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Trapped modes in the linearized problem of a potential flow about semisubmerged bodies

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Introduction

A forward stationary motion of surface-piercing bodies in an ideal, incompressible, heavy fluid of infinite depth is considered. In the framework of the linear theory of surface waves the motion is described by the so-called Neumann-Kelvin problem for a velocity potential. In [3] and [10] this problem was proved to be well-posed in the case of totally submerged bodies. At the same time, for surface-piercing bodies the Neumann–Kelvin problem is known to have a family of solutions depending on $2N$ parameters, where N is the number of bodies (see e.g. [9]). Thus, the initial set of equations should to be augmented by supplementary conditions. Some versions of such conditions were proposed in [9], [6], [4] and [5]. We use the statement of the paper [9] delivering the so-called "least singular" solution. The velocity field of such potential is bounded in the corner points, where the contour of body meets the free surface of fluid.

In [4] the "least singular" statement was proved to be uniquely solvable for all values of the forward velocity with possible exception for a sequence tending to zero. Our purpose is to demonstrate that these irregular values do exist. Following [7] where a non-uniqueness example was obtain for the 2D sea-keeping problem, we use the so-called inverse procedure for simultaneous construction of surface-piercing bodies and of the potential of mode with finite energy, trapped by these bodies. Namely, we fix a value of the forward velocity and construct a potential as a sum of source and sink of the same strength placed in the free surface and separated by a distance of some wavelengths. The special choice of distance guarantees absence of waves at infinity for the potential and, hence, the finiteness of energy. Investigation of potential's streamlines shows that there is a pair of them with the both ends in the free surface and enclosing sources inside. These streamlines are interpreted as body contours. Thus, a geometry is obtained for which the homogeneous problem has a nontrivial solution. The trapped modes are supposed to be related to resonances in some more general initial value problem (see [8]).

In [7] example was only constructed with two bodies. Indeed, more that two homotopical families of contours can be obtained in enlarging the distance between singularities. We prove that for any fixed value of forward velocity and a caravan, consisting of two and more bodies, the geometry of these bodies can be fixed so that the homogeneous Neumann–Kelvin problem has a non-trivial solution with finite energy. Proof of the assertion is based on description of the streamlines of level zero, which separate homotopical families of streamlines, and on a relationship connecting values of the stream function in the fluid and in the free surface at fixed horizontal coordinate.

1. Statement of the problem

We describe the boundary value problem for one semisubmerged body. The notations are introduced in fig. 1 where W is the domain occupied by fluid, $F = F_+ \cup F_-$ is the free surface of fluid, D and S are the cross-section and the wetted surface of the cylinder. Also, we denote by U and by g the constant forward speed of the body and the gravity acceleration respectively. It is assumed that the contours S are not tangent to the free surface, i.e. $\beta_\pm \neq 0, \pi$. The unit normal vector n is directed into the domain occupied by fluid. From now on we use the dimensionless coordinate $x = Xg/U^2$ and $y = Yg/U^2$.

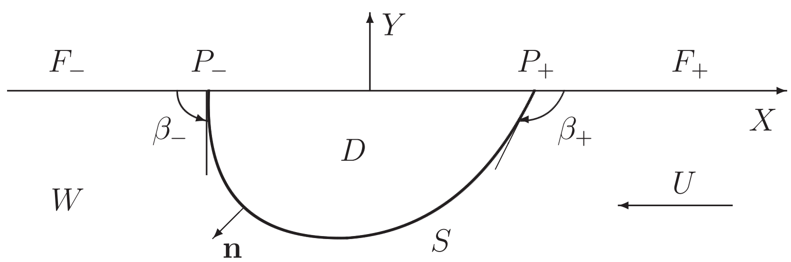

Fig. 1

In a coordinate system attached to the body the Neumann–Kelvin problem for the velocity potential $u \in H^1_{loc}(W)$ is as follows:

$$\nabla^2 u = 0 \quad \text{in} \quad W \tag{1.1}$$

$$u_{xx} + u_y = 0 \quad \text{on} \quad F \setminus \{P_+, P_-\} \tag{1.2}$$

$$\partial u/\partial n = f \quad \text{on} \quad S \setminus \{P_+, P_-\} \tag{1.3}$$

$$\lim_{x \to +\infty} |\nabla u| = 0 \tag{1.4}$$

$$\sup\{|\nabla u| : (x, y) \in W \setminus E\} < \infty \tag{1.5}$$

where E is a compact set such that $\overline{D} \subset E$ and $E \cap (F_\pm \setminus \{P_\pm\}) \neq \emptyset$.

The problem (1.1)–(1.5) is known to have two-parameter family of solutions and should be augmented by two supplementary conditions. For this purpose we use the asymptotics of a solution to the problem near the corner points $P_\pm$. This asymptotics was established in [4]. Introduce the polar coordinate $(\rho_\pm, \theta_\pm)$ with origin in the point $P_\pm$ and polar axes

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directed along the ray $F_\pm$. The angle θ_+ (θ_-) is measured clockwise (counterclockwise) so that $0 \le \theta_\pm \le \beta_\pm$. Then, when $\rho_\pm \to 0$ we have

$$u = C_\pm + B_\pm \rho_\pm^{\pi/(2\beta_\pm)} \sin\left(\pi\theta_\pm/(2\beta_\pm)\right) + +A_\pm \rho_\pm \cos(\theta_\pm - \alpha_\pm) + O\big(\rho_\pm^{\lambda_\pm}\big), \quad \text{when} \quad \beta_\pm \ne \pi/2$$

$$u = C_\pm + B_\pm \left[\rho_\pm \log \rho_\pm \sin\theta_\pm + \rho_\pm(\theta_\pm - \pi/2)\cos\theta_\pm\right] + \\ + A_\pm \rho_\pm \cos(\theta_\pm - \alpha_\pm) + O\big(\rho_\pm^{\lambda_\pm}\big), \quad \text{when} \quad \beta_\pm = \pi/2 \tag{1.6}$$

Here $\alpha_\pm$ and $\lambda_\pm$ are constants, $1 < \lambda_\pm < 2$ when $\beta_\pm \ge \pi/2$ and $\lambda_\pm = 2$ when $\beta_\pm < \pi/2$.

In [9] F. Ursell introduced for a semicircle a statement which delivers the so-called "least singular" solution with bounded in the corner points vector field. This statement was considered in [4] for arbitrary contours with $\beta_\pm \ge \pi/2$ when, as it follows from (1.6), the vector field is allowed to have singularities. Here we generalize the least singular statement so that it works for the contours with $\beta_+ < \pi/2$ or $\beta_- < \pi/2$ as well.

Definition 1**.** We speak that a potential u is "least singular", if it satisfies (1.1)–(1.5) and is submitted to the following conditions

$$B_+ = B_- = 0, \tag{1.7}$$

where $B_\pm$ are the coefficients in (1.6).

This statement is naturally defined for any number of semisubmerged bodies. Note that the results of the paper [4] concerning the "least singular" statement are true for the solution introduced in the last definition.

In the work we shall construct trapped modes of the problem under consideration. For this purpose we use the source function of the problem (1.1)–(1.5). The Green function $G(x, y; \xi, \eta)$ satisfying conditions (1.2), (1.4) and submitted to the source equation

$$\nabla_{x,y} G(x, y; \xi, \eta) = -\delta(x - \xi, y - \eta) \quad \text{when} \quad y < 0,\ \eta < 0$$

and to the condition

$$\limsup_{|x+iy|\to\infty} |\nabla_{x,y} G| < \infty$$

can be written as follows

$$G(z; \zeta) = -(2\pi)^{-1} \operatorname{Re}\Big\{\log\big((z-\zeta)(z-\overline{\zeta})\big) - 2\,\mathrm{e}^{-i(z-\overline{\zeta})}\big[\mathrm{Ei}\big(i(z-\overline{\zeta})\big) - i\pi\big]\Big\} \tag{1.8}$$

Here $z = x + iy$, $\zeta = \xi + i\eta$ and $\mathrm{Ei}(z)$ is the exponential integral.

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For our purposes it is sufficient to consider the case when the source is situated in the free surface. Introduce a stream function of the source $H(z;\zeta)$ which is complex conjugated to the Green function with respect to z. By 8.212.5 in [2] we write:

$$H(x,y;\xi,0) = -\pi^{-1}\arg(z-\xi) + \pi^{-1}\,\text{v.p.}\int_0^\infty \frac{\mathrm{e}^{ky}\sin k(x-\xi)}{k-1}\,\mathrm{d}k - \mathrm{e}^{y}\cos(x-\xi) \qquad (1.9)$$

where $\arg(z)\in[-\pi,0]$ when $y\le 0$. The relationship

$$\arg(z) = -\frac{\pi}{2} + \int_0^\infty \frac{\mathrm{e}^{ky}\sin kx}{k}\,\mathrm{d}k, \quad y\le 0$$

follows from 3.941.1 in [2]. The latter leads to another representation

$$H(x,y;\xi,0) = \pi^{-1}\left(\text{v.p.}\int_0^\infty \frac{\mathrm{e}^{ky}\sin k(x-\xi)}{k(k-1)}\,\mathrm{d}k + \frac{\pi}{2}\right) - \mathrm{e}^{y}\cos(x-\xi) \qquad (1.10)$$

In view of the asymptotics $\mathrm{Ei}(z)$ as $z\to 0$ (see. 5.1.10 in [1]) it is easy to see that the function $H(x,y;\xi,0)$ is continuous in $(\xi,0)$.

2. Non-uniqueness examples

Consider a family of potentials u_n which are combinations of source and sink located in the free surface of fluid at distance of some wavelength. Let the corresponding stream function be defined as follows

$$v_n(x,y) = \pi H(x,y;\pi n,0) - \pi H(x,y;-\pi n,0) \qquad (2.11)$$

It is to note that the stream functions are even functions of x.

Remark 1. When $|z|\to\infty$ and $|\zeta|<C<\infty$ we have (see e.g. [4])

$$G(z;\zeta) = -\pi^{-1}\log|z| - \vartheta(-x)2\,\mathrm{e}^{y+\eta}\sin(x-\xi) + \mathrm{O}(|z|^{-1})$$

where ϑ is the Heaviside function. Thus, the functions u_n are defined in such a way that they do not have the logarithmic and the wave components in asymptotics at infinity downstream.

Denote by $\mathcal{R}_0$ the set of the function $v_n(x,y)$ streamlines whose both endpoints are located in the free surface. Let the all streamlines be parameterized by $t\in[0,1]$. Consider the set $\mathcal{R}_1 = \big\{\big(x(t),y(t)\big)\in\mathcal{R}_0 : \exists\big(x_i(t),y_i(t)\big)\in\mathcal{R}_0, \big[x_i(0),x_i(1)\big]\subset\big(x(0),x(1)\big), i=1,2, \big(x_1(0),x_1(1)\big)\cap\big(x_2(0),x_2(1)\big)=\emptyset\big\}$, defined as the set of streamlines enclosing more than one family of streamlines.

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We introduce $\mathcal{R} = \mathcal{R}_0 \setminus \mathcal{R}_1$ and the homotopical equivalence $\rho \subset \mathcal{R} \times \mathcal{R}$. We speak that $\gamma(t)$ and $\gamma'(t)$ are homotopical, i.e. $(\gamma, \gamma') \in \rho$, if there exists a function $\Phi(t,s)$ such that: $\Phi(t,s)$ is continuous for $t \in [0,1]$ and $s \in [0,1]$; $\Phi(t,s) \in \mathcal{R}$ for all $s \in [0,1]$; $\Phi(t,0) = \gamma(t)$, $\Phi(t,1) = \gamma'(t)$.

In the Section 4 we shall prove the following assertion

Theorem 1. The number of elements of the factor-set $\mathcal{R}/\rho$ is equal to $2n+1$.

Let T_1 and T_2 be the sets of all streamlines enclosing the sources. Below we shall prove that $T_1 \neq \emptyset$ and $T_2 \neq \emptyset$. Denote by Q_1, Q_2 and T_i $(3 \leq i \leq 2n+1)$ sets of homotopically equivalent contours so that $Q_1 \supset T_1$ and $Q_2 \supset T_2$. Then, Theorem 1 states that $\mathcal{R}/\rho = \{Q_1, Q_2, T_3 \ldots, T_{2n+1}\}$. We fix some contours $S = \gamma_1 \cup \gamma_2 \cup \ldots \gamma_N$ where $2 \leq N \leq 2n+1$ and $\gamma_i \in T_i$. It is easy to see that if the contours S are fixed as contours of bodies in the problem (1.1)–(1.5), (1.7), then the corresponding potentials u_n are solutions for the problem with homogeneous condition (1.3).

3. Properties of the functions $\mathbf{v_n}$

Comparing the presentations (1.9) and (1.10), we get the differential equations $\partial v_n/\partial y = v_n - V_n$. Thus, at fixed x we have

$$v_n(x,y) = \mathrm{e}^y \left(v_n(x,0) + \int_y^0 V_n(x,t)\,\mathrm{e}^{-t}\,\mathrm{d}t \right) \tag{3.12}$$

Here

$$V_n(x,y) = \arg(x + \pi n + iy) - \arg(x - \pi n + iy)$$

and, obviously,

$$0 \leq V_n(x,y) \leq \pi, \quad \text{when} \quad y < 0 \tag{3.13}$$

Hence, the integral in (3.12) is a monotonical function of y.

The formulas (2.11) and (1.9) leads to the representation

$$v_n(x,0) = \pi + \int_0^\infty \frac{\sin k(x-\pi n) - \sin k(x+\pi n)}{k-1}\,\mathrm{d}k$$

By 3.722.5 and 3.354.1 from [2], we have

$$v_n(x,0) = \pi\big(1 - 2\cos(x - \pi n)\big) + \int_0^\infty \frac{\mathrm{e}^{(x-\pi n)k} + \mathrm{e}^{-(x+\pi n)k}}{1+k^2}\,\mathrm{d}k \tag{3.14}$$

when $x \in [0, \pi n]$ and

$$v_n(x,0) = \int_0^\infty \frac{\mathrm{e}^{-(x+\pi n)k} - \mathrm{e}^{-(x-\pi n)k}}{1+k^2}\,\mathrm{d}k. \tag{3.15}$$

when $x > \pi n$.

Using the last representations and taking into account the symmetry of the functions v_n with respect to x, one can easily prove two following assertions.

Lemma 1. The function $v_n(x,0)$ has $2n$ zeros $\xi_1 < \xi_2 < \ldots < \xi_{2n}$ and $\xi_i \in (-\pi n, \pi n)$ for $i = 1, 2, \ldots, 2n$.

Lemma 2. There are $2n+1$ local extremums of the function $v_n(x,0)$ located in the points $\chi_1 < \chi_2 < \ldots < \chi_{2n+1}$ where $\chi_1 \in \big(-\pi n, \xi_1\big)$, $\chi_{2n+1} \in \big(\xi_{2n}, \pi n\big)$ and $\chi_i \in \big(\xi_{i-1}, \xi_i\big)$ $(i = 2, 3, \ldots, 2n)$.

4. Non-uniqueness examples geometry

First we note that properties of harmonical functions yield that endpoints of streamlines can not be located inside the fluid, particularly there are no isolated points $v_n(x,y) = c$. A proof of this fact can be found e.g. in [7].

Consider the lines of level zero. In view of (3.12) a solution of the equation $v_n(x^*, y) = 0$ at fixed x^* satisfies the relationship

$$v_n(x^*, 0) = -\int_y^0 V_n(x^*, t)\,\mathrm{e}^{-t}\,\mathrm{d}t \tag{4.16}$$

From the definition of V_n and the inequalities (3.13) it follows that the right-hand side of the last equation is a negative, unbounded and monotonically decreasing function of the depth $|y|$. Thus, the unique solution of (4.16) only exists when $v_n(x^*, 0) \le 0$.

From (3.15) and Lemma 1 we see that $v_n(x,0) \le 0$ only for $x \in F_i$ $(i = 1, 2, \ldots, n+1)$, where $F_1 = (-\infty, \xi_1]$, $F_{n+1} = [\xi_{2n}, +\infty)$, $F_j = [\xi_{2j}, \xi_{2j+1}]$ $(j = 1, 2, \ldots, n-1)$. Denote by $\gamma_0^{(i)}$ the lines of level zero so that $F_i = \mathrm{pr}_x\big(\gamma_0^{(i)}\big)$ $(i = 1, 2, \ldots, n+1)$. Then, the lines $\gamma_0^{(1)}$ and $\gamma_0^{(n+1)}$ do to infinity and the contours $\gamma_0^{(j)}$ $(j = 2, 3, \ldots, n-1)$ are bounded.

Obviously, any line of negative level with endpoint in F_i is confined in the contour $\gamma_0^{(i)}$ $(1 \le i \le n+1)$ and is homotopically equivalent to the latter. This fact is true in view of Lemma 2 which forbids existence of two lines of the same level with endpoints in F_i. Lines of non-zero level emanating from F_1, F_{n+1} can not go to infinity (see Remark 1) and, thus, their second endpoints are also situated in the free surface. Since $\xi_1 > -\pi n$ and $\xi_{2n} < \pi n$ then there exist contours enclosing the sources.

Consider the contour $R\,\mathrm{e}^{i\theta}$, $-\pi \le \theta \le 0$ as $R \to \infty$ and make of use (1.8) and the following asymptotic representation of $\mathrm{Ei}(z)$ as $\mathrm{Re}(z) > 0$ and $|z| \to \infty$:

$$\mathrm{Ei}(z) = i\pi\,\mathrm{sign}\,(\mathrm{Im}(z)) + \mathrm{e}^z\left\{\sum_{k=1}^{N}(k-1)!z^{-k} + \mathrm{O}\left(|z|^{-N-1}\right)\right\}$$

(see 5.1.7 and 5.1.51 in [1]). It is easy to compute that $v_n(x,y) = -2\pi n\sin(\theta)/R + \mathrm{O}\,(R^{-2})$ and, thus, the line $\gamma_0^{(1)}$ ($\gamma_0^{(n+1)}$) as $x \to -\infty$ ($x \to +\infty$) coinsides with the line $y = -1$.

By (3.12) and (3.13) we establish

$$v_n(\xi_i, y) = \mathrm{e}^y\int_y^0 V_n(\xi_i, t)\,\mathrm{e}^{-t}\,\mathrm{d}t \le \pi\,\mathrm{e}^y\left(\mathrm{e}^{-y} - 1\right) < \pi, \quad i = 1, 2, \ldots, 2n$$

At the same time, (3.14) yields that $v_n(\chi_{2j}, 0) > 3\pi$ ($j = 1, 2, \ldots, n$). Consider a line emanating from the free surface in one of the intervals which form the set $\{x : v_n(x, 0) > \pi\}$. The line has to end in the same interval of the free surface, because it can not intersect the rays $x = \xi_i$ ($i = 1, 2, \ldots, 2n$) and can not go to infinity. In view of Lemma 2 two lines of the same level can not co-exist in one of the intervals in question.

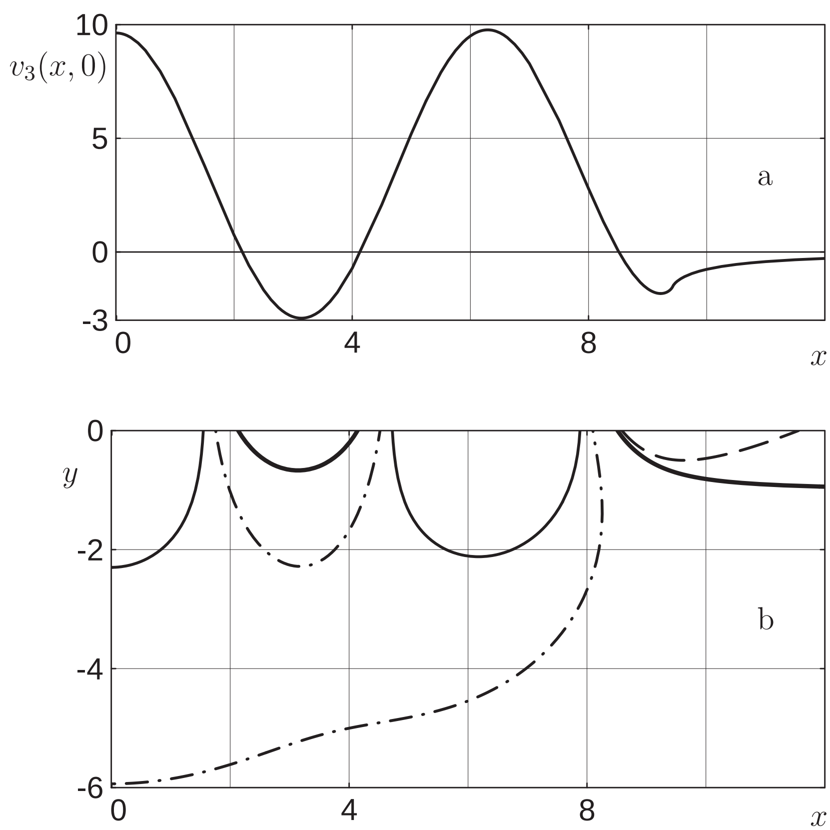

Fig. 2

Thus, we prove the assertion fo Theorem 1. Namely, the maximal number of semisubmerged bodies for which the function v_n delivers non-uniqueness example to the problem (1.1)–(1.5), (1.7) is equal to $2n+1$ and corresponds to the number of local extremums of the function $v_n(x,0)$.

Shown in fig. 2 are results of computation for $n=3$. Fig. 2(a) shows the function $v_3(x,0)$. Fig. 2(b) demonstrates streamlines $v_3(x,y)=c$, where the solid, dashed and dashed with points lines correspond to $c=3.5; -1/3$ and 2.2 respectively. The set $v_3(x,0)=0$ is plotted with bold lines. The graphs are symmetric with respect to y-axis.

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