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About Using Moment of Momentum and Angular Velocity Vectors for Description of Rotational Motions of a Rigid Body

Two sets of vector variables for the analysis of rotational motions of a rigid body are presented. The first set consists of the vector projections of the angular velocity on the eigenvectors of the inertia tensor, the second one is based on the angular velocity and moment of momentum vectors. Vector analogues of the dynamic Euler equations are obtained for the considered variables. The presented equations allow to determine the orientation of the rigid body in space, whereas analysis of the classic Euler equations gives only scalar projections of the angular velocity on the body-fixed basis. However, the similarity of the proposed vector equations to the classic scalar ones allows application of the similar mathematical methods for the problem analysis. Example problems were solved using the proposed methods.

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1. Introduction

Let us consider a rigid body with a fixed point under the action of an external moment. The motion of the body can by described by the classic system of Euler-Poisson equations

$$\begin{aligned} &\theta_1\dot{\omega}_1 + (\theta_3 - \theta_2)\,\omega_2\omega_3 = M_1\,, &\qquad \dot{\gamma}_1 = \omega_3\gamma_2 - \omega_2\gamma_3\,,\\ &\theta_2\dot{\omega}_2 + (\theta_1 - \theta_3)\,\omega_3\omega_1 = M_2\,, &\qquad \dot{\gamma}_2 = \omega_1\gamma_3 - \omega_3\gamma_1\,,\\ &\theta_3\dot{\omega}_3 + (\theta_2 - \theta_1)\,\omega_1\omega_2 = M_3\,, &\qquad \dot{\gamma}_3 = \omega_2\gamma_1 - \omega_1\gamma_2\,, \end{aligned} \tag{1), (2}$$

where θ_k are the principal moments of inertia, ω_k, M_k, γ_k are projections of the angular velocity, the external moment, and the vertical unit vector on principal directions of the inertia tensor ($k = 1, 2, 3$).

The above system of equations is divided into two subsystems: dynamic Euler equations (1) and kinematic Poisson equations (2). Let us consider the case, when the external moment depends only on the angular velocities: $M_k = M_k(\omega_1, \omega_2, \omega_3)$ (a dissipative moment for example). Then subsystem (1) is separated and it forms a closed system for the angular velocity projections ω_k. If subsystem (1) is solved then the obtained functions $\omega_k(t)$ can be substituted in subsystem (2), which forms a linear system with variable coefficients for the projections γ_k. If this linear system is solved then the space orientation of the rigid body can be determined finally by integration of a known function of time.

Thus, the problem is decomposed into two subproblems, namely the calculations of ω_k and γ_k. *The first* subproblem, although it requires the solution of nonlinear equations, generally is less complicated than the second one. In fact, a system (1) is autonomous, the coefficients are constant, and the equations are symmetric and relatively simple. Such system is appropriate for analytical and semianalytical methods. *The second* subproblem is linear, but with variable coefficients, which should be obtained from the solution of the first subproblem. Such coefficients can have a very complicated form, therefore generally subsystem (2) cannot be solved analytically. That is why investigations usually stop after finding a solution of subsystem (1), which gives only projections ω_k of the angular velocity on a mobile basis. However, the quantities ω_k give very little description of the rigid body motion. Since these quantities are projections on a *mobile* basis, they do not allow to determine even the direction of the angular velocity vector in a fixed frame of reference, not mentioning the orientation of the rigid body in space. Thus, the solution of the first subproblem does not give even a half of the whole problem solution.

In the presented paper equations analogous to the dynamic Euler equations, but written in the terms of *vector* projections of the angular velocity, will be obtained. The advantage of the proposed equations is that their integration solves the problem completely, without a necessity to solve any additional equations. Alternative equations for the moment of momentum and the angular velocity vectors will be also obtained. Below the direct tensor calculus is used [1, 2]. A brief description of the main designations and identities common for the direct tensor calculus is given in Appendix A. Application of this technique to the analysis of rotational motions of rigid bodies is considered in [3, 4].

2. Vector analogue of dynamic Euler equations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be unit eigenvectors of the inertia tensor of a rigid body. Then the inertia tensor $\boldsymbol{\theta}$ can be expressed as

$$\boldsymbol{\theta} = \theta_1\mathbf{e}_1\mathbf{e}_1 + \theta_2\mathbf{e}_2\mathbf{e}_2 + \theta_3\mathbf{e}_3\mathbf{e}_3\,, \tag{3}$$

where $\mathbf{e}_1\mathbf{e}_1$, $\mathbf{e}_2\mathbf{e}_2$, and $\mathbf{e}_3\mathbf{e}_3$ are tensor products of the corresponding eigenvectors. The eigenvectors $\mathbf{e}_k$ form a mobile Cartesian basis. The scalar projections ω_k of the angular velocity vector $\boldsymbol{\omega}$ on the mobile basis satisfy the identities

$\omega_k = \mathbf{e}_k \cdot \boldsymbol{\omega}$ $(k = 1, 2, 3)$. Let us introduce *vector projections of the angular velocity vector* as

$$\boldsymbol{\omega}_1 \stackrel{\text{def}}{=} \omega_1 \mathbf{e}_1 , \qquad \boldsymbol{\omega}_2 \stackrel{\text{def}}{=} \omega_2 \mathbf{e}_2 , \qquad \boldsymbol{\omega}_3 \stackrel{\text{def}}{=} \omega_3 \mathbf{e}_3 . \tag{4}$$

The derivative of the vector projection $\boldsymbol{\omega}_1$ with respect to time is

$$\dot{\boldsymbol{\omega}}_1 = (\omega_1 \mathbf{e}_1)^{\cdot} = \dot{\omega}_1 \mathbf{e}_1 + \omega_1 \dot{\mathbf{e}}_1 . \tag{5}$$

To obtain the vector analogue of the dynamic Euler equations let us express the derivative $\dot{\omega}_1$ from the Euler equations (1) and the derivative $\dot{\mathbf{e}}_1$ from the Poisson equations (2), which in vector form can be rewritten as $\dot{\mathbf{e}}_k = \boldsymbol{\omega} \times \mathbf{e}_k$. Then formula (5) takes the form

$$\theta_1 \dot{\boldsymbol{\omega}}_1 = (\theta_2 - \theta_3)\, \omega_2 \omega_3 \mathbf{e}_1 + M_1 \mathbf{e}_1 + \theta_1 \omega_1 \boldsymbol{\omega} \times \mathbf{e}_1 .$$

Using the vector projections (4) instead of the scalar ones the above formula can be rewritten as

$$\theta_1 \dot{\boldsymbol{\omega}}_1 = (\theta_2 - \theta_3)\, \boldsymbol{\omega}_2 \times \boldsymbol{\omega}_3 + \theta_1 (\boldsymbol{\omega}_2 + \boldsymbol{\omega}_3) \times \boldsymbol{\omega}_1 + \mathbf{M}_1 , \tag{6}$$

where $\mathbf{M}_1 \stackrel{\text{def}}{=} M_1 \mathbf{e}_1$ is the vector projection of the external moment. The derivatives of $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$ can be calculated analogously, which gives the desired system of differential equations for the vector projections $\boldsymbol{\omega}_k$:

$$\begin{aligned} &\theta_1 \dot{\boldsymbol{\omega}}_1 + (\theta_3 - \theta_2)\, \boldsymbol{\omega}_2 \times \boldsymbol{\omega}_3 + \theta_1 \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_2 + \boldsymbol{\omega}_3) = \mathbf{M}_1 , \\ &\theta_2 \dot{\boldsymbol{\omega}}_2 + (\theta_1 - \theta_3)\, \boldsymbol{\omega}_3 \times \boldsymbol{\omega}_1 + \theta_2 \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_3 + \boldsymbol{\omega}_1) = \mathbf{M}_2 , \\ &\theta_3 \dot{\boldsymbol{\omega}}_3 + (\theta_2 - \theta_1)\, \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 + \theta_3 \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) = \mathbf{M}_3 . \end{aligned} \tag{7}$$

The obtained equations have a similar form as the dynamic Euler equations

$$\begin{aligned} &\theta_1 \dot{\omega}_1 + (\theta_3 - \theta_2)\, \omega_2 \omega_3 = M_1 , \\ &\theta_2 \dot{\omega}_2 + (\theta_1 - \theta_3)\, \omega_3 \omega_1 = M_2 , \\ &\theta_3 \dot{\omega}_3 + (\theta_2 - \theta_1)\, \omega_1 \omega_2 = M_3 . \end{aligned} \tag{8}$$

Hereafter we shall call name the equations (7) and (8) *vector and scalar Euler equations*, respectively. The solution of the vector equations gives the solution of the total problem as opposed to the scalar equations. In fact, if the vectors $\boldsymbol{\omega}_k$ are obtained as solution of system (7) then dividing them by their absolute values one can obtain the eigenvectors, $\mathbf{e}_k$ of the inertia tensor, and therefore the body orientation in the fixed frame of reference is known[1]). Due to the similarity of the vector and scalar Euler equations, most of the methods known for the scalar equations can be used to solve the vector ones. Note that the vectors $\boldsymbol{\omega}_k$ contain information about the space orientation of the body, hence the right sides of eqs. (7) always can be expressed by the vectors $\boldsymbol{\omega}_k$, even in the case when the external moment depends not only on the angular velocities, but on the body orientation as well.

A rigid body with a fixed point has 3 degrees of freedom, so a set of equations of the 6-th order is required. However, the vector equations (7) are of the 9-th order. This is due to the excessiveness of the variables $\boldsymbol{\omega}_k$. In fact, the vectors $\boldsymbol{\omega}_k$ are orthogonal, so they satisfy the three additional scalar identities

$$\boldsymbol{\omega}_1 \cdot \boldsymbol{\omega}_2 = 0 , \qquad \boldsymbol{\omega}_2 \cdot \boldsymbol{\omega}_3 = 0 , \qquad \boldsymbol{\omega}_3 \cdot \boldsymbol{\omega}_1 = 0 . \tag{9}$$

Replacement of one of the vector differential equations in system (7) by the identities (9) leads to a system of 6-th order. But in most cases the excessive system (7) is more convenient for an analytical analysis.

3. The dynamic variables

One of the advantages of Euler equations (7) and (8) is their symmetric form. However, for an analytical solution it can be more suitable to use another form of the equations, which allows to take into account the physical characteristics of the problem. Let us introduce *vector dynamic variables* $\mathbf{u}_k$ and *scalar dynamic variables* H_k:

$$\mathbf{u}_k \stackrel{\text{def}}{=} \boldsymbol{\theta}^k \cdot \boldsymbol{\omega} , \qquad H_k \stackrel{\text{def}}{=} \boldsymbol{\omega} \cdot \boldsymbol{\theta}^k \cdot \boldsymbol{\omega} . \tag{10}$$

Here k is an integer, $\boldsymbol{\theta}^k$ is the power k-th of the inertia tensor, dot stands for the scalar product. The quantities $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$ and ω_1, ω_2, ω_3 further will be called as *kinematic variables*. Substitution of the inertia tensor (3) in the above definitions transforms them to the following non-tensor form:

$$\mathbf{u}_k = \theta_1^k \boldsymbol{\omega}_1 + \theta_2^k \boldsymbol{\omega}_2 + \theta_3^k \boldsymbol{\omega}_3 , \qquad H_k = \theta_1^k \omega_1^2 + \theta_2^k \omega_2^2 + \theta_3^k \omega_3^2 , \tag{11}$$

[1]) Some difficulties can arrive only if the rigid body has a kind of dynamic symmetry (equality of two or three moments of inertia). In this case an infinite number of principal directions of the inertia tensor exists and it is necessary to choose only the directions fixed with the rigid body.

which also can be considered as a bound between the dynamic and the kinematic variables. Besides, according to definition (10), the following constraint between the variables $\mathbf{u}_k$ and H_k holds:

$$\mathbf{u}_k \cdot \mathbf{u}_n = H_{k+n}\,, \tag{12}$$

which allows to express the scalar dynamic variables in terms of vector dynamic variables.

Let us use the Hamilton-Cayley identity [2] for the inertia tensor

$$\boldsymbol{\theta}^3 - I_1\boldsymbol{\theta}^2 + I_2\boldsymbol{\theta} - I_3\mathbf{E} = 0\,, \tag{13}$$

where $\mathbf{E}$ is unit tensor and I_1, I_2, I_3 are the principal invariants of the inertia tensor:

$$I_1 = \theta_1 + \theta_2 + \theta_3\,, \qquad I_2 = \theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1\,, \qquad I_3 = \theta_1\theta_2\theta_3\,.$$

Identity (13) allows to express any power of the tensor $\boldsymbol{\theta}$ in terms of $\mathbf{E}$, $\boldsymbol{\theta}$, $\boldsymbol{\theta}^2$. Analogously the variables $\mathbf{u}_k$ and H_k for any integer index k can be expressed, respectively, in terms of the variables $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$ and H_0, H_1, H_2, which we shall name *principal dynamic variables.* In particular, direct substitution of (13) into (10) gives

$$\mathbf{u}_3 = I_1\mathbf{u}_2 - I_2\mathbf{u}_1 + I_3\mathbf{u}_0\,, \qquad H_3 = I_1H_2 - I_2H_1 + I_3H_0\,.$$

The principal dynamic variables have the following physical meaning:

$H_0 = \omega^2$	Square of angular velocity
$H_1 = 2T$	Double kinetic energy
$H_2 = L^2$	Square of moment of momentum
$\mathbf{u}_0 = \boldsymbol{\omega}$	Angular velocity vector
$\mathbf{u}_1 = \mathbf{L}$	Moment of momentum vector
$\mathbf{u}_2 = \boldsymbol{\theta}\cdot\mathbf{L}$	–

The constraint (11) between the dynamic and the kinematic variables for the principal dynamic variables can be represented in the following matrix form:

$$\mathbf{u} \stackrel{\text{def}}{=} \begin{bmatrix}\mathbf{u}_0\\ \mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix} = \mathcal{A}\begin{bmatrix}\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\\ \boldsymbol{\omega}_3\end{bmatrix}, \qquad H \stackrel{\text{def}}{=} \begin{bmatrix}H_0\\ H_1\\ H_2\end{bmatrix} = \mathcal{A}\begin{bmatrix}\omega_1^2\\ \omega_2^2\\ \omega_3^2\end{bmatrix}; \qquad \mathcal{A} \stackrel{\text{def}}{=} \begin{bmatrix}1 & 1 & 1\\ \theta_1 & \theta_2 & \theta_3\\ \theta_1^2 & \theta_2^2 & \theta_3^2\end{bmatrix}. \tag{14}$$

Determinant of the matrix $\mathcal{A}$ is the Vandermonde determinant

$$\det\mathcal{A} = (\theta_1 - \theta_2)\,(\theta_2 - \theta_3)\,(\theta_3 - \theta_1) \stackrel{\text{def}}{=} \Delta(\mathcal{A})\,. \tag{15}$$

If all principal moments of inertia are different then $\det\mathcal{A} \neq 0$ and relations (14) can be inverted and written in the following form:

$$\boldsymbol{\omega}_k = P_k(\mathbf{u})/\Pi_k\,, \qquad \omega_k^2 = P_k(H)/\Pi_k \qquad (k = 1, 2, 3)\,, \tag{16}$$

where $\mathbf{u}$ and H are the column matrices determined by (14), P_k are functions of an arbitrary column matrix

$$x = [x_0, x_1, x_2]^T \Rightarrow P_k(x) = x_2 - (I_1 - \theta_k)\,x_1 + (I_3/\theta_k)\,x_0 \qquad (k = 1, 2, 3)\,, \tag{17}$$

and Π_k are the following inertia constants:

$$\Pi_1 = (\theta_1 - \theta_2)\,(\theta_1 - \theta_3)\,, \qquad \Pi_2 = (\theta_2 - \theta_3)\,(\theta_2 - \theta_1)\,, \qquad \Pi_3 = (\theta_3 - \theta_1)\,(\theta_3 - \theta_2)\,. \tag{18}$$

Thus, the principal dynamic variables are bijectively connected with the kinematic variables by identities (14) and (16).

4. Differential equations for the vector dynamic variables

Applying relations (16) in the vector Euler equations (7) gives (after some transformations) the *differential equations for the vector dynamic variables*

$$\begin{aligned}
&\dot{\mathbf{u}}_0 + \frac{1}{I_3}\,\mathbf{u}_1 \times \mathbf{u}_2 = \boldsymbol{\theta}^{-1}\cdot\mathbf{M}\,,\\
&\dot{\mathbf{u}}_1 = \mathbf{M}\,,\\
&\dot{\mathbf{u}}_2 + I_1\mathbf{u}_0 \times \mathbf{u}_1 + 2\mathbf{u}_2 \times \mathbf{u}_0 = \boldsymbol{\theta}\cdot\mathbf{M}\,.
\end{aligned} \tag{19}$$

The above equations are less symmetric than the vector Euler equations (7), but they have less nonlinear terms in their left parts. The right parts of eqs. (19) can be decomposed in the terms of the vectors $\mathbf{u}_k$ (see example from Section 9). The above equations can be obtained directly from the equation of the moment of momentum balance, without use of the vector Euler equations — see Appendix B.

5. Rigid body orientation in terms of dynamic variables

If the vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$ are known as solution of system (19) then formulae (16) for a nonsymmetric rigid body give vectors $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$, which uniquely determine the orientation of the rigid body in fixed space. Let us show that explicitly. According to (4) and (16) the eigenvectors of the inertia tensor can be expressed as

$$\mathbf{e}_k = \frac{1}{\omega_k}\,\boldsymbol{\omega}_k = \pm\frac{P_k(\mathbf{u})}{\sqrt{\Pi_k P_k(H)}}\,. \tag{20}$$

Plus-minus in the above expression is due to the eigenvectors ambiguity. Applying of (20) in eq. (3) gives the inertia tensor in terms of vector dynamic variables

$$\boldsymbol{\theta} = \sum_{k=1}^{3} \theta_k \mathbf{e}_k \mathbf{e}_k = \sum_{k=1}^{3} \frac{\theta_k}{\Pi_k P_k(H)}\, P_k(\mathbf{u})\, P_k(\mathbf{u})\,. \tag{21}$$

The orientation of a rigid body can be effectively represented by means of the turn tensor [3], which is determined as

$$\mathbf{P} = \mathbf{e}_1\mathbf{e}_1^0 + \mathbf{e}_2\mathbf{e}_2^0 + \mathbf{e}_3\mathbf{e}_3^0\,, \tag{22}$$

where $\mathbf{e}_k^0$ are the eigenvectors in the reference position of the rigid body. If the turn tensor is known as a function of time then any vector, $\mathbf{a}$, attached to the rigid body can be found at any moment of time as $\mathbf{a}(t) = \mathbf{P}(t)\cdot\mathbf{a}^0$, where $\mathbf{a}^0$ is the value of the vector $\mathbf{a}$ in the reference position. This allows to describe the motion of every point of the rigid body by means of the single tensor function of time $\mathbf{P}(t)$. Returning to the dynamic variables the tensor of turn can be expressed as

$$\mathbf{P} = \sum_{k=1}^{3} \mathbf{e}_k\mathbf{e}_k^0 = \sum_{k=1}^{3} \frac{1}{\Pi_k\sqrt{P_k(H^0)\,P_k(H)}}\, P_k(\mathbf{u})\, P_k(\mathbf{u}^0)\,, \tag{23}$$

where the upper index “0” denotes the quantities in the reference position.

Let us consider the singular case of the axially symmetric rigid body. Assume that $\mathbf{e}_3$ coincides with the symmetry axis, then $\theta_1 = \theta_2 \stackrel{\text{def}}{=} \theta_{12}$ and from (18) it follows that $\Pi_1 = \Pi_2 = 0$, $\Pi_3 = (\theta_3 - \theta_{12})^2 \neq 0$. Hence in this case formula (20) allows to find only the unit vector $\mathbf{e}_3$ of the symmetry axis. But in the case of axial symmetry the inertia tensor can be expressed in terms of $\mathbf{e}_3$ only (see Appendix A), which gives

$$\boldsymbol{\theta} = \theta_{12}\mathbf{E} + (\theta_3 - \theta_{12})\,\mathbf{e}_3\mathbf{e}_3 = \theta_{12}\mathbf{E} + \frac{\theta_3}{(\theta_3 - \theta_{12})\,P_3(H)}\, P_3(\mathbf{u})\, P_3(\mathbf{u})\,.$$

The turn tensor in this case can be determined accurately to turn around the symmetry axis only, but usually this is sufficient. For a spherically symmetric rigid body $\theta_1 = \theta_2 = \theta_3 \stackrel{\text{def}}{=} \theta$, and hence the inertia tensor is known always: $\boldsymbol{\theta} = \theta(\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3) = \theta\mathbf{E}$, but the turn tensor cannot be determined in terms of dynamic variables in this case.

Thus, *the vector dynamic variables uniquely determine the inertia tensor of a rigid body; the turn tensor can be determined accurately to the symmetry group of the inertia tensor.*

6. Description of the motion based on the moment of momentum and angular velocity vectors

Above a description of a rigid body motion based on the vector dynamic variables $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$ was introduced. Remind that $\mathbf{u}_0$ is the vector of angular velocity, $\mathbf{u}_1$ is the vector of moment of momentum, the vector $\mathbf{u}_2$ does not have analogous physical interpretation. However it appears that two vectors $\mathbf{u}_0$ and $\mathbf{u}_1$ are sufficient for the motion description. Let us show that. For a nonsymmetric rigid body the vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$ are linearly independent (see eqs. (14)–(15) or Appendix C). Nevertheless *vector* $\mathbf{u}_2$ *can be expressed in terms of* $\mathbf{u}_0$ *and* $\mathbf{u}_1$ *in nonlinear way*:

$$\mathbf{u}_2 = [(H_0H_3 - H_1H_2)\,\mathbf{u}_1 - (H_1H_3 - H_2^2)\,\mathbf{u}_0 \pm \sqrt{\Delta(H)}\,\mathbf{u}_0\times\mathbf{u}_1]/(H_0H_2 - H_1^2)\,. \tag{24}$$

The above formula is proved in Appendix D. The formula allows to eliminate the variable $\mathbf{u}_2$ from the differential equations (19) for the vector dynamic variables. Then the first two equations from (19) can be represented as

$$\dot{\boldsymbol{\omega}} - \frac{1}{I_3}\,\frac{hH_3 - L^4}{\omega^2L^2 - h^2}\,\mathbf{L}\times\boldsymbol{\omega} \pm \frac{1}{I_3}\,\frac{\sqrt{\Delta(H)}}{\omega^2L^2 - h^2}\,\mathbf{L}\times(\mathbf{L}\times\boldsymbol{\omega}) = \boldsymbol{\theta}^{-1}\cdot\mathbf{M}\,, \qquad \dot{\mathbf{L}} = \mathbf{M}\,, \tag{25}$$

where $\boldsymbol{\omega} = \mathbf{u}_0$ is the angular velocity, $\mathbf{L} = \mathbf{u}_1$ is the moment of momentum, and the sign "±" should be chosen from the conditions of continuity — see Appendix C. Thus, the above system represents the desired *equations of motion of a rigid body in terms of the moment of momentum and angular velocity vectors.* Eqs. (25) are of 6-th order, as it should be for the rotational motions of a rigid body. The quantities h, H_3, and Δ(H) in eqs. (25) can be expressed in terms of $\mathbf{L}$ and $\boldsymbol{\omega}$ by the formulae

$$h = \mathbf{L}\cdot\boldsymbol{\omega}\,, \qquad H_3 = I_1 L^2 - I_2 h + I_3\omega^2\,,$$
$$\Delta(H) = -(L^2 - (\theta_2+\theta_3)\,h + \theta_2\theta_3\omega^2)\,(L^2 - (\theta_3+\theta_1)\,h + \theta_3\theta_1\omega^2)\,(L^2 - (\theta_1+\theta_2)\,h + \theta_1\theta_2\omega^2)\,.$$

Assume that the vectors $\mathbf{L}$ and $\boldsymbol{\omega}$ are known as the solution of system (25). Then, the vector $\mathbf{u}_2$ can be found from formula (24), and then formula (21) gives the inertia tensor of the rigid body. Thus, the following important conclusion can be deduced: the *space orientation of the inertia tensor of a rigid body is determined uniquely by the instantaneous values of the angular velocity vector and the moment of momentum vector.* Let us remark that for a translational motion the vectors of velocity and momentum never allow to determine the position of a body without integration. In Section 5 it was deduced that the vector dynamic variables determine the orientation of a rigid body accurately to the symmetry group of its inertia tensor. The same result holds for the vectors $\boldsymbol{\omega}$ and $\mathbf{L}$. Thus, *algebraic determination of the orientation of a rigid body using the vectors of the angular velocity and moment of momentum is only possible due to the anisotropy of the inertia properties with respect to rotation.*

7. Differential equations for the scalar dynamic variables

The scalar dynamic variables represent the most important scalar dynamic characteristics of a rigid body motion, namely the kinetic energy and the absolute values of the angular velocity and moment of momentum vectors. Let us obtain equations of motion in terms of these variables.

Using identity (12) the principal scalar variables can be expressed in terms of vectors $\mathbf{u}_0$ and $\mathbf{u}_1$:

$$H_0 = \mathbf{u}_0\cdot\mathbf{u}_0\,, \qquad H_1 = \mathbf{u}_0\cdot\mathbf{u}_1\,, \qquad H_2 = \mathbf{u}_1\cdot\mathbf{u}_1\,. \tag{26}$$

Let us find derivatives of the scalar variables (26) using differential equations (19) for $\mathbf{u}_k$:

$$\dot H_0 = 2\mathbf{u}_0\cdot\dot{\mathbf{u}}_0 = -\frac{2}{I_3}\,\mathbf{u}_0\cdot(\mathbf{u}_1\times\mathbf{u}_2) + 2\mathbf{u}_0\cdot\boldsymbol{\theta}^{-1}\cdot\mathbf{M} = \mp\frac{2}{I_3}\,\sqrt{\Delta(H)} + 2\mathbf{u}_{-1}\cdot\mathbf{M}\,,$$
$$\dot H_1 = \dot{\mathbf{u}}_0\cdot\mathbf{u}_1 + \mathbf{u}_0\cdot\dot{\mathbf{u}}_1 = \mathbf{u}_1\cdot\boldsymbol{\theta}^{-1}\cdot\mathbf{M} + \mathbf{u}_0\cdot\mathbf{M} = 2\mathbf{u}_0\cdot\mathbf{M}\,,$$
$$\dot H_2 = 2\mathbf{u}_1\cdot\dot{\mathbf{u}}_1 = 2\mathbf{u}_1\cdot\mathbf{M}\,,$$

where

$$\Delta(H) \overset{\text{def}}{=} -\,P_1(H)\;P_2(H)\;P_3(H) \tag{27}$$

and formula (46) from Appendix C is used to calculate the mixed product of $\mathbf{u}_k$ vectors. Summarizing the above one can obtain the following *differential equations for the scalar dynamic variables*:

$$\begin{cases} \dot H_0 \pm \dfrac{2}{I_3}\,\sqrt{\Delta(H)} = 2\mathbf{u}_{-1}\cdot\mathbf{M}\,, \\ \dot H_1 = 2\mathbf{u}_0\cdot\mathbf{M}\,, \\ \dot H_2 = 2\mathbf{u}_1\cdot\mathbf{M}\,. \end{cases} \tag{28}$$

It can be shown that if the moments M_k in the Euler equations (1) are functions of the angular velocities ω_k, then the right parts of the above equations are functions of the dynamic variables H_k. System (28) does not have the symmetry of the Euler equations (1), but it deals with the variables which are much more useful from the physical point of view. In the next section this will be shown on the example of the Euler problem. System (28) is only of third order and similarly to the Euler system (1) it cannot solve the problem completely (three additonal scalar equations are required), but it allows to find important scalar characteristics of the process, such as the kinetic energy and the absolute value of the moment of momentum, and therefore this system can be helpful for an analytical solution.

8. The Euler problem

Let us consider the Euler problem about free rotations of a rigid body: $\mathbf{M}\equiv 0$. In this case eqs. (28) for the scalar dynamic variables take the form

$$\dot H_0 \pm \frac{2}{I_3}\,\sqrt{\Delta(H)} = 0\,, \qquad \dot H_1 = 0\,, \qquad \dot H_2 = 0\,, \tag{29}$$

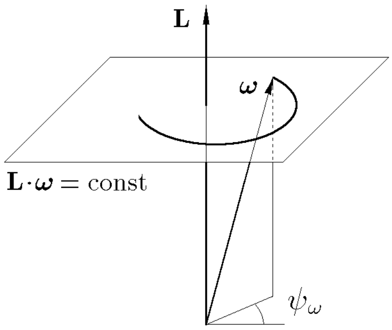

Fig. 1.

where H_0 is the square of the angular velocity, H_1 the double kinetic energy, and H_2 the square of the moment of momentum, respectively. Hence, the 2-nd and the 3-rd equations from (29) give the integral of energy and the integral of moment of momentum, respectively. Consider the 1-st equation. The quantity under the square root has the form

$$\Delta(H) = -P_1(H)\, P_2(H)\, P_3(H)\,, \qquad P_k(H) = H_2 - (I_1 - \theta_k)\, H_1 + (I_3/\theta_k)\, H_0\,.$$

Since the quantities H_1 and H_2 are constant, the functions $P_k(H)$ are linear functions of H_0 with constant coefficients. These functions can be represented in the form

$$P_k(H) = (I_3/\theta_k)\,(H_0 - H_{0k})\,, \qquad H_{0k} \stackrel{\text{def}}{=} \frac{(I_1 - \theta_k)\, H_1 - H_2}{I_3/\theta_k}\,.$$

Then $\Delta(H) = -I_3^2(H_0 - H_{01})\,(H_0 - H_{02})\,(H_0 - H_{03})$ and substituting that in the first equation (29) gives the elliptic integral for H_0

$$\int \frac{dH_0}{\sqrt{-(H_0 - H_{01})\,(H_0 - H_{02})\,(H_0 - H_{03})}} = \mp 2 \int dt\,. \tag{30}$$

Since H_0 is the square of the angular velocity, the above equation is a classical solution of the Euler problem. It was obtained almost automatically from eqs. (29), however it is necessary to perform unwieldy transformations to obtain it from the Euler equations (1).

To visualize the motion let us use system (19) for the vector dynamic variables

$$\dot{\mathbf{u}}_0 + \frac{1}{I_3}\,\mathbf{u}_1 \times \mathbf{u}_2 = 0\,, \qquad \dot{\mathbf{u}}_1 = 0\,, \qquad \dot{\mathbf{u}}_2 + I_1\mathbf{u}_0 \times \mathbf{u}_1 + 2\mathbf{u}_2 \times \mathbf{u}_0 = 0\,. \tag{31}$$

From the second equation it immediately follows that $\mathbf{u}_1$ is constant, that is the moment of momentum does not depend on time. Scalar multiplication of the first equation (31) by vector $\mathbf{u}_1$ gives $(\mathbf{u}_1 \cdot \mathbf{u}_0)^{\cdot} = 0$. Hence, the projection of the angular velocity on the direction of the moment of momentum is constant. This practically means, that the vertex of the angular velocity vector is moving in a constant plane, which is orthogonal to $\mathbf{u}_1$, as depicted in Fig. 1.

Let ψ_ω be the angle of the precession of the angular velocity around the moment of momentum vector. Then, using (31) a time derivative of this angle can be calculated as

$$\dot{\psi}_\omega = |\mathbf{u}_1|\,\frac{\mathbf{u}_1 \cdot (\mathbf{u}_0 \times \dot{\mathbf{u}}_0)}{(\mathbf{u}_1 \times \mathbf{u}_0)^2} = -\frac{1}{I_3}\,|\mathbf{u}_1|\,\frac{(\mathbf{u}_1 \times \mathbf{u}_0) \cdot (\mathbf{u}_1 \times \mathbf{u}_2)}{(\mathbf{u}_1 \times \mathbf{u}_0)^2} = \frac{1}{I_3}\,\sqrt{H_2}\,\frac{H_1 H_3 - H_2^2}{H_0 H_2 - H_1^2}\,. \tag{32}$$

To determine $\psi_\omega(t)$ one more integral should be calculated, which can be taken in terms of elliptic functions.

Thus, the vector $\mathbf{u}_1$ is known and constant, the vector $\mathbf{u}_0$ performs precession around $\mathbf{u}_1$, the modulus of $\mathbf{u}_0$ and the angle of precession are determined by (30) and (32). Since the vector $\mathbf{u}_0$ is found, the vector $\mathbf{u}_2$ can be obtained from (24). Then, the inertia tensor and turn tensor of the rigid body can be calculated algebraically from (21) and (23).

9. Motion of a rigid body in a medium with linear resistance

Let us consider the motion of a nonsymmetric rigid body under the action of a dissipative moment, which is linearly proportional to the angular velocity:

$$\mathbf{M} = -\mathbf{B} \cdot \boldsymbol{\omega}\,, \qquad \mathbf{B} = \sum_{k=1}^{3} B_k \mathbf{e}_k \mathbf{e}_k\,, \qquad B_k \geq 0\,, \tag{33}$$

where $\mathbf{B}$ is the dissipation tensor which for simplicity is chosen to be coaxial with the inertia tensor of the rigid body, B_k are the coefficients of dissipation. In non tensor form the above formula can be represented as

$$\mathbf{M} = -(B_1\boldsymbol{\omega}_1 + B_2\boldsymbol{\omega}_2 + B_3\boldsymbol{\omega}_3)\,.$$

The vector Euler equations (7) for this problem take the form

$$\begin{aligned}
&\theta_1\dot{\boldsymbol{\omega}}_1 + B_1\boldsymbol{\omega}_1 + (\theta_3 - \theta_2)\,\boldsymbol{\omega}_2 \times \boldsymbol{\omega}_3 + \theta_1\boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_2 + \boldsymbol{\omega}_3) = 0\,,\\
&\theta_2\dot{\boldsymbol{\omega}}_2 + B_2\boldsymbol{\omega}_2 + (\theta_1 - \theta_3)\,\boldsymbol{\omega}_3 \times \boldsymbol{\omega}_1 + \theta_2\boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_3 + \boldsymbol{\omega}_1) = 0\,,\\
&\theta_3\dot{\boldsymbol{\omega}}_3 + B_3\boldsymbol{\omega}_3 + (\theta_2 - \theta_1)\,\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 + \theta_3\boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) = 0\,.
\end{aligned} \tag{34}$$

Since the tensors $\mathbf{B}$ and $\boldsymbol{\theta}$ are coaxial, the tensor $\mathbf{B}$ can be expressed as

$$\mathbf{B} = \beta_2\mathbf{E} + \beta_1\boldsymbol{\theta} + \beta_0\boldsymbol{\theta}^2 \Rightarrow \mathbf{M} = -\mathbf{B}\cdot\boldsymbol{\omega} = -(\beta_2\mathbf{u}_0 + \beta_1\mathbf{u}_1 + \beta_0\mathbf{u}_2)\,,$$

where β_k are constants, which can be expressed in terms of B_k and θ_k. Then the eqs. (19) for the vector dynamic variables can be written in the form

$$\begin{aligned}
&\dot{\mathbf{u}}_0 + \beta_2\mathbf{u}_{-1} + \beta_1\mathbf{u}_0 + \beta_0\mathbf{u}_1 + \frac{1}{I_3}\,\mathbf{u}_1 \times \mathbf{u}_2 = 0\,,\\
&\dot{\mathbf{u}}_1 + \beta_2\mathbf{u}_0 + \beta_1\mathbf{u}_1 + \beta_0\mathbf{u}_2 = 0\,,\\
&\dot{\mathbf{u}}_2 + \beta_2\mathbf{u}_1 + \beta_1\mathbf{u}_2 + \beta_0\mathbf{u}_3 + I_1\mathbf{u}_0 \times \mathbf{u}_1 + 2\mathbf{u}_2 \times \mathbf{u}_0 = 0\,.
\end{aligned} \tag{35}$$

The eqs. (28) for the scalar dynamic variables for this problem are

$$\begin{aligned}
&\frac{1}{2}\,\dot{H}_0 + \beta_2 H_{-1} + \beta_1 H_0 + \beta_0 H_1 \pm \frac{1}{I_3}\,\sqrt{\Delta(H)} = 0\,,\\
&\frac{1}{2}\,\dot{H}_1 + \beta_2 H_0 + \beta_1 H_1 + \beta_0 H_2 = 0\,,\\
&\frac{1}{2}\,\dot{H}_2 + \beta_2 H_1 + \beta_1 H_2 + \beta_0 H_3 = 0\,.
\end{aligned} \tag{36}$$

The quantities $\mathbf{u}_{-1}$, $\mathbf{u}_3$; H_{-1}, H_3 in the above systems are to be expressed by the Hamilton-Cayley identity (13) as

$$\begin{aligned}
&I_3\mathbf{u}_{-1} = I_2\mathbf{u}_0 - I_1\mathbf{u}_1 + \mathbf{u}_2\,, \qquad \mathbf{u}_3 = I_1\mathbf{u}_2 - I_2\mathbf{u}_1 + I_3\mathbf{u}_0\,,\\
&I_3 H_{-1} = I_2 H_0 - I_1 H_1 + H_2\,, \qquad H_3 = I_1 H_2 - I_2 H_1 + I_3 H_0\,.
\end{aligned}$$

There are two types of the nondifferential terms in systems (34)–(36): the linear dissipative terms and the nonlinear inertia terms. For the big β values the dissipative terms dominate and the motion becomes close to an exponentially damped motion. For the small β values the inertia terms dominate and the motion can be described as a slowly changing solution of the Euler problem.

In [5, 6] a method of analytical analysis of the considered problem, based on an expansion of the Euler-Poisson equations (1)–(2) into an exponential series, is offered. This method can be used for the analysis of the vector equations (34)–(35) as well. The difference is that the use of the vector equations (34)–(35) allows to obtain the solution directly from the equations of motion, using single expansion into series. The approach, based on the Euler-Poisson equations needs sequential solution of two systems of equations, so it is necessary to substitute series into series that leads to far more complicated equations.

Let us consider the solution of the problem in the simplest case: $\beta_0 = 0$, $\beta_2 = 0$; $\beta_1 \overset{\text{def}}{=} \beta \neq 0$, which practically means that the dissipative moment is proportional to the moment of momentum. Then system (35) takes the form

$$\begin{aligned}
&\dot{\mathbf{u}}_0 + \beta\mathbf{u}_0 + \frac{1}{I_3}\,\mathbf{u}_1 \times \mathbf{u}_2 = 0\,,\\
&\dot{\mathbf{u}}_1 + \beta\mathbf{u}_1 = 0\,,\\
&\dot{\mathbf{u}}_2 + \beta\mathbf{u}_2 + I_1\mathbf{u}_0 \times \mathbf{u}_1 + 2\mathbf{u}_2 \times \mathbf{u}_0 = 0\,.
\end{aligned}$$

The above system can be rewritten as

$$\begin{aligned}
&\mathrm{e}^{-\beta t}(\mathrm{e}^{\beta t}\,\mathbf{u}_0)^{\cdot} + \frac{1}{I_3}\,\mathbf{u}_1 \times \mathbf{u}_2 = 0\,,\\
&\mathrm{e}^{-\beta t}(\mathrm{e}^{\beta t}\,\mathbf{u}_0)^{\cdot} = 0\,,\\
&\mathrm{e}^{-\beta t}(\mathrm{e}^{\beta t}\,\mathbf{u}_0)^{\cdot} + I_1\mathbf{u}_0 \times \mathbf{u}_1 + 2\mathbf{u}_2 \times \mathbf{u}_0 = 0\,.
\end{aligned} \tag{37}$$

Introduce the time-like variable τ, such as

$$\frac{d}{dt} = \mathrm{e}^{-\beta t}\,\frac{d}{d\tau}\,. \tag{38}$$

Then system (37) takes the form

$$(\mathrm{e}^{\beta t}\,\mathbf{u}_0)' + \frac{1}{I_3}\,(\mathrm{e}^{\beta t}\,\mathbf{u}_1) \times (\mathrm{e}^{\beta t}\,\mathbf{u}_2) = 0\,,$$

$$(\mathrm{e}^{\beta t}\,\mathbf{u}_0)' = 0\,,$$

$$(\mathrm{e}^{\beta t}\,\mathbf{u}_0)' + I_1(\mathrm{e}^{\beta t}\,\mathbf{u}_0) \times (\mathrm{e}^{\beta t}\,\mathbf{u}_1) + 2(\mathrm{e}^{\beta t}\,\mathbf{u}_2) \times (\mathrm{e}^{\beta t}\,\mathbf{u}_0) = 0\,.$$

Here prime denotes derivation with respect to τ. Change of variables $\mathbf{U}_k(\tau) = \mathrm{e}^{\beta t}\,\mathbf{u}_k(t)$ transforms the above equations to the eqs. (31) of the Euler problem. Integration of formula (38) with the initial condition $\tau|_{t=0} = 0$ gives $\tau = (1/\beta)\,(1 - \mathrm{e}^{-\beta t})$. Then the general solution of the problem can be represented in the form

$$\mathbf{u}_k(t) = \mathrm{e}^{-\beta t}\,\mathbf{U}_k\left(\frac{1}{\beta}\,(1 - \mathrm{e}^{-\beta t})\right),$$

where $\mathbf{U}_k(t)$ is the general solution of the Euler problem.

10. Concluding remarks

The general problem of rotations of a rigid body under the action of an external moment is considered in the paper. Equations analogous to the *scalar* dynamic Euler equations, but expressed in terms of *vector* variables, are introduced. This makes it possible to obtain a closed form solution of the problem, whereas the solution of the classic Euler equations gives only scalar projections of the angular velocity, not to mention the orientation of the rigid body in space. However, a similarity of the proposed vector equations to the classic scalar ones allows application of the similar mathematical methods for the problem analysis.

Two sets of the vector variables are considered. The first set (*kinematic variables*) includes three vector projections of the angular velocity onto the eigenvectors of the inertia tensor. This gives vector equations with similar symmetry properties as the scalar Euler equations. The second set (*dynamic variables*) includes products of the angular velocity and three different powers of the inertia tensor. The equations of motion in terms of dynamic variables do not have symmetry of the Euler equations, but they are shorter and more convenient for applications, because the dynamic variables express such essential physical characteristics as the angular velocity and moment of momentum of the rigid body.

In addition, it is shown that the dynamics of a rigid body can be described in terms of the angular velocity and moment of momentum vectors only. Two vector equations of the first order in terms of these variables form a closed set of equations for the rigid body dynamics. It is shown that the instantaneous values of the angular velocity vector and the moment of momentum vector uniquely determine the space orientation for the inertia tensor of a rigid body. In other words, these vectors determine the inertia tensor in a purely algebraic way, without any integration. Let us remark that for a translational motion the vectors of velocity and momentum never allow to determine the position of a body without integration. This confirms once more a huge difference between rotational and translational motions. It is proved in the paper that the vectors of the angular velocity and moment of momentum determine the orientation of a rigid body accurately to the symmetry group of its inertia tensor. Thus, algebraic determination of the orientation of a rigid body using the vectors of the angular velocity and moment of momentum is only possible due to the anisotropy of the inertia properties with respect to rotation. In the case of spherical symmetry the orientation cannot be determined in an algebraic way, so in this case there is a complete analogy with translational motion.

To illustrate the suggested methods two problems from the rigid body dynamics are considered. It is shown that the known solution of the Euler problem can be obtained directly from the presented equations. The equations of motion in the form admitting an analytical solution are obtained for the rotation of a rigid body in a linear viscous medium. For a particular case the exact solution is presented.

Appendixes

A Vectors and tensors

The language of direct tensor calculus is used in this paper. In this appendix a brief description of the main designations and identities is presented. In detail tensor calculus is considered in [1, 2]. An application of the direct tensor calculus to the analysis of rotational motions of rigid bodies is presented in [3, 4]. For better visibility the following analogy between the tensor and matrix algebra can be remarked. In a Cartesian basis a vector can be represented by a column matrix 1×3 and a tensor can be represented by a square matrix 3×3. Then the scalar products of vectors and tensors can be represented by matrix products. But of course this analogy is very limited, since matrix representation is only an image of a vector or tensor object, and this representation strongly depends on the choose of the basis. In direct tensor calculus the vectors and tensors are considered independently of a basis and as indivisible objects, not as a set of coordinates.

Vectors and tensors in the presented paper are marked by bold font: **a**, **b**, **C**, **D**. A scalar, vector, and tensor product of the vectors **a** and **b** is a scalar, vector, and tensor, which is denoted by $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \times \mathbf{b}$, and **ab**, respectively. A scalar product of the tensor

$\mathbf{C}$ and the vector $\mathbf{b}$ is a vector denoted by $\mathbf{C}\cdot\mathbf{b}$. A scalar product of the tensors $\mathbf{C}$ and $\mathbf{D}$ is a tensor $\mathbf{C}\cdot\mathbf{D}$. An integer power n of the tensor $\mathbf{C}$ is a tensor $\mathbf{C}^n$, which is defined by the sequential scalar product of n tensors $\mathbf{C}$. A unit tensor is denoted by $\mathbf{E}$ and for any vector $\mathbf{a}$ the following identity holds: $\mathbf{a}\cdot\mathbf{E}=\mathbf{E}\cdot\mathbf{a}=\mathbf{a}$. If $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are unit vectors of a Cartesian basis then the unit tensor can be represented as $\mathbf{E}=\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2+\mathbf{e}_3\mathbf{e}_3$. An inverse of the tensor $\mathbf{C}$ is a tensor denoted by $\mathbf{C}^{-1}$, which satisfies the identity $\mathbf{C}\cdot\mathbf{C}^{-1}=\mathbf{C}^{-1}\cdot\mathbf{C}=\mathbf{E}$. A tensor $\mathbf{C}$ is symmetric if for any vector $\mathbf{b}$ the following condition holds: $\mathbf{b}\cdot\mathbf{C}=\mathbf{C}\cdot\mathbf{b}$. The inertia tensor $\boldsymbol{\theta}$ of a rigid body is an example of a symmetric tensor. An eigenvalue λ and an eigenvector $\mathbf{e}$ of a tensor $\mathbf{C}$ satisfy the identity $\mathbf{e}\cdot\mathbf{C}=\lambda\mathbf{C}$. Eigenvalues of a symmetric tensor are positive and unit eigenvectors form a Cartesian basis. Any symmetric tensor $\mathbf{C}$ can be represented by the spectral decomposition

$$\mathbf{C}=\lambda_1\mathbf{e}_1\mathbf{e}_1+\lambda_2\mathbf{e}_2\mathbf{e}_2+\lambda_3\mathbf{e}_3\mathbf{e}_3\,, \tag{39}$$

where λ_k and $\mathbf{e}_k$ are respectively the eigenvalues and the unit eigenvectors of the tensor $\mathbf{C}$ (there can be three different eigenvalues for a symmetric tensor at maximum). If two eigenvalues of a symmetric tensor are equal, say $\lambda_1=\lambda_2\stackrel{\text{def}}{=}\lambda_{12}$, then the tensor is said to have a space symmetry with respect to the $\mathbf{e}_3$ axis. In this case identity (39) can be rewritten as

$$\mathbf{C}=\lambda_{12}(\mathbf{E}-\mathbf{e}_3\mathbf{e}_3)+\lambda_3\mathbf{e}_3\mathbf{e}_3\,. \tag{40}$$

If all eigenvalues are equal, $\lambda_1=\lambda_2=\lambda_3\stackrel{\text{def}}{=}\lambda$, then the tensor is spherical and is proportional to the unit tensor:

$$\mathbf{C}=\lambda\mathbf{E}\,. \tag{41}$$

A power n of a symmetric tensor (39) can be represented as

$$\mathbf{C}^n=\lambda_1^n\mathbf{e}_1\mathbf{e}_1+\lambda_2^n\mathbf{e}_2\mathbf{e}_2+\lambda_3^n\mathbf{e}_3\mathbf{e}_3\,. \tag{42}$$

B Derivation of the differential equations for the vector dynamic variables

Let us show how to obtain eqs. (19) for the vector dynamic variables $\mathbf{u}_0$, $\mathbf{u}_1$, and $\mathbf{u}_2$ directly from the equation of the moment of momentum balance

$$\dot{\mathbf{L}}=\mathbf{M}\,, \tag{43}$$

where $\mathbf{L}$ is the moment of momentum of the rigid body, $\mathbf{M}$ is the external moment. Since $\mathbf{L}=\mathbf{u}_1$, then the above equation immediately gives the second equation for system (19). To obtain the equation for $\mathbf{u}_0$ let us represent the moment of momentum as $\mathbf{L}=\boldsymbol{\theta}\cdot\boldsymbol{\omega}=\boldsymbol{\theta}\cdot\mathbf{u}_0$, where $\boldsymbol{\theta}$ is the inertia tensor, and $\boldsymbol{\omega}$ is the angular velocity of the rigid body. Then eq. (43) can be transformed as following:

$$(\boldsymbol{\theta}\cdot\mathbf{u}_0)^{\cdot}=\mathbf{M}\Rightarrow\boldsymbol{\theta}\cdot\dot{\mathbf{u}}_0+\mathbf{u}_0\times(\boldsymbol{\theta}\cdot\mathbf{u}_0)=\mathbf{M}\Rightarrow\dot{\mathbf{u}}_0+\boldsymbol{\theta}^{-1}\cdot(\mathbf{u}_0\times\mathbf{u}_1)=\boldsymbol{\theta}^{-1}\cdot\mathbf{M}\,. \tag{44}$$

The term $\boldsymbol{\theta}^{-1}\cdot(\mathbf{u}_0\times\mathbf{u}_1)$ in the above equation can be reduced as

$$\boldsymbol{\theta}^{-1}\cdot(\mathbf{u}_0\times\mathbf{u}_1)=\frac{1}{\det\boldsymbol{\theta}}\,(\boldsymbol{\theta}\cdot\mathbf{u}_0)\times(\boldsymbol{\theta}\cdot\mathbf{u}_1)=\frac{1}{I_3}\,\mathbf{u}_1\times\mathbf{u}_2\,, \tag{45}$$

where the identity $\mathbf{A}\cdot(\mathbf{a}\times\mathbf{b})=(\det\mathbf{A})\,(\mathbf{A}^{-1}\cdot\mathbf{a})\times(\mathbf{A}^{-1}\cdot\mathbf{b})$, which is valid for any symmetric tensor $\mathbf{A}$ and any vectors $\mathbf{a}$, $\mathbf{b}$, is used [3]. Substitution of (45) in eq. (44) gives the first equation for system (19). Finally, let us represent the moment of momentum as $\mathbf{L}=\boldsymbol{\theta}^{-1}\cdot\mathbf{u}_2$. Transformations analogous to (44)–(45) give the third equation for system (19).

C Formula for the mixed product of the vector dynamic variables

The mixed product of the vector dynamic variables $\mathbf{u}_0$, $\mathbf{u}_1$, and $\mathbf{u}_2$ can be expressed in terms of the scalar dynamic variables H_0, H_1, and H_2 by the following identity:

$$\mathbf{u}_0\cdot(\mathbf{u}_1\times\mathbf{u}_2)=\pm\sqrt{\Delta(H)}\,,\qquad \Delta(H)\stackrel{\text{def}}{=}-\,P_1(H)\,P_2(H)\,P_3(H)\,, \tag{46}$$

where cofactors $P_k(H)$ are defined by formulae (14), (17).

P r o o f: Vector dynamic variables $\mathbf{u}_k$ can be decomposed in the eigenbasis of the inertia tensor as following: $\mathbf{u}_k=\theta_1^k\omega_1\mathbf{e}_1+\theta_2^k\omega_2\mathbf{e}_2+\theta_3^k\omega_3\mathbf{e}_3$. Then the mixed product can be calculated as

$$\mathbf{u}_0\cdot(\mathbf{u}_1\times\mathbf{u}_2)=\begin{vmatrix}\omega_1&\omega_2&\omega_3\\ \theta_1\omega_1&\theta_2\omega_2&\theta_3\omega_3\\ \theta_1^2\omega_1&\theta_2^2\omega_2&\theta_3^2\omega_3\end{vmatrix}=\begin{vmatrix}1&1&1\\ \theta_1&\theta_2&\theta_3\\ \theta_1^2&\theta_2^2&\theta_3^2\end{vmatrix}\omega_1\omega_2\omega_3=\Delta(\mathcal{A})\,\omega_1\omega_2\omega_3\,. \tag{47}$$

Express now $\omega_1\omega_2\omega_3$ in terms of H_k, using formulae (16):

$$\omega_k^2=\frac{1}{\Pi_k}\,P_k(H)\quad\Rightarrow\quad\omega_1^2\omega_2^2\omega_3^2=\frac{1}{\Pi_1\Pi_2\Pi_3}\,P_1(H)\,P_2(H)\,P_3(H)\,. \tag{48}$$

According to definition (18), $\Pi_1\Pi_2\Pi_3=-\Delta^2(\mathcal{A})$. Substitution of (48) in (47) gives the desired formula. □

D Vector $\mathbf{u}_2$ in terms of $\mathbf{u}_0$ and $\mathbf{u}_1$

Consider eq. (47). Since $\Delta(\mathcal{A})=(\theta_1-\theta_2)\,(\theta_2-\theta_3)\,(\theta_3-\theta_1)$, for a nonsymmetric rigid body the mixed product $\mathbf{u}_0\cdot(\mathbf{u}_1\times\mathbf{u}_2)$ is non zero and the dynamic vector variables $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$ are linearly independent (apart from some singular values of the angular velocity). However, *vector* $\mathbf{u}_2$ *can be expressed in terms of* $\mathbf{u}_0$ *and* $\mathbf{u}_1$ *in nonlinear way.* Let us show that. The scalar dynamic variables H_k can be expressed in terms of the vectors $\mathbf{u}_0$ and $\mathbf{u}_1$ by formulae (26). The scalar products

$$\mathbf{u}_2\cdot\mathbf{u}_0=H_2\,,\qquad \mathbf{u}_2\cdot\mathbf{u}_1=H_3\,,\qquad \mathbf{u}_2\cdot\mathbf{u}_2=H_4 \tag{49}$$

result in scalar variables H_k and therefore they can be expressed in terms of the vectors $\mathbf{u}_0$, $\mathbf{u}_1$ as well. But this means that the vector $\mathbf{u}_2$ can be expressed in terms of $\mathbf{u}_0$, $\mathbf{u}_1$ (except, maybe, of some singular cases). In fact: if the vectors $\mathbf{u}_0$, $\mathbf{u}_1$ are known, then from (49) the modulus of $\mathbf{u}_2$ is known and the angles between $\mathbf{u}_2$ and $\mathbf{u}_0$ and between $\mathbf{u}_2$ and $\mathbf{u}_1$ are known. But this is sufficient for the determination of $\mathbf{u}_2$. This is shown strictly as follows:

Proof: For any three vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ the following identity holds:

$$\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)\,\mathbf{E} = \mathbf{a}_2 \times \mathbf{a}_3\mathbf{a}_1 + \mathbf{a}_3 \times \mathbf{a}_1\mathbf{a}_2 + \mathbf{a}_1 \times \mathbf{a}_2\mathbf{a}_3\,. \tag{50}$$

This identity represents a decomposition of the unit tensor $\mathbf{E}$ in terms of the nonorthogonal basis $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ [2]. It can be shown that in case $\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3) = 0$ this identity is also correct. For the vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$ the above identity takes the form

$$\pm\sqrt{\varDelta(H)}\,\mathbf{E} = \mathbf{u}_1 \times \mathbf{u}_2\mathbf{u}_0 + \mathbf{u}_2 \times \mathbf{u}_0\mathbf{u}_1 + \mathbf{u}_0 \times \mathbf{u}_1\mathbf{u}_2\,,$$

where identity (46) is used to represent the mixed product of $\mathbf{u}_k$. Multiply scalarly this relation by the vector $(\mathbf{u}_0 \times \mathbf{u}_1)$. After reducing the double vector products we obtain

$$\pm\sqrt{\varDelta(H)}\,\mathbf{u}_0 \times \mathbf{u}_1 = (H_1H_3 - H_2^2)\,\mathbf{u}_0 - (H_0H_3 - H_1H_2)\,\mathbf{u}_1 + (H_0H_2 - H_1^2)\,\mathbf{u}_2\,. \tag{51}$$

The above relation gives the desired constraint between the principal vectors. The coefficient of $\mathbf{u}_2$ can be represented in the form $H_0H_2 - H_1^2 = (\theta_1 - \theta_2)^2\,\omega_1^2\omega_2^2 + (\theta_2 - \theta_3)^2\,\omega_2^2\omega_3^2 + (\theta_3 - \theta_1)^2\,\omega_3^2\omega_1^2$, hence it can be zero only for spherical inertia tensor or zero angular velocity. For any other case vector $\mathbf{u}_2$ can be expressed from (51) in terms $\mathbf{u}_0$ and $\mathbf{u}_1$ as following:

$$\mathbf{u}_2 = [(H_0H_3 - H_1H_2)\,\mathbf{u}_1 - (H_1H_3 - H_2^2)\,\mathbf{u}_0 \pm \sqrt{\varDelta(H)}\,\mathbf{u}_0 \times \mathbf{u}_1]/(H_0H_2 - H_1^2)\,. \tag{52}$$

The sign "±" in the above formula should be equal to the sign of the mixed product $\mathbf{u}_0 \cdot (\mathbf{u}_1 \times \mathbf{u}_2)$ in the initial conditions. The sign can change only if this mixed product vanishes. □

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