot{\mathbf{n}}\ddot{\mathbf{n}}\cdot\ddot{\mathbf{n}} - (\dot{\mathbf{n}}\cdot\ddot{\mathbf{n}})^2 - \dot{\mathbf{n}}\cdot\dot{\mathbf{n}}(\dot{\mathbf{n}}\cdot\ddot{\mathbf{n}})^2 = \nu^2(\ddot{\mathbf{n}}\cdot\ddot{\mathbf{n}} - \dot{\nu}^2 - \nu^4),$$

which implies

$$\ddot{\mathbf{n}}\cdot\ddot{\mathbf{n}} = \frac{1}{\nu^2}[\mathbf{n}\cdot(\dot{\mathbf{n}}\times\ddot{\mathbf{n}})]^2 + \dot{\nu}^2 + \nu^4 = \frac{f^6}{\nu^2} + \dot{\nu}^2 + \nu^4 \overset{\text{def}}{=} F^4(t).$$

Hence, the system of (3.6) can be represented in the form $\dot{\mathbf{n}}\cdot\dot{\mathbf{n}} = \nu^2(t)$, $\ddot{\mathbf{n}}\cdot\ddot{\mathbf{n}} = F^4(t)$. In such a way, the problem is reduced to that of the identification of the unit vector the absolute values of whose first and the second derivatives are prescribed functions of time t. To solve this problem is substantially more complicated than to obtain all results presented above. This problem is invariant, in the sense that there are no some distinguishing directions in space. At the same time, it is the property that complicates the problem, since it is not clear what direction is to be taken as a reference one when describing rotations. The introduction of any distinguishing direction breaks the symmetry of this problem.

Note that integrals similar to those of (3.3) have been obtained in [2, 3, 4]. In particular, in the cited works, the square of the total angular velocity has been found which, with reference to (3.3), can be expressed as

$$\boldsymbol{\omega}\cdot\boldsymbol{\omega} = (\mathbf{n}\times\dot{\mathbf{n}} + \Omega\mathbf{n})^2 = \dot{\mathbf{n}}\cdot\dot{\mathbf{n}} + \Omega^2 = \nu_0^2\exp\left(-\frac{2B_{12}t}{\theta_{12}}\right) + \Omega_0^2\exp\left(-\frac{2B_3t}{\theta_3}\right).$$

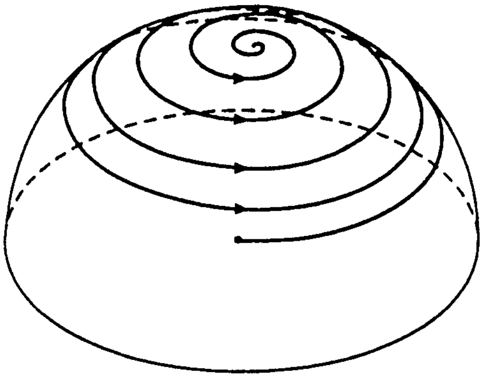

Fig. 2

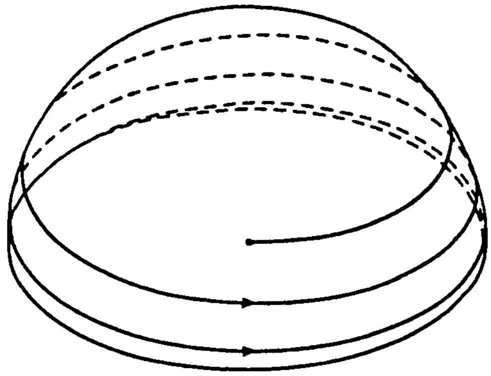

Fig. 3

4. SPHERICAL CURVATURE

From the aforementioned, it follows that to solve the problem, one needs to describe the motion of $\mathbf{n}$. The end of this vector describes a trajectory on the unit sphere. We introduce the dimensionless coordinate along this trajectory, s, such that $|ds| = |d\mathbf{n}|$, and the angle of rotation of this trajectory in the plane tangent to the sphere, ξ. Let $\boldsymbol{\tau}$ be the unit vector of the tangent to the trajectory, i.e., $\boldsymbol{\tau} = \mathbf{n}'$. (Here and in what follows, the prime stands for differentiation with respect to s.) We define as $d\xi$ the angle between the vectors $\boldsymbol{\tau}$ and $(\mathbf{E}-\mathbf{nn})\cdot(\boldsymbol{\tau}+d\boldsymbol{\tau})$, i.e., the angle between the vector $\boldsymbol{\tau}$ and the projection of the vector $\boldsymbol{\tau} + d\boldsymbol{\tau}$ onto the tangent plane. This definition can be expressed by the relation

$$\mathbf{n}\, d\xi \overset{\text{def}}{=} \boldsymbol{\tau} \times [(\mathbf{E}-\mathbf{nn})\cdot(\boldsymbol{\tau}+d\boldsymbol{\tau})]. \tag{4.1}$$

Multiply this relation scalarly by $\mathbf{n}$ to obtain

$$d\xi = \mathbf{n}\cdot(\boldsymbol{\tau}\times d\boldsymbol{\tau}) \quad \Longrightarrow \quad \xi' = \mathbf{n}\cdot(\boldsymbol{\tau}\times\boldsymbol{\tau}') = \mathbf{n}\cdot(\mathbf{n}'\times\mathbf{n}''). \tag{4.2}$$

The quantity $k \overset{\text{def}}{=} \xi' = d\xi/ds$ is the geodesic curvature of the trajectory on the sphere. For brevity, we will refer to this quantity as the spherical curvature of the trajectory. In accordance with (4.2), this quantity is related to the vector $\mathbf{n}$ by

$$k \overset{\text{def}}{=} \xi' = \mathbf{n}\cdot(\mathbf{n}'\times\mathbf{n}''). \tag{4.3}$$

One can show that the total curvature of the trajectory, K, is related to the spherical curvature by $K^2 = 1 + k^2$. Whence, in particular, it follows that $K = 1$ whenever the spherical curvature vanishes. Hence, in this case, the trajectory is a great circle on the sphere. Note also that K^2 can be expressed by the relation $K^2 = \mathbf{n}''\cdot\mathbf{n}''$, which is an analogue of Eq. (4.3).

From (3.6), we know the scalar triple product $\mathbf{n}\cdot(\dot{\mathbf{n}}\times\ddot{\mathbf{n}})$. Let us establish the relationship between this product and the spherical curvature k. We have $ds^2 = d\mathbf{n}\cdot d\mathbf{n} \quad \Longrightarrow \quad \dot{s}^2 = \dot{\mathbf{n}}\cdot\dot{\mathbf{n}} = \nu^2$. Recall that $\nu(t)$ is a prescribed function of time. Let $\dot{s} = \nu$. Then we have $d/dt = \nu\, d/ds$. Using this definition, we can represent the triple product $\mathbf{n}\cdot(\dot{\mathbf{n}}\times\ddot{\mathbf{n}})$ as

$$\mathbf{n}\cdot(\dot{\mathbf{n}}\times\ddot{\mathbf{n}}) = \mathbf{n}\cdot[\nu\mathbf{n}'\times(\nu\nu'\mathbf{n}' + \nu^2\mathbf{n}'')] = \nu^3\mathbf{n}\cdot(\mathbf{n}'\times\mathbf{n}') = \nu^3 k. \tag{4.4}$$

Then, using (4.4) and (3.5) we determine the spherical curvature k as a function of time:

$$k = \frac{f^3(t)}{\nu^3(t)} = \frac{\theta_3}{\theta_{12}}\frac{\Omega(t)}{\nu(t)} = \frac{\theta_3\Omega_0}{\theta_{12}\nu_0}\exp\left[\left(\frac{B_{12}}{\theta_{12}} - \frac{B_3}{\theta_3}\right)t\right]. \tag{4.5}$$

This expression implies the following three conclusions concerning the motion:

1. If $B_{12}/\theta_{12} > B_3/\theta_3$, then $k \to \infty$. In this case, the spherical curvature monotonically increases tending to infinity and, hence, the trajectory of the end of the vector $\mathbf{n}$ on the unit sphere is a spiral with decreasing radii of turns. In the limit, this trajectory converges to a point. See Fig. 2.
2. If $B_{12}/\theta_{12} < B_3/\theta_3$, then $k \to 0$. In this case, the trajectory is an unwinding spiral converging to a great circle on the sphere. See Fig. 3.
3. If $B_{12}/\theta_{12} = B_3/\theta_3$, then $k = $ const. In this case, the trajectory degenerates into a circle. It is the particular case (1.3) where the viscosity tensor is proportional to the inertia tensor.

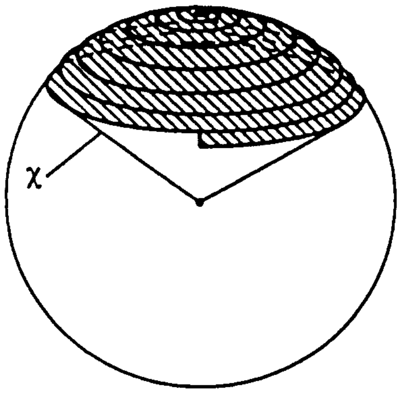

Fig. 4

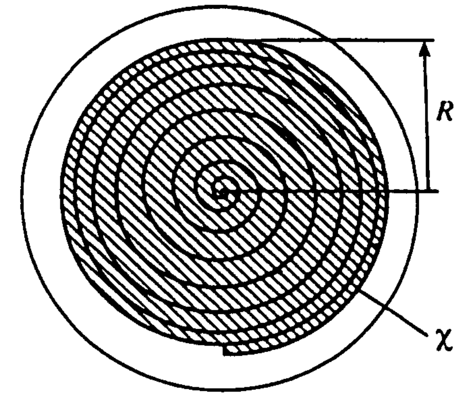

Fig. 5

5. QUASICOORDINATES

From the previous section, we know the derivatives

$$\dot{s} = \nu(t), \quad \dot{\xi} = \nu\xi' = \nu k = \frac{\theta_3}{\theta_{12}}\Omega(t).$$

The quantities s and ξ define the path passed along the trajectory on the sphere and the angle of rotation of this trajectory. In a sense, these can be interpreted as the coordinates of the vector $\mathbf{n}$. The introduction of an additional quantity ζ such that $\dot{\zeta} = \Omega$ permits one to solve the problem. According to (3.4), we have

$$\dot{s} = \nu(t) = \nu_0 \exp\left(-\frac{B_{12}t}{\theta_{12}}\right), \quad \dot{\xi} = \frac{\theta_3}{\theta_{12}}\Omega(t) = \frac{\theta_3}{\theta_{12}}\Omega_0 \exp\left(-\frac{B_3 t}{\theta_3}\right), \quad \dot{\zeta} = \Omega(t) = \Omega_0 \exp\left(-\frac{B_3 t}{\theta_3}\right).$$

Integrating these relations with zero initial conditions yields

$$s = \frac{\theta_{12}}{B_{12}}\nu_0\left[1 - \exp\left(-\frac{B_{12}t}{\theta_{12}}\right)\right], \quad \xi = \frac{\theta_3^2\Omega_0}{B_{12}\theta_{12}}\left[1 - \exp\left(-\frac{B_3 t}{\theta_3}\right)\right], \quad \zeta = \frac{\theta_3}{B_3}\Omega_0\left[1 - \exp\left(-\frac{B_3 t}{\theta_3}\right)\right]. \tag{5.1}$$

The quasicoordinates s, ξ, and ζ are not instantaneous characteristics of the position of the body. The knowledge of these variables at some current instant t does not permit one to identify the position of the body at this time instant. For this identification, it is necessary to know the entire time history of the quasicoordinates. However, these quasicoordinates have an important advantageous property—they preserve the symmetry of the problem, which substantially facilitates its solution.

The relations of (5.1) cannot be considered the solution of the problem in the proper sense of this term. However, these relations permit one to give an answer to virtually any question which may arise in this problem. As has already been mentioned, the trajectory of the end of the unit vector $\mathbf{n}$ on the unit sphere is a spiral. The limiting values of the variables s, ξ, and ζ as $t \to \infty$ define the length of this spiral, the total angle of the spiral winding, and the number of revolutions of the body about the axis of symmetry. The radius of the spiral turn, R, at an instant t can be expressed as

$$R = \frac{1}{K} = \frac{1}{\sqrt{1+k^2}}, \qquad k = \frac{d\xi}{ds} = \frac{\theta_3\Omega_0}{\theta_{12}\nu_0}\exp\left[\left(\frac{B_{12}}{\theta_{12}} - \frac{B_3}{\theta_3}\right)t\right]. \tag{5.2}$$

The angle of rotation of the body about its axis in the absolute space, $\Delta\varphi$, for the time during which the unit vector $\mathbf{n}$ passes one turn of the spiral can be calculated using the solid angle theorem [8]. This calculation yields

$$\Delta\varphi = \chi + \Delta\zeta,$$

where $\Delta\zeta$ is the increment of the angular quasicoordinate ζ per one turn and χ is the solid angle bounded by the spiral. See Figs. 4 and 5. If the spiral turns lie fairly close to each other, then the solid angle χ can be approximately expressed in terms of the radius of the turn defined by Eq. (5.2) as $\chi \approx 2\pi\left(1 - \sqrt{1-R^2}\right)$.

Thus, the expressions of (5.1) provide a fairly simple, graphic, and practically complete description for the motion of the system. It is well known that for a large number of relatively simple systems it is impossible to construct a closed-form solution in terms of instantaneous coordinates. In the cases where such a solution can be constructed, one often has to deal with extremely lengthy relations which do not admit a graphic physical interpretation. In such cases, it may be advisable to utilize quasicoordinates, as has been done in the present paper.

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