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On Non-Uniqueness in the 2D Linear Problem of a Two-Layer Flow about Interface-Piercing Bodies

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The two-dimensional Neumann–Kelvin problem describing the steady-state forward motion of a totally submerged tandem is considered in the case when fluid consists of two superposed layers of different densities and bodies intersect the interface between them. For the so-called least singular solution examples of non-uniqueness (trapped modes) are constructed using the inverse procedure. This procedure was previously applied by McIver [7] to the problem of time-harmonic water waves and by Motygin [8] and Kuznetsov & Motygin [6] to the least singular and resistanceless statements of the Neumann–Kelvin problem involving a surface-piercing tandem in a homogenious fluid. In the situation under consideration the inverse method involves investigation of stream lines generated by two vortices placed in the interface. The spacing of vortices delivering trapped modes depends on the forward velocity.

Introduction

In the present note we consider the Neumann–Kelvin problem describing in the framework of the linearized water wave theory the steady-state two-dimensional motion of cylindrical bodies in an inviscid, incompressible fluid under gravity. The fluid consists of two superposed layers having different densities and the bodies intersect the interface between the layers. This case reveals new features in comparison with that of a body totally immersed in one of the layers considered by Motygin & Kuznetsov [9]. As for a body piercing the free surface of a homogeneous fluid (see Ursell [11] and Kuznetsov & Maz'ya [5]) the Neumann–Kelvin problem for interface-piercing bodies turns out to be under-defined and should be augmented by some supplementary conditions. Here we apply a statement leading to the "least singular" solution proposed by Ursell [11] for a circular cylinder semi-immersed in water. This statement requires the velocity field to be bounded throughout fluid. For an arbitrary body in a homogenious fluid the solvability theorem was proven for this statement by Kuznetsov & Maz'ya [5], but it holds not for all values of the forward velocity. A certain sequence of values could be excluded, and the existence of the exceptional values was demonstrated for special surface-piercing bodies by Kuznetsov & Motygin [6].

The main purpose of the note is to show that similar examples of totally submerged interface-piercing bodies do exist delivering non-uniqueness under the least singular supplementary conditions for isolated values of the forward velocity. The corresponding velocity potentials having finite energy ("trapped modes") are constructed by means of the so-called inverse procedure previously applied to the problem of time-harmonic water waves by McIver [7], and to the Neumann–Kelvin problem by Motygin [8] and Kuznetsov & Motygin [6]. This method uses the stream line pattern of a system of singularities placed on the boundary (the interface between the layers in this work) so that waves cancel at infinity pairwise. If stream lines enclosing the singularities are found, they can be interpreted as contours of bodies, and the potential represents a mode trapped by them. Since the potentials constructed here are waveless, the non-uniqueness examples work also for a statement of the Neumann–Kelvin problem prescribing amplitude and phase of waves at infinity (see Kuznetsov & Motygin [6]). A statement having no trapped modes does exist (see Klimenko [3] in this volume).

Statement of the problem

The geometric notation is given in fig. 1. The upper (lower) fluid of the density $\rho^{(+)}$ $\left(\rho^{(-)} > \rho^{(+)}\right)$ occupies domain $W^{(+)}$ $\left(W^{(-)}\right)$ which is bounded from above by the free surface $\{y = h\}$ (by the interface between the layers $\{y = 0\}$ from which two segments are removed). It is assumed that there are two immersed cylinders and the contour $S_i = \overline{S_i^{(+)}} \cup \overline{S_i^{(-)}}$ of each cylinder intersects the interface only at two points P_{2i-1}, P_{2i}, $i = 1, 2$ (the ends of segments removed from the interface). Here $S_i^{(+)}$ $\left(S_i^{(-)}\right)$ is the smooth part of contour wetted by the upper (lower) fluid. The contours are allowed to have corners at P_i so that the angles $\beta_i^{(\pm)}$ are not equal to 0 or π.

The fluid motion in the upper (lower) layer is described by a velocity potential $u^{(+)}$ $\left(u^{(-)}\right)$, which must satisfy the boundary value problem:

$$\nabla^2 u^{(\pm)} = 0 \quad \text{in} \quad W^{(\pm)}, \tag{1}$$

$$u_{xx}^{(+)} + \nu u_y^{(+)} = 0 \quad \text{when} \quad y = h, \tag{2}$$

$$u_y^{(+)} = u_y^{(-)} \quad \text{when } y = 0 \text{ outside the bodies}, \tag{3}$$

$$\rho^{(+)} \left[u_{xx}^{(+)} + \nu u_y^{(+)}\right] = \rho^{(-)} \left[u_{xx}^{(-)} + \nu u_y^{(-)}\right] \quad \text{when } y = 0 \text{ outside the bodies}, \tag{4}$$

$$\partial u^{(\pm)}/\partial n = f^{(\pm)} \text{ on } S_i^{(\pm)}, \text{ bounding } W^{(\pm)} \text{ internally}, \tag{5}$$

$$\sup_{W^{(\pm)} \setminus E} |\nabla u^{(\pm)}| < \infty, \quad \lim_{x \to +\infty} |\nabla u^{(\pm)}| = 0, \quad \int_{W \cap E} |\nabla u^{(\pm)}|^2 \, \mathrm{d}x \, \mathrm{d}y < \infty. \tag{6}$$

Here E is a compact set including an open vicinity of $\{P_1, P_2, P_3, P_4\}$, $\nu = g\, U^{-2}$, U is the constant speed of bodies, and g is the acceleration due to gravity. The functions $u^{(\pm)}$ are defined up to constant terms.

There exist two regimes of flow about the bodies (see Motygin & Kuznetsov [9]). If $\nu > \nu_*$, where

$$\nu_* = (1 + \varepsilon)/\varepsilon h, \quad \varepsilon = \rho^{(-)}/\rho^{(+)} - 1,$$

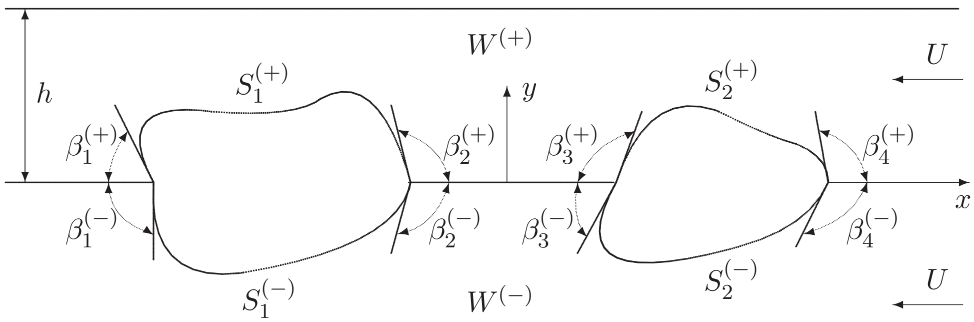

Figure 1: A sketch of geometrical notations

then there exist a superposition of waves with wavenumbers ν and ν_0 behind the body. The former (latter) are "surface" ("internal") waves manifesting themselves mainly on the free surface (interface). Here ν_0 is the only positive zero of

$$Q(k) = (1+\varepsilon)k + (k - \varepsilon\nu)\tanh kh. \tag{7}$$

existing for $\nu > \nu_*$. If $\nu < \nu_*$, then there are only surface waves at infinity downstream.

The third condition in (6) allows to avoid strong singularities at the corner points P_i. The local asymptotics near the corner point P_i (similar to that given by Kuznetsov & Maz'ya [5]) can be derived. Let (r, θ) be the polar coordinate system with origin at P_i, and $\theta \in \big(-\beta_i^{(+)}, \beta_i^{(-)}\big)$ measured from $\{y = 0\}$ clockwise (anticlockwise) at P_2, P_4 (P_1, P_3). Then, by separation of variables one finds that

$$\begin{aligned} u^{(\pm)}(r,\theta) = u^{(\pm)}(P_i) + \frac{f^{(\pm)}(P_i)}{\cos\beta_i^{(\pm)}\sin\lambda_i\beta_i^{(\pm)}} r\sin\theta + \frac{A_i}{\sin\lambda_i\beta_i^{(\pm)}} r^{\lambda_i}\cos\lambda_i(\beta_i^{(\pm)} \pm \theta) \\ + \Phi^{(\pm)}(r,\theta) + O(r^{1+\delta}), \quad \delta > 0 \end{aligned} \tag{8}$$

as $r \to 0$. Here $\lambda_i \in (0, 2)$ is the smallest positive root of

$$\rho^{(-)}\cot\lambda_i\beta_i^{(-)} = -\rho^{(+)}\cot\lambda_i\beta_i^{(+)}, \tag{9}$$

and if $\rho^{(-)}\cot\beta_i^{(-)} = -\rho^{(+)}\cot\beta_i^{(+)}$, then

$$\Phi^{(\pm)}(r,\theta) = \frac{A_i r}{\sin\lambda_i\beta_i^{(\pm)}}\left[-\log r\cos\left(\beta_i^{(\pm)} \pm \theta\right) + \frac{\beta_i^{(\pm)}\cos\theta}{\sin\beta_i^{(\pm)}} + \theta\sin\left(\theta \pm \beta_i^{(\pm)}\right)\right],$$

(hence, $\lambda_i = 1$), and $\Phi^{(\pm)}(r,\theta) = 0$ otherwise. The asymptotics (8) can be justified using results of Kondratyev [4] (see also the book [10] by Nazarov & Plamenevsky).

By (8) the velocity field can be singular at a point P_i when $\beta_i^{(+)}$ and $\beta_i^{(-)}$ satisfy

$$\rho^{(-)}\cot\beta_i^{(-)} \le -\rho^{(+)}\cot\beta_i^{(+)}. \tag{10}$$

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If (10) holds at P_i, $i = 1, 2, 3, 4$, then we introduce (following Ursell [11]) the least singular solution $u^{(+)}$ and $u^{(-)}$ satisfying (1)–(6) and having bounded derivatives throughout the fluid, that is, $A_i = 0$ in (8) $(i = 1, 2, 3, 4)$. The solvability theorem for this statement can be obtained following the scheme developed by Kuznetsov & Maz'ya [5] for a surface-piercing body in a homogeneous fluid. In that paper the existence of the least singular solution was proven for all values of ν with possible exception for a sequence tending to infinity.

Non-uniqueness examples

First we consider the supercritical regime ($\nu < \nu_*$) when there are only surface waves at infinity downstream. We define potentials $u^{(+)}$ and $u^{(-)}$ in terms of Green's function G (given by Motygin & Kuznetsov [9]) as follows:

$$u^{(\pm)} = u_0^{(\pm)}(x - \pi/\nu, y) - u_0^{(\pm)}(x + \pi/\nu, y), \quad \pm y > 0 \qquad (11)$$
$$u_0^{(\pm)} = \pi\nu^{-1}\left[G_x^{(\pm)}(z, +\mathrm{i}0) - G_x^{(\pm)}(z, -\mathrm{i}0)\right], \quad z = x + \mathrm{i}y$$

where $G^{(\pm)}(z, \zeta)$ is a potential of the source located at $\zeta \in W^{(\pm)}$ and the last combination of $G_x^{(\pm)}$ corresponds to a horizontal vortex. Using the asymptotics of Green's function derived by Motygin & Kuznetsov [9] and presented in the paper by Klimenko [3], it is easy to see that the potentials $u^{(+)}$ and $u^{(-)}$ are waveless at infinity. Further, denote by $v^{(\pm)}$ and $v_0^{(\pm)}$ stream functions corresponding to $u^{(\pm)}$ and $u_0^{(\pm)}$ respectively. From the Cauchy-Riemann equations we find that

$$v_0^{(\pm)} = \pi\nu^{-1}\left[G_y^{(\pm)}(z, -\mathrm{i}0) - G_y^{(\pm)}(z, +\mathrm{i}0)\right]. \qquad (12)$$

Then (12) and (3), which holds for $G^{(\pm)}(z, \zeta)$ as a function of z, imply

$$v^{(+)} = v^{(-)}, \quad \text{when} \quad y = 0, \quad x \neq \pm\pi/\nu. \qquad (13)$$

By (13) a stream line $v^{(+)} = c$ is extended below the interface as a stream line $v^{(-)} = c$. Further, the representation of Green's function given by Motygin & Kuznetsov [9] yields

$$v_0^{(\pm)}(x, y) = -\gamma \log|z| + I^{(\pm)}(x, y). \qquad (14)$$

Here

$$\gamma = \frac{\varepsilon}{2 + \varepsilon}, \qquad I^{(\pm)}(x, y) = \int_0^\infty \left\{ D^{(\pm)}(k, y)\cos kx + \frac{\gamma \mathrm{e}^{-k}}{k} \right\} \mathrm{d}k, \qquad (15)$$

where $D^{(+)} = D_1 \mathrm{e}^{k(y-h)} + D_2 \mathrm{e}^{-ky}$, $D^{(-)} = D_3 \mathrm{e}^{ky}$ and

$$D_1 = \frac{\varepsilon(k + \nu)}{2(k - \nu)Q(k)\cosh kh}, \quad D_2 = \frac{\varepsilon \mathrm{e}^{kh}}{2Q(k)\cosh kh} - \frac{\gamma}{k}, \quad D_3 = \frac{\varepsilon(k - \nu\tanh kh)}{(k - \nu)Q(k)} - \frac{\gamma}{k}.$$

The functions $D^{(\pm)}(k, y)$ have simple poles at $k = \nu$, and the integrals in (15) are understood as the Cauchy principal value.

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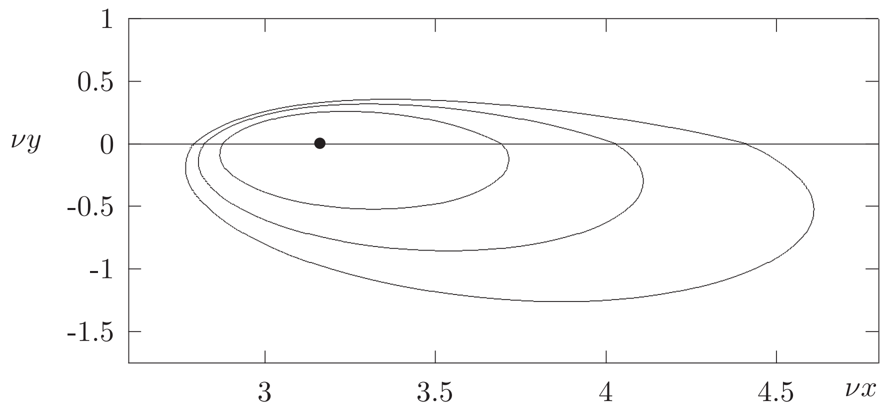

Figure 2: Stream lines $v^{(\pm)}(x, y) = 0.05,\ 0.1,\ 0.2$ for $\nu h = 1$, $\varepsilon = 0.5$, delivering examples of a forebody for a wave-trapping tandem in the supercritical regime ($\nu < \nu_*$).

c	$\beta_3^{(+)}$	$\beta_3^{(-)}$	$\beta_4^{(+)}$	$\beta_4^{(-)}$	λ_3	λ_4
0.2	108.81°	79.04°	125.60°	71.83°	0.9881	0.9572
0.1	111.69°	76.18°	137.34°	61.19°	0.9933	0.9622
0.05	113.25°	74.56°	146.38°	51.04°	0.9964	0.9701

Table 1: Values of $\beta_i^{(\pm)}$ and λ_i obtained numerically for the contours shown in fig. 2.

Since $D_i(k) = O(k^{-2})$ as $k \to +\infty$, $I^{(+)}$ $\left(I^{(-)}\right)$ converge uniformly in (x, y) belonging to the closed upper (lower) layer. Thus we see that

$$I^{(\pm)}(x, y) = O(1) \quad \text{as} \quad z \to 0. \tag{16}$$

This and the equalities (13) and (14) guarantee that stream lines $v^{(\pm)} = c$ are close to circles for sufficiently large c, and enclose the singularities placed at $(\pm\pi/\nu, 0)$. Some of stream lines enclosing the right singularity are shown in fig. 2. The pattern of stream lines is symmetric about the y-axis. Numerical scheme used for computation of $v_0^{(\pm)}$ is described in the next section.

Interpreting two stream lines surrounding $(\pm\pi/\nu, 0)$ as contours of bodies S_1 and S_2 respectively (see fig. 1), we obtain a geometry for which the potentials (11) deliver a non-trivial solution to the homogeneous problem (1)–(6). Since the derivatives of (11) have no singularities at P_i, these potentials provide a non-uniqueness example for the least singular statement if (10) holds for pairs of angles $\beta_i^{(+)}$, $\beta_i^{(-)}$ $(i = 1, 2, 3, 4)$. This means that λ_i obtained from (9) is less than one, and hence, in general, solutions to (1)–(6) admit singularities of the velocity field for contours satisfying (10).

Since (3), (4) hold for $G^{(\pm)}(z, \zeta)$ as a function of z, we get from (12)

$$\rho^{(+)}\left[v_y^{(+)} - \nu v^{(+)}\right] = \rho^{(-)}\left[v_y^{(-)} - \nu v^{(-)}\right] \quad \text{for} \quad y = 0, \quad x \neq \pm\pi/\nu.$$

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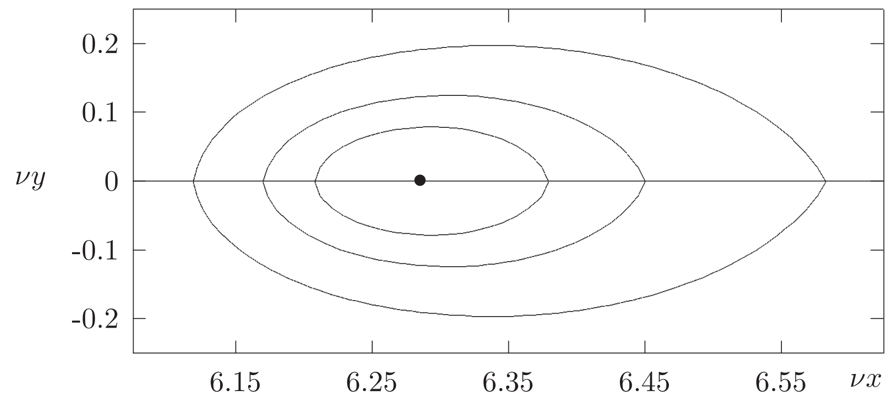

Figure 3: Stream lines $v^{(\pm)}(x, y) = 1.5, 2.0, 2.5$ for $\nu_0/\nu = 2$, $\nu h = 15$, $\varepsilon = 2$, delivering examples of a forebody for a wave-trapping tandem in the subcritical regime ($\nu > \nu_*$).

c	$\beta_3^{(+)}$	$\beta_3^{(-)}$	$\beta_4^{(+)}$	$\beta_4^{(-)}$	λ_3	λ_4
2.5	94.12°	94.12°	97.12°	97.10°	0.9562	0.9267
2.0	95.00°	94.98°	100.35°	100.65°	0.9473	0.8948
1.5	95.05°	95.04°	105.30°	105.36°	0.9468	0.8542

Table 2: Values of $\beta_i^{(\pm)}$ and λ_i obtained numerically for the contours shown in fig. 3.

Using (3) and

$$\cot \beta_i^{(\pm)} = \pm(-1)^i v_y^{(\pm)}/v_x^{(\pm)}, \ i = 1, 2, 3, 4, \tag{17}$$

in (4) one obtains

$$\rho^{(+)} \cot \beta_i^{(+)} + \rho^{(-)} \cot \beta_i^{(-)} = (-1)^i \left(\rho^{(-)} - \rho^{(+)}\right) v^{(+)}(P_i)/v_x^{(+)}(P_i).$$

Thus (10) holds when

$$(-1)^i v^{(+)}(P_i)/v_x^{(+)}(P_i) \leq 0.$$

From (14) and (16) we obtain that $\partial v_0^{(\pm)}(x, 0)/\partial x \sim -\gamma x^{-1}$ as $x \to 0$. So we see that the last inequality takes place for the stream lines $v^{(\pm)} = c$ when c is large enough. Numerical results for $\beta_i^{(\pm)}$ and the exponent λ_i in (8) are presented in table 1 for contours shown in fig. 2. The angles $\beta_i^{(\pm)}$ are calculated using (17) where $v_x^{(\pm)}$ and $v_y^{(\pm)}$ are evaluated in the same way as $v^{(\pm)}$ (see the next section); λ_i are obtained by solving (9) numerically.

A similar method allows to construct examples of non-uniqueness for the subcritical case when internal waves exist at infinity downstream. Using (11) we define waveless potentials as follows

$$u_*^{(\pm)}(x, y) = u^{(\pm)}(x - \pi/\nu_0, y) - u^{(\pm)}(x + \pi/\nu_0, y).$$

Level lines of the stream function $v_*^{(\pm)}(x,y)$ corresponding to $u_*^{(\pm)}$ enclose four points $\big((-1)^i\pi/\nu + (-1)^j\pi/\nu_0, 0\big)$, $i,j = 1,2$. Thus, the potentials $u_*^{(\pm)}$ provide non-uniqueness examples with four interface-piercing bodies. The existence of contours enclosing singularities is again a consequence of (14) and (16).

It is worth noting that if $\nu > \nu_*$ and ν/ν_0 is rational, then non-uniqueness examples can be constructed using two singularities similarly to the supercritical case. If the equality

$$\frac{\nu_0}{\nu} = \frac{m}{n}, \text{ where } m,n \in \{1,2,3,\dots\},$$

holds, waveless potentials can be defined as follows

$$u^{(\pm)}(x,y) = u_0^{(\pm)}(x - n\pi/m\nu, y) - u_0^{(\pm)}(x + n\pi/m\nu, y)$$

A pattern of stream lines for this case of the subcritical regime is shown in fig. 3 and the corresponding values of $\beta_i^{(\pm)}$ and λ_i are given in table 2.

Computational procedure

In this section we describe a scheme for calculation $v_0^{(-)}$ when the difficulty arises because of infinite upper limit of $I^{(-)}$ (see (15)). Moreover, its integrand is an oscillating function having a simple pole at $k = \nu$.

First we regularize the integral. The function

$$\mathcal{D}_3(k) = D_3(k) - \frac{\gamma}{k} - \frac{2\varepsilon\nu}{(\varepsilon + \mathrm{e}^{2\nu h})(k-\nu)},$$

is analytic on $(-\sigma, +\infty)$, $\sigma > 0$. Then $I^{(-)}$ can be rewritten as follows:

$$I^{(-)}(x,y) = \int_0^{+\infty} \mathcal{D}_3(k)\mathrm{e}^{ky}\cos kx\,\mathrm{d}k - \gamma\log|x + \mathrm{i}y| + \frac{2\varepsilon\nu}{\varepsilon + \mathrm{e}^{2kh}}\,\mathrm{F}_0(x,y,\nu), \qquad (18)$$

where the well-known formula

$$\log|x + \mathrm{i}y| = -\int_0^{+\infty} \frac{\mathrm{e}^{ky}\cos kx - \mathrm{e}^{-k}}{k}\,\mathrm{d}k.$$

is used together with

$$\mathrm{F}_0(x,y,a) = \int_0^{+\infty} \frac{\mathrm{e}^{ky}\cos kx}{k-a}\,\mathrm{d}k = \mathrm{Re}\left\{\mathrm{e}^{a(y-\mathrm{i}x)}\,\mathrm{Ei}(-a(y-\mathrm{i}x))\right\}$$

(see 8.212.5 in Gradshteyn & Ryzhik [2]). Futhermore, we split the integral in (18) into a sum

$$\int_0^{+\infty} \mathcal{D}_3(k)\mathrm{e}^{ky}\cos kx\,\mathrm{d}k = \left(\int_0^b + \int_b^{+\infty}\right) = J_1(x,y,b) + J_2(x,y,b). \qquad (19)$$

Since

$$\mathcal{D}_3(k) = -\frac{4\gamma}{k} - \left(1 - \frac{(2+4\varepsilon+\varepsilon^2)\mathrm{e}^{-2kh}}{2+\varepsilon}\right)\frac{2\gamma}{k-\gamma} + \frac{4\gamma^2(1+\varepsilon)\mathrm{e}^{-2kh}}{(2+\varepsilon)(k-\gamma)^2}$$
$$- \frac{2\varepsilon\nu}{(\varepsilon+\mathrm{e}^{2\nu h})(k-\nu)} + \frac{2\varepsilon\nu\mathrm{e}^{-2kh}}{(k-\nu)} + O(\mathrm{e}^{-3kh}) \quad \text{as} \quad k \to \infty,$$

we have for sufficiently large b:

$$J_2(x,y,b) \approx -4\gamma\,\mathrm{F}_1(x,y,0,b) + 2\gamma\,\mathrm{F}_1(x,y,\gamma,b) - \frac{2\varepsilon\nu}{(\varepsilon+\mathrm{e}^{2\nu h})}\mathrm{F}_1(x,y,\nu,b)$$
$$+ 2\varepsilon\nu\,\mathrm{F}_1(x,y-2h,\nu,b) - \frac{2\gamma(2+4\varepsilon+\varepsilon^2)}{2+\varepsilon}\mathrm{F}_1(x,y-2h,\gamma,b)$$
$$+ \frac{4\gamma^2(1+\varepsilon)}{(2+\varepsilon)}\mathrm{F}_2(x,y-2h,\gamma,b), \tag{20}$$

where

$$\mathrm{F}_\ell(x,y,a,b) = \int_b^{+\infty} \frac{\mathrm{e}^{ky}\cos kx}{(k-a)^\ell}\,\mathrm{d}k = (b-a)^{1-\ell}\,\mathrm{Re}\left\{\mathrm{e}^{a(y-\mathrm{i}x)}\,\mathrm{E}_\ell((a-b)(y-\mathrm{i}x))\right\}.$$

The last equality follows from 5.1.4 in Abramowitz & Stegun [1]. In order to compute J_1 in (19) a weighted integration scheme of middle rectangles is applied with the weight $\rho(x,y,k) = \exp\{ky\}\cos kx$. Then, $J_1(x,y,b) \approx L_N(x,y,b)$, where

$$L_N = \sum_{j=1}^{N} \mathcal{D}_3(\xi_j) \int_{k_{j-1}}^{k_j} \rho(x,y,k)\,\mathrm{d}k. \tag{21}$$

Here $k_j = j\Delta$, $\xi_j = (k_j + k_{j-1})/2$ and $\Delta = b/N$. Substituting the integrals in (21) we obtain

$$J_1 \approx L_N = \mathrm{Re}\left\{\frac{1-\mathrm{e}^{-\mathrm{i}z\delta}}{\mathrm{i}z}\sum_{j=1}^{N}\mathcal{D}_3(\xi_j)\mathrm{e}^{-\mathrm{i}zk_{j-1}}\right\}, \quad z = x + \mathrm{i}y. \tag{22}$$

Let us estimate the difference $J_1 - L_N$. Using the Lagrange formula

$$\mathcal{D}_3(k) = \mathcal{D}_3(\xi_j) + \mathcal{D}_3'(\theta(k))(k-\xi_j),$$

where $\theta(k) \in (\xi_j, k)$, in

$$R_j = \int_{\xi_j}^{k_j} \rho(x,y,k)\,\{\mathcal{D}_3(k) - \mathcal{D}_3(\xi_j)\}\,\mathrm{d}k,$$

and taking into account that $|\rho(k,x,y)| \leq 1$ when $y < 0$, we find

$$|R_j| = \left|\int_{\xi_j}^{k_j} \rho(k,x,y)\mathcal{D}_3'(\theta(k))(k-\xi_j)\,\mathrm{d}k\right| \leq M_j \frac{(k-\xi_j)^2}{2}\Bigg|_{\xi_j}^{k_j} = \frac{M_j\Delta^2}{8},$$

where $M_j = \max\{|\mathcal{D}_3'(k)|\}$ for $k \in [\xi_j, k_j]$. Hence,

$$|J_1(x,y,b) - L_N(x,y,b)| \leq \max_{k\in[0,b]} \{|\mathcal{D}_3'(k)|\} \frac{N\Delta^2}{4}.$$

It is worth mentioning that the estimate is uniform with respect to x and y. Finally, we compute $v_0^{(-)}$ combining (18)–(20) and (22). A similar scheme is used for calculation of $v_0^{(+)}$.

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