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Sloshing problem in a half-plane covered by a dock with two gaps: monotonicity and asymptotics of eigenvalues

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Abstract

The two-dimensional sloshing problem is considered in a half-plane covered by a rigid dock with two symmetric gaps. It is proved that the antisymmetric (symmetric) sloshing eigenvalues are monotonically decreasing (increasing) functions of spacing between gaps and formulae for their derivatives are obtained.

sloshing problem / eigenvalue / eigenfunction / integral operator / asymptotic formula

Résumé. Les oscillations libres d'une fluide dans un demi-plan sous un couvercle rigid avec deux ouvertures: la monotonie des valeurs propres

oscillations libres d'une fluide / valeur propre / fonction propre / opérateur intégral / formule asymptotique

1 Introduction

Sloshing problem in a half-plane covered by a rigid dock with a single gap has received much consideration (see [2] and references cited therein) because eigenvalues of this problem furnish universal upper bounds for sloshing frequencies in the two-dimensional domains having the same free surface. The aim of the present note is to consider the problem for a dock with two symmetric gaps and to establish that the corresponding eigenvalues are monotonic functions of spacing between gaps. Some other properties of eigenvalues are also obtained.

Let an inviscid, incompressible, heavy fluid occupy the half-plane $y < 0$ and be covered by a rigid dock so that the free surface consists of two gaps $\{b < |x| < b + 1,\ y = 0\}$ (it is convenient to use non-dimensional Cartesian coordinates such that each gap has a unit length). Neglecting the surface tension, we consider free, small-amplitude, time-harmonic oscillations of fluid and its motion is assumed to be irrotational. Since the fluid domain is symmetric about the y-axis, sloshing modes are either *symmetric* or *antisymmetric*, that is, are even or odd functions of x respectively, and so we restrict our considerations to the quadrant $\{x \geqslant 0,\ y \leqslant 0\}$.

2 Antisymmetric modes

In the present section, we are concerned only with antisymmetric modes, and the corresponding boundary value problem for a time-independent velocity potential $u^{(-)}(x, y)$ (without loss of generality, $u^{(-)}$ can be assumed to be a real function) is as follows:

$$\nabla^2 u^{(-)} = 0, \qquad x > 0,\ y < 0, \tag{1}$$

$$u^{(-)} = 0, \qquad x = 0,\ y < 0, \tag{2}$$

$$u^{(-)}_y = 0, \qquad 0 < x < b,\ x > b + 1,\ y = 0, \tag{3}$$

$$u^{(-)}_y - \nu^{(-)} u^{(-)} = 0, \qquad b < x < b + 1,\ y = 0. \tag{4}$$

Our aim is to investigate properties of eigenvalues $\nu^{(-)}(b)$ and eigenfunctions $u^{(-)}(x, y; b)$ (sloshing modes) as functions of b, but, unless it is necessary, we do not indicate this dependence for the sake of brevity. Solutions to (1)–(4) are sought in the natural class of functions having finite kinetic energy, that is,

$$\int_{-\infty}^{0} \int_{0}^{+\infty} |\nabla u^{(-)}|^2 \, dx \, dy < \infty. \tag{5}$$

This condition provides that $u^{(-)}$ is continuous up to the x-axis and $\nabla u^{(-)}$ has a logarithmic singularity at the dock tips (an asymptotic formula representing $u^{(-)}$ near the tips can be obtained using methods described in [4]).

PROPOSITION 1. – *By virtue of*

$$u^{(-)}(x, y) = \frac{1}{2\pi} \int_{b}^{b+1} w^{(-)}(\xi - b) \log \frac{(x+\xi)^2 + y^2}{(x-\xi)^2 + y^2} \, d\xi, \tag{6}$$

(1)–(5) *is equivalent to the following spectral problem:*

$$w^{(-)}(x) = \frac{\nu^{(-)}}{\pi} \int_{0}^{1} \Big[\log(x + \xi + 2b) - \log|x - \xi|\Big] \, w^{(-)}(\xi) \, d\xi, \quad x \in (0, 1), \tag{7}$$

where the integral operator $\boldsymbol{K}^{(-)}$ in the right-hand side is a compact, selfadjoint, positive operator in $L_2(0, 1)$. The null-space of $\boldsymbol{K}^{(-)}$ is trivial.

In order to prove the equivalence of (1)–(5) and (7), one has to use properties of the single layer potential (6). The triviality of the null-space follows from the inversion formula obtained in [3].

Applying known results for weakly singular, selfadjoint, positive, integral operators including Jentsch's theorem (see, for example, [8], sect. 20), one immediately arrives at the following

COROLLARY 1. – *For a fixed $b \geqslant 0$, there exists a sequence of eigenvalues*

$$0 < \nu_1^{(-)} < \nu_2^{(-)} \leqslant \ldots \leqslant \nu_n^{(-)} \leqslant \ldots \quad \textit{such that} \quad \nu_n^{(-)} \to \infty \quad \text{as} \quad n \to \infty;$$

the eigenvalue $\nu_1^{(-)}$ is simple and the corresponding eigenfunction $w_1^{(-)}(x)$ is continuous and non-negative in $[0, 1]$.

Further results on the simplicity of eigenvalues are formulated in Proposition 7. Since $\boldsymbol{K}^{(-)}$ depends on $b \geqslant 0$ continuously and the kernel of $\boldsymbol{K}^{(-)}$ is a monotonically increasing function of b, the known results (see, for example, [7], sect. 95) lead to the following

PROPOSITION 2. – *For each $n = 1, 2, \ldots$, either $\nu_n^{(-)}$ and $w_n^{(-)}(x)$ are continuous functions of $b \geqslant 0$, and $\nu_n^{(-)}$ decreases with b.*

Quantitatively, the rate of decreasing is characterized as follows.

PROPOSITION 3. – *For each $n = 1, 2, \ldots$, the identity*

$$\frac{d\nu_n^{(-)}}{db} = - \int_{-\infty}^{0} \Big|\partial_x \, u_n^{(-)}(0, y)\Big|^2 \, dy \Big/ \int_{b}^{b+1} \Big|u_n^{(-)}(x, 0)\Big|^2 \, dx, \quad b > 0.$$

holds.

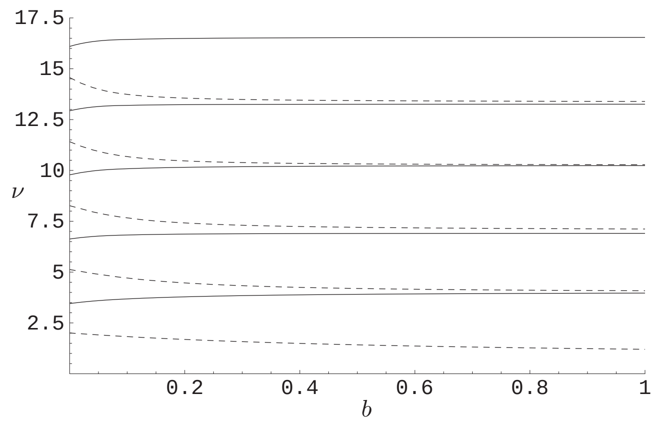

Figure 1. $\nu_1^{(-)}, \ldots, \nu_5^{(-)}$ (dashed curves) and $\nu_1^{(+)}, \ldots, \nu_5^{(+)}$ (solid curves) are plotted as functions of $b \in [0,1]$.

Figure 1. $\nu_1^{(-)}, \ldots, \nu_5^{(-)}$ (courbes traits) et $\nu_1^{(+)}, \ldots, \nu_5^{(+)}$ (courbes continues) en fonctions de $b \in [0,1]$.

The proof of this proposition is similar to that of Proposition 6 below. More information can be obtained for large values of b.

Proposition 4. – *(i) As $b \to \infty$, the asymptotic formulae are valid:*

$$\frac{\pi}{\nu_1^{(-)}} = \log(2b) + \frac{3}{2} + O\left(1/\log b\right),$$

$$1 - w_1^{(-)}(x) = \frac{\nu_1^{(-)}}{\pi}\left[\frac{3}{2} + \int_0^1 \log|x-\xi|\,\mathrm{d}\xi\right] + O\left(1/\log^2 b\right).$$

Here the eigensolution $w_1^{(-)} \geqslant 0$ is normalized so that $\int_0^1 w_1^{(-)}(x)\,\mathrm{d}x = 1$, and the formula holds uniformly in $[0,1]$.

(ii) The integral operator $\boldsymbol{K}^{(-)}$, acting in the subspace of $L_2(0,1)$ orthogonal to $w_1^{(-)}$, tends strongly to $\boldsymbol{K}_\infty^{(-)}$ as $b \to \infty$, and the latter operator has $-\pi^{-1}\log|x-\xi|$ as its kernel.

Numerical computations show that the eigenvalues of $\boldsymbol{K}_\infty^{(-)}$ corresponding to the eigenfunctions orthogonal to constants are equal to the antisymmetric sloshing eigenvalues in a half-plane covered by a dock with a single gap of unit length (the latter eigenvalues are given in [2]). This means that if the spacing between gaps is sufficiently large, then fluid oscillations in each gap take place as if there is no the other gap. Fig. 1 shows that for $n = 2, 4$ the values $\nu_n^{(-)}(b)$ obtained from (7) (they are shown by dashed lines) are sufficiently close to the described limit values even for $b = 1$.

3 Symmetric modes

A real velocity potential $u^{(+)}(x,y)$ of symmetric sloshing modes satisfies the same conditions (1) and (3)–(5) as $u^{(-)}(x,y)$, and the spectral parameter $\nu^{(+)}$ replaces $\nu^{(-)}$ in (4), but

$$u_x^{(+)} = 0, \quad x = 0, \ y < 0, \tag{8}$$

must be imposed instead of (2). There is a trivial symmetric sloshing mode $u_0^{(+)}(x,y) \equiv 1$ corresponding to $\nu_0^{(+)} = 0$. It follows from Green's formula that the symmetric eigensolutions satisfy

$$\int_b^{b+1} u^{(+)}(x,0)\,\mathrm{d}x = 0. \tag{9}$$

The following representation:

$$u^{(+)}(x,y) = -\frac{1}{2\pi}\int_b^{b+1} w^{(+)}(\xi - b)\,\log\Big[\big((x+\xi)^2+y^2\big)\big((x-\xi)^2+y^2\big)\Big]\,\mathrm{d}\xi, \tag{10}$$

which is similar to (6), leads to the spectral problem:

$$w^{(+)}(x) = -\nu^{(+)}\pi^{-1}\int_0^1 \Big[\log(x+\xi+2b) + \log|x-\xi|\Big]\; w^{(+)}(\xi)\,\mathrm{d}\xi, \quad x\in(0,1).$$

Unfortunately, it has no solution satisfying (9), and so $u^{(+)}$ obtained from (10) violates the condition of energy finiteness (5). Nevertheless, a spectral problem involving an integral operator with more complicated kernel can be obtained in the present case. The starting point is the function

$$W(z;\xi) = -\frac{1}{\pi}\left\{\log\frac{4(z-\xi)}{(1-2z)(1-2\xi)} - \frac{1+2z}{2}\log\left(\frac{1+2z}{1-2z}\right) - \frac{1+2\xi}{2}\log\left(\frac{1+2\xi}{1-2\xi}\right) + \frac{1}{2} - \pi\mathrm{i}(z+\xi)\right\}, \quad z = x+\mathrm{i}y,$$

derived in [1], where the kernel $\operatorname{Re} W(x,\xi)$ appeared in the integral equation equivalent to sloshing problem and providing the non-trivial symmetric modes for the dock with the single gap

$$\{-1/2 < x < +1/2,\ y=0\}.$$

Let

$$\begin{aligned} G(x,y;\xi) = 2^{-1}\operatorname{Re}\big\{ & W(z+b+1/2;\xi+b+1/2) + W(z-b-1/2;\xi-b-1/2) \\ & + W(z+b+1/2;-\xi+b+1/2) + W(z-b-1/2;-\xi-b-1/2) \\ & - \pi^{-1}\big[2b^2\log(2b) + 2(1+b^2)\log[2(1+b)] - (1+2b)^2\log(1+2b)\big]\big\} \end{aligned}$$

and considering G as a function of (x,y) one immediately verifies that (1), (3), (5) and (8) hold. Besides, we have $\int_b^{b+1} G(x,0;\xi)\,\mathrm{d}\xi = 0$.

PROPOSITION 5. – *By virtue of*

$$u^{(+)}(x,y) = \int_b^{b+1} w^{(+)}(\xi-b)\; G(x,y;\xi)\,\mathrm{d}\xi,$$

sloshing problem for non-trivial symmetric modes is equivalent to the following spectral problem:

$$w^{(+)}(x) = \nu^{(+)}\int_0^1 G(x+b,0;\xi+b)\; w^{(+)}(\xi)\,\mathrm{d}\xi, \quad x\in(0,1), \tag{11}$$

where the integral operator $\boldsymbol{K}^{(+)}$ in the right-hand side is a compact, selfadjoint operator in the subspace of $\mathrm{L}_2(0,1)$ consisting of functions which satisfy (9).

As in sect. 2, this proposition yields

Corollary 2. – *There exists a sequence of eigenvalues*

$$0 < \nu_1^{(+)} \leqslant \nu_2^{(+)} \leqslant \ldots \leqslant \nu_n^{(+)} \leqslant \ldots \quad \textit{such that} \quad \nu_n^{(+)} \to \infty \quad \text{as} \quad n \to \infty;$$

for each $n = 1, 2, \ldots$, *either* $\nu_n^{(+)}$ *and* $w_n^{(+)}(x)$ *are continuous functions of* $b \geqslant 0$ *and these functions are analytic for* $b > 0$.

The main result of the present section is as follows.

Proposition 6. – *The identity*

$$\frac{\mathrm{d}\nu_n^{(+)}}{\mathrm{d}b} = \int_{-\infty}^{0} \left|\partial_y\, u_n^{(+)}(0, y)\right|^2 \mathrm{d}y \Bigg/ \int_b^{b+1} \left|u_n^{(+)}(x, 0)\right|^2 \mathrm{d}x, \quad b > 0, \tag{12}$$

holds for each $n = 1, 2, \ldots$, *and so* $\nu_n^{(+)}$ *is a monotonically increasing function of* $b \geqslant 0$.

Proof. – Let $u_n^{(+)}(x, y; b)$ be a symmetric eigenmode corresponding to the sloshing eigenvalue $\nu_n^{(+)}(b)$. Proposition 5 implies that $\nu_n^{(+)}(b)$ is a differentiable function of $b > 0$. Let Δ be a sufficiently small number (such that $b + \Delta > 0$). After extending $u_n^{(+)}(x, y; b + \Delta)$ to the whole half-plane $y < 0$ by means of the Schwarz Reflection Principle, we consider $u_n^{(+)}(x + \Delta, y; b + \Delta)$ defined in the closed quadrant $\{x \geqslant 0,\ y \leqslant 0\}$ even when $\Delta < 0$. The latter function satisfies the similar boundary conditions as $u_n^{(+)}(x, y; b)$ on

$$\{0 < x < b,\ y = 0\}, \quad \{b < x < b + 1,\ y = 0\} \quad \text{and} \quad \{b + 1 < x < +\infty,\ y = 0\}$$

respectively. Let us apply the second Green's formula to $u_n^{(+)}(x, y; b)$ and $u_n^{(+)}(x + \Delta, y; b + \Delta)$ in $\{x > 0,\ y < 0\}$. This gives

$$\int_b^{b+1} \left[u_n^{(+)}(x, 0; b)\, \partial_y\, u_n^{(+)}(x + \Delta, 0; b + \Delta) - u_n^{(+)}(x + \Delta, 0; b + \Delta)\, \partial_y\, u_n^{(+)}(x, 0; b)\right] \mathrm{d}x$$
$$= \int_{-\infty}^{0} \left[u_n^{(+)}(0, y; b)\, \partial_x\, u_n^{(+)}(\Delta, y; b + \Delta) - u_n^{(+)}(\Delta, y; b + \Delta)\, \partial_x\, u_n^{(+)}(0, y; b)\right] \mathrm{d}y$$

because (5) guarantees that the integral over a large quarter-circle tends to zero as its radius goes to infinity; the homogeneous Neumann condition on the dock is also applied here. Using (8) and the Lagrange theorem in the second integral, and the free surface conditions in the first one, we get

$$\left[\nu_n^{(+)}(b + \Delta) - \nu_n^{(+)}(b)\right] \int_b^{b+1} u_n^{(+)}(x, 0; b)\ u_n^{(+)}(x + \Delta, 0; b + \Delta)\, \mathrm{d}x$$
$$= \Delta \int_{-\infty}^{0} u_n^{(+)}(0, y; b)\, \partial_x^2\, u_n^{(+)}(\theta(y)\Delta, y; b + \Delta)\, \mathrm{d}y,$$

where $0 < \theta(y) < 1$ for $y \in (-\infty, 0)$. Letting $\Delta \to 0$ in this equation divided by Δ produces

$$\frac{\mathrm{d}\nu_n^{(+)}}{\mathrm{d}b} \int_b^{b+1} \left|u_n^{(+)}(x, 0; b)\right|^2 \mathrm{d}x = \int_{-\infty}^{0} u_n^{(+)}(0, y; b)\, \partial_x^2\, u_n^{(+)}(0, y; b)\, \mathrm{d}y.$$

In order to obtain (12), it remains to transform the last integral using the Laplace equation and then applying integration by parts. The out of integral terms vanish because $\partial_y\, u_n^{(+)}(0, y; b)$ satisfies the no flow condition on the dock and decays at infinity.

As in the antisymmetric case, numerical computations show that for large b the value of $\nu_n^{(+)}(b)$ obtained from (11) asymptotes the n-th sloshing eigenvalue in a half-plane covered by a dock with a single gap of unit length. Fig. 1 shows that for $n = 1, 3, 5$ the values $\nu_n^{(+)}(b)$ (they are shown by solid lines) are sufficiently close to the described limit values even for $b = 1$.

We conclude the note with the following result.

PROPOSITION 7. – *All symmetric eigenvalues are simple for any $b \geqslant 0$. For antisymmetric eigenvalues this property holds at least for $b = 0$ and sufficiently small positive b.*

To the authors' knowledge, there are only two papers treating the question of simplicity of the sloshing eigenvalues. In [5], it is demonstrated that the first eigenvalue is simple, and a condition guaranteeing that the second eigenvalue is simple is obtained in [6].

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