

A uniqueness criterion for linear problems of wave-body interaction

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Abstract

The question of uniqueness for linearized problems describing interaction of submerged bodies with an ideal unbounded fluid is far from its final resolution. In the present work a new criterion of uniqueness is suggested based on Green's integral identity and maximum principles for elliptic differential equations. The criterion is formulated as an inequality involving integrals of the Green function over the bodies' wetted surfaces. This criterion is quite general and applicable for any number of submerged bodies of fairly arbitrary shape (satisfying an exterior sphere condition) and in any dimension; it can also be generalised to more complicated elliptic problems. Very simple bounds are also derived from the criterion, which deliver uniqueness sets in the space of parameters defined by submergence of the system of bodies and the frequency of oscillation. Results of numerical investigation and comparison with known uniqueness criteria are presented.

Introduction

This article is concerned with the two and three-dimensional linear boundary-value problems of the interaction between an ideal unbounded fluid and bodies located under the free surface of the fluid. Two classes of problem, describing the radiation of waves by the forced motion of rigid bodies and the diffraction of waves by fixed rigid bodies, appear within the framework of the surface wave theory under the usual assumptions that the motion is steady-state, irrotational and the oscillations have small amplitudes.

Our interest here is in the question of uniqueness. For these seemingly simple problems very few criteria of uniqueness are known despite the long history of

surface-wave theory and the importance of the uniqueness property (see [Ursell \(1992\)](#), where this question is placed first in a list of unfinished problems).

In order to explain the difficulty in investigating the uniqueness question we note that uniqueness in the problems under consideration is known to be equivalent to non-existence of so-called trapped modes, that is non-trivial localised solutions of the homogeneous boundary-value problem. At the same time, the problems in the absence of bodies allow wave solutions propagating to infinity, and hence trapped modes are found within the space of functions having infinite energy and the frequencies of trapped modes can be considered as point eigenvalues embedded in the continuous spectrum. This case is much more difficult to analyse than that of non-embedded eigenvalues.

An extensive review of existing results on uniqueness can be found, e.g. in [Kuznetsov et al. \(2002, part I\)](#), but two main results should be mentioned here. In [Vainberg & Maz'ya \(1973\)](#) the so-called Maz'ya's identity was introduced (see also [Maz'ya, 1978](#); [Hulme, 1984](#)), which allows uniqueness to be proven at all frequencies for particular body geometries such that some special vector fields no-where enter the wetted surface of the bodies. Another scheme was used for the two-dimensional problem by [Simon & Ursell \(1984\)](#) who extended the approach of [John \(1950\)](#) to prove uniqueness for any system of submerged bodies which are contained between lines which intersect the free surface at the same point and at angle 45° to the horizontal; for lines that are not straight the possible inclinations are greater. The technique used by [Simon & Ursell \(1984\)](#) also shows that the potential will be unique provided that the frequency is small or large enough. In either limit, the method gives a bound on how small, or large, the frequency must be for uniqueness to be guaranteed. An improvement of the result by [Simon & Ursell \(1984\)](#) for the case when the submerged structure is symmetric about a vertical line (two-dimensional problem) or axisymmetric (three-dimensional problem) can be found in [Kuznetsov & Simon \(1999\)](#).

One might expect that it would be possible to improve the results of [Vainberg & Maz'ya \(1973\)](#) or [Simon & Ursell \(1984\)](#) to include much wider, if not all, classes of geometries, at least for the case of totally submerged bodies. However, in recent years a number of examples of non-uniqueness (that is structures that support trapped modes) have been constructed. This was first done for particular surface-piercing bodies by [McIver \(1996\)](#) and for totally submerged bodies by [McIver \(2000\)](#) by using the so-called inverse scheme, in which the geometry of a trapping configuration is sought for a given potential. Other examples of non-uniqueness for totally submerged bodies are constructed in [McIver & Porter \(2002\)](#) and [Porter \(2002\)](#), where numerical techniques are developed for finding trapped modes for given configurations and results are presented for submerged elliptical toroids and for a variety of symmetrical pairs of submerged obstacles. Thus, it is now clear that no general uniqueness proof is possible and, in particular, the angle in the proof by [Simon & Ursell \(1984\)](#) can not be decreased to zero; structures found in [Porter \(2002\)](#) have a maximum angle of about 6.7° .

In the present work we shall present a new criterion for uniqueness based on use of the Green identity and maximum principles. The basic approach has a quite general nature and can be applied to elliptic problems with more complicated operators than that in the present paper and in dimensions higher than three. The criterion is given in the form of an inequality that involves integrals of a Green's function on the wetted part of submerged bodies and in this way includes the frequency of oscillations and the geometry. It is applicable for any number of bodies of arbitrary shape with sufficiently regular boundary, e.g. for any bodies whose wetted surfaces belong to the class C^2 . More precise formulation admits surfaces of the class C^1 satisfying the so-called external sphere condition. The latter means that any point of the wetted surface should lie on boundary of a ball in the fluid domain. We shall also show that at least in the two-dimensional case the criterion is applicable to piecewise C^2 -contours satisfying the external ball condition.

The inequality of the criterion can be checked numerically for fixed geometrical configurations and frequencies and if it is satisfied (with the accuracy of the computation taken into account) this proves uniqueness for the particular geometry and frequency. Examples of such computations will be given in this paper. In addition, estimates for the gradient of the Green function obtained in the present work allow us to obtain very simple bounds (similar to those by [Simon & Ursell \(1984\)](#), but, unlike the latter, also applicable to the three-dimensional problem) for trapped modes in the space of parameters defined by submergence of the system of bodies and the frequency of oscillation; in particular, the bounds guarantee uniqueness for sufficiently large values of frequency for any system of bodies. Importantly, these simple bounds do not depend on the horizontal spacing between the bodies (for some bounds dependence on the geometry is reduced to include only the area of the wetted surface of the bodies) and, thus, for the case when the bodies are widely separated, the bounds have a significant advantage when compared with those of [Simon & Ursell \(1984\)](#).

Now we give a brief outline of the paper. In § 1 we introduce notations and present the mathematical problem. A criterion of uniqueness for the problem is derived in § 2. Bounds for uniqueness set in the space of frequency and submergence of bodies are obtained in § 3; some examples of the bounds computed numerically are presented. Improvement of the uniqueness criterion with the help of auxiliary potentials is suggested in § 4, where reduction of the criterion to a finite part of the free surface is also done. A set of auxiliary potentials is suggested in § 5. Results of numerical computation for the criterion and comparison with known uniqueness results are presented in § 6. The paper has two appendices where some representations and estimates for derivatives of Green's function are obtained.

1. Statement of the problem

We consider the linearized problem describing motion of an inviscid incompressible heavy fluid with a free surface in the presence of a system of bodies which are totally submerged to the fluid. Under the usual assumptions of linearized surface wave theory (see e.g., [Lamb, 1932](#)), the time-harmonic motion of the fluid can be described by a velocity potential

$$\operatorname{Re}\{\exp(-i\omega t) u(\mathbf{x}, y)\},$$

where t is time, ω is the frequency of the oscillations, (\mathbf{x}, y) are spacial coordinates with the origin of the coordinate system in the unperturbed free surface, y is the vertical coordinate that decreases with depth, $\mathbf{x} = (x, z)$ when the three-dimensional problem is considered and $\mathbf{x} = x$ for the two-dimensional problem, corresponding to the situation of normal incidence of surface waves to a system of infinitely long cylinders with generators along the z -axis. The potential u satisfies the equations

$$\Delta u = 0 \quad \text{in} \quad W, \tag{1.1a}$$

$$\partial_y u - Ku = 0 \quad \text{on} \quad F, \tag{1.1b}$$

and

$$\partial_n u = f \quad \text{on} \quad S.$$

where W is the domain occupied by fluid, $F = \{y = 0\}$ is the mean free surface, S is the wetted surface of the submerged obstruction, we use the notation $\partial_a^n = \frac{\partial^n}{\partial a^n}$ and the Laplace operator $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ in the three-dimensional case and $\Delta = \partial_x^2 + \partial_y^2$ for the two-dimensional problem. Further, $K = \omega^2/g$, g is the acceleration due to gravity and \mathbf{n} is the unit normal vector directed inside the fluid. For the two-dimensional problem W , F and S will denote the cross-sections of the corresponding three-dimensional domains. The standard radiation condition at infinity should also be imposed upon u , and the function f in the last formula is defined by the type of the oscillations.

We shall assume that S is a collection of C^1 -surfaces satisfying also some regularity condition (e.g. the exterior sphere condition) which will be discussed later. Besides, the statement above presumes that the potential u has regular normal derivative on the boundary ∂W of the domain occupied by the fluid and that u belongs to $C^2(W) \cap C(\overline{W})$.

Further, in the paper we shall also consider some geometrical configurations with singular points (more precisely, corner points for the two-dimensional case). Then, to exclude strong singularities of u at non-regular points on S a condition of local finiteness of energy should be added to the statement of the problem.

Uniqueness of the problem is known (see e.g., [Kuznetsov et al., 2002](#), sect. 2.2.1) to be equivalent to non-existence of the so-called trapped modes, that is unforced oscillations of the fluid which are adjacent to submerged

obstructions and have finite energy. A trapped mode potential u satisfies the conditions (1.1a), (1.1b) and, in addition,

$$\partial_n u = 0 \quad \text{on} \quad S, \quad (1.1c)$$

$$\int_W \{|\nabla u|^2 + K^2|u|^2\} d\mathbf{x} dy < \infty. \quad (1.1d)$$

Without loss of generality, the trapped mode potential u can be assumed to be real.

2. A criterion for uniqueness

We start with the definition of the Green function for the three-dimensional problem. Let $G(P, Q)$ be the potential of a source located at the point $Q(x_0, y_0, z_0)$ and computed at the point $P(x, y, z)$. Then for $y_0 < 0$ the function $G(P, Q)$ as a function of P satisfies the equations

$$(\partial_x^2 + \partial_y^2 + \partial_z^2) G = -\delta(x - x_0, y - y_0, z - z_0), \quad y < 0, \quad (2.1a)$$

$$\partial_y G - Ku = 0, \quad y = 0, \quad (2.1b)$$

where δ is the Dirac delta function. The function should also be bounded in the domain $y < 0$ with a neighbourhood of the source Q excluded.

We introduce the notations

$$r = [|\mathbf{x} - \mathbf{x}_0|^2 + (y - y_0)^2]^{1/2}, \quad r' = [|\mathbf{x} - \mathbf{x}_0|^2 + (y + y_0)^2]^{1/2}, \quad (2.2)$$

where $\mathbf{x}_0 = (x_0, z_0)$. For our purposes it is immaterial which type of radiation condition at infinity is imposed upon the Green function, however it will be convenient to define the function to be real. Then, the three-dimensional Green function is known to have the following representation (see e.g. [Wehausen & Laitone, 1960](#))

$$G(P; Q) = \frac{1}{4\pi} \left[\frac{1}{r} - \frac{1}{r'} \right] + \frac{1}{2\pi} \int_0^\infty \frac{\mu}{\mu - K} e^{\mu(y+y_0)} J_0(\mu|\mathbf{x} - \mathbf{x}_0|) d\mu, \quad (2.3)$$

where J_0 is the Bessel function of the first kind defined e.g. in ([Gradsteyn & Ryzhik, 1994](#), sect. 8.40) and the notation \int here and subsequently denotes principal value integration.

Analogously, in the two-dimensional case, when $Q = Q(x_0, y_0)$, $P = P(x, y)$, $\mathbf{x} = x$, $\mathbf{x}_0 = x_0$ in (2.2) and the dependence in z should be omitted in the conditions (2.1), we have (see e.g. [Wehausen & Laitone, 1960](#))

$$G(P, Q) = \frac{1}{2\pi} [\log r' - \log r] + \frac{1}{\pi} \int_0^\infty \frac{e^{\mu(y+y_0)}}{\mu - K} \cos \mu(x - x_0) d\mu. \quad (2.4)$$

Then, in both two- and three-dimensional problems we shall apply the third

Green identity to a hypothetical trapped mode, that is a solution to the problem (1.1), and to the Green function G . Taking into account conditions (1.1a)–(1.1d), (2.1a) and (2.1b) we find

$$u(Q) = \int_S u(P) \partial_{n(P)} G(P, Q) \, ds_{(P)}, \quad Q \in W. \quad (2.5)$$

From the latter formula it follows that

$$|u(Q)| \leq \max_{P \in S} \{|u(P)|\} \int_S |\partial_{n(P)} G(P, Q)| \, ds_{(P)}. \quad (2.6)$$

We note that the Green function can be extended analytically from the domain $y < 0$ to the domain $y < -y_0$ (it follows in particular from the representations (2.3) and (2.4)). Thus, in the case under consideration, when bodies are submerged, the representation (2.5) is clearly valid for a point Q belonging to the free surface. Thus, from (2.6) we arrive at the inequality

$$\sup_F |u| \leq \max_S |u| \sup_{Q \in F} \left\{ \int_S |\partial_{n(P)} G(P, Q)| \, ds_{(P)} \right\}. \quad (2.7)$$

Further we can make use of maximum principles to compare the values $\sup_F |u|$ and $\max_S |u|$. The *strong maximum principle* of E. Hopf (see e.g., Gilbarg & Trudinger, 1983, Theorem 3.5) guarantees that a non-constant potential u satisfying (1.1a) cannot attain a minimum or maximum at an interior point of any finite subset of W . The theorem is also valid for the unbounded domain W ; we should only note that the value of $|u|$ decays at infinity (see e.g., Kuznetsov et al., 2002, sect. 2.2.1).

Consider first the case of wetted surfaces without singular points. Let S belong to the class C^1 and let any point of S satisfy the *exterior sphere condition*. The latter requires that the point should lie on the boundary of a ball in W . In this case Lemma 3.4 from Gilbarg & Trudinger (1983) is applicable and it states that any outward directional derivative at a boundary point where u attains its maximum (minimum) is strictly positive (negative). In view of the condition (1.1c), the assertion guarantees that the maximum/minimum is located in F but not in S . We note that if u is constant, then $u = 0$ by (1.1b). Thus, if the geometrical restriction on S is satisfied, we find that either $u \equiv 0$ in W or

$$\max_S |u| < \sup_F |u|. \quad (2.8)$$

Combining the inequalities (2.7) and (2.8) we arrive at the following assertion.

THEOREM 2.1: *If for a given geometry S and a given value of the parameter K the inequality*

$$\sup_{Q \in F} \left\{ \int_S |\partial_{n(P)} G(K; P, Q)| \, ds_{(P)} \right\} \leq 1 \quad (2.9)$$

holds, then the problem (1.1) has only the trivial solution.

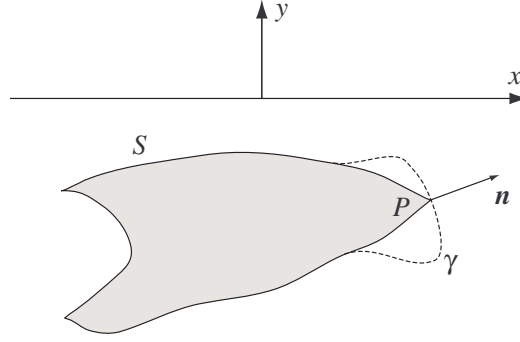


Figure 1: A contour with corner points

It should be noted that in general the statement of Lemma 3.4 in [Gilbarg & Trudinger \(1983\)](#) is not true for surfaces of the class C^1 without additional regularity conditions, e.g. the sphere condition. A counter-example with a C^1 -contour not satisfying the sphere condition can be found in [Gilbarg & Trudinger \(1983\)](#) in comments to Lemma 3.4. In the commentary to [Gilbarg & Trudinger \(1983, ch. 3\)](#) some references to generalizations of the statement of Lemma 3.4 are given which weaken the regularity requirement. It is also notable that surfaces of the class C^2 satisfy the sphere condition (see e.g., [Gilbarg & Trudinger, 1983, ch. 2](#)). Thus, the statement of Theorem 2.1 holds for any geometry S belonging to C^2 .

Lemma 3.4 from [Gilbarg & Trudinger \(1983\)](#) is not applicable to the general case of surfaces with singular points and detailed analysis of the cases when (2.8) still holds for such surfaces is beyond of the scope of the present work. We only give an example of the case when Theorem 2.1 is applicable to contours with corner points. Consider the two-dimensional problem for a system of contours S , which consists of a finite number of arcs belonging to C^2 including the endpoints (an example of such a configuration is shown in fig. 1). Let P be a corner point and α be the angle between the tangents at the corner measured through the fluid. We shall only consider the contours satisfying the exterior ball condition, i.e. the case of corners protruding into the fluid when $\alpha \in (\pi, 2\pi]$.

By known results on asymptotics of solution to elliptic problems (see e.g., [Nazarov & Plamenevsky, 1994](#)), taking into account (1.1c) and (1.1d), one can find the expansion

$$u(\rho, \theta) = u(P) + c\rho^{\pi/\alpha} \cos(\pi\alpha^{-1}\theta) + \varphi_1(\rho, \theta) + \varphi_2(\rho, \theta), \quad (2.10)$$

where (ρ, θ) is a polar coordinate system with origin at P such that θ varies in the fluid from 0 on one tangent at P to α on the other. Further, $\varphi_1 = d\rho \cos \theta$ when $\alpha = 2\pi$ and $\varphi_1 = 0$ otherwise; c and d are some coefficients. The estimates $\varphi_2 = O(\rho^{1+\delta})$, $|\nabla \varphi_2| = O(\rho^\delta)$ hold as $\rho \rightarrow 0$ with some $\delta > 0$.

For contours with corner points the maximum principle holds and the maximum of $|u| = M$ is achieved on the boundary of W . Assume that the maximum is at a regular point of S . Then Lemma 3.4 from [Gilbarg & Trudinger \(1983\)](#) guarantees that at that point $\partial_n u \neq 0$ which contradicts the condition (1.1c).

Now suppose that the maximum is located at the corner point P . We consider a body augmented by a “bulb” (see fig. 1); let the curve γ belong to C^2 and the tangent at P be defined as $\theta = \beta$, where $\beta \in (0, \pi/2)$. The maximum of $|u|$ in the new water domain is still situated at P ($|u| < M$ on $\gamma \setminus \{P\}$ since the curve is located inside the domain W). Let $c \neq 0$ in the asymptotics (2.10). Lemma 3.4 from Gilbarg & Trudinger (1983) guarantees that for any fixed $\theta \in (\beta, \beta + \pi)$

$$\liminf_{\rho \rightarrow 0} \rho^{-1} (u(\rho, \theta) - u(P)) > 0. \quad (2.11)$$

At the same time, if $c \neq 0$, $\rho^{-1} (u(\rho, \theta) - u(P)) \sim c \rho^{\pi/\alpha - 1} \cos(\pi \alpha^{-1} \theta)$, and the factor $\cos(\pi \alpha^{-1} \theta)$ has different sign for $\theta \in (\beta, \alpha/2)$ and $\theta \in (\alpha/2, \beta + \pi)$. Thus, we conclude that $c = 0$. For $\alpha = 2\pi$, the limit (2.11) is equal to $d \cos \theta$, and $d = 0$ since the term changes sign in the interval $\theta \in (\beta, \beta + \pi)$, when β belongs to $(0, \pi/2)$. Finally, on the extended contour the normal derivative at P is defined and equal to zero. The latter in view of Lemma 3.4 from Gilbarg & Trudinger (1983) contradicts the assumption that maximum of $|u|$ is located at P . Thus, we conclude that (2.8) holds.

3. Uniqueness bounds in frequency and submergence

Consider a fixed system of bodies $S(Y)$ located below the depth

$$Y = \max\{y : (\mathbf{x}, y) \in S\}, \quad Y < 0.$$

We can use the representation (A.8) and estimates (A.9) and (B.17) to find a subset Ω of the (K, Y) parameter space such that uniqueness of the problem (1.1) is guaranteed in Ω . We start with the criterion (2.9) and write

$$\begin{aligned} \int_S |\partial_{n(\mathbf{x}, y)} G(\mathbf{x}, y, \mathbf{x}_0, 0)| ds_{(\mathbf{x}, y)} &\leq \int_S |\nabla_{\mathbf{x}, y} G(\mathbf{x}, y, \mathbf{x}_0, 0)| ds_{(\mathbf{x}, y)} \\ &\leq \int_S g(K, y) ds_{(\mathbf{x}, y)} \end{aligned} \quad (3.1)$$

where by (B.17) for the three-dimensional case

$$\begin{aligned} g(K, y) = K^2 \left[\left(2^{-1} e^{Ky} + \pi^{-1} \min \left\{ \frac{\sqrt{2}}{(Ky)^2}, \frac{5}{(K|y|)^3} \right\} \right)^2 \right. \\ \left. + \left(\pi^{-1} e^{Ky} + \pi^{-2} \min \left\{ \frac{\sqrt{2}}{(Ky)^2}, \frac{5}{(K|y|)^3} \right\} \right)^2 \right]^{1/2}, \end{aligned} \quad (3.2)$$

and by (A.8) and (A.9) for the two-dimensional case

$$g(K, y) = K e^{Ky} + \pi^{-1} \min \left\{ \frac{2}{|y|}, \frac{K(2\sqrt{2} + 1)}{(Ky)^2} \right\}. \quad (3.3)$$

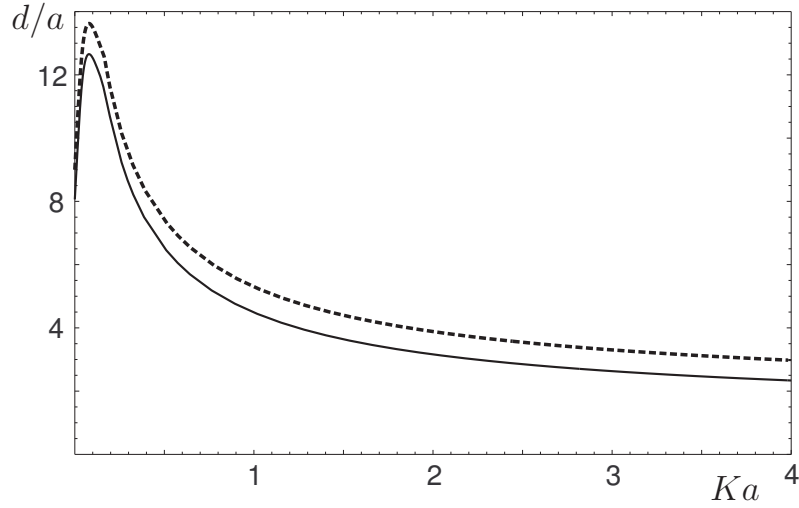


Figure 2: Bounds given by the formulae (3.4) (solid line) and (3.5) (dashed line) for two circular cylinders of radius a and whose centres are located at a depth d . The uniqueness domain is located above the curves.

In each case the function $g(K, y)$ is monotonically decreasing in the depth y .

Thus, the boundary of the region in the (K, Y) parameter space for which uniqueness is guaranteed is given by

$$\int_{S(Y)} g(K, y) \, ds_{(\mathbf{x}, y)} = 1; \quad (3.4)$$

a weaker condition is

$$\sum_{i=1}^N g(K, Y_i) |S_i| = 1, \quad (3.5)$$

where S_i is one of the bodies forming the N -body system S , $|S_i|$ is the area of its surface, and $Y_i = \max\{y : (\mathbf{x}, y) \in S_i\}$.

For any system of bodies of given submergence and geometry uniqueness is guaranteed by (2.9) and (3.1) for sufficiently large values of $K \geq K^*$. This follows from the estimates $O(K^{-1})$ as $K \rightarrow \infty$ for the functions g defined in (3.2) and (3.3). The bound K^* specific to the geometry and submergence of the bodies, but not depending of their relative horizontal positions, can be found from (3.4) or (3.5). It is notable that, unlike the bounds obtained by Simon & Ursell (1984) and Kuznetsov & Simon (1999), the bounds obtained from (3.4) and (3.5) do not depend on the horizontal spacing between the bodies, and, thus, give an improvement when the spacing is large. This will be seen later in § 6, where the bounds (3.4) and the bounds of Kuznetsov & Simon (1999) are compared.

Results of numerical evaluation of the curves in parameter space given by (3.4) and (3.5) are shown in fig. 2 and 3. The computations are for a system consisting of two circles (fig. 2) and two spheres (fig. 3) of radius a and whose centres are located at a distance l from the line $\mathbf{x} = 0$ and at a depth d . Non-existence of trapped modes for the geometry is guaranteed in the sets of parameters located in fig. 2 and 3 above the curves.

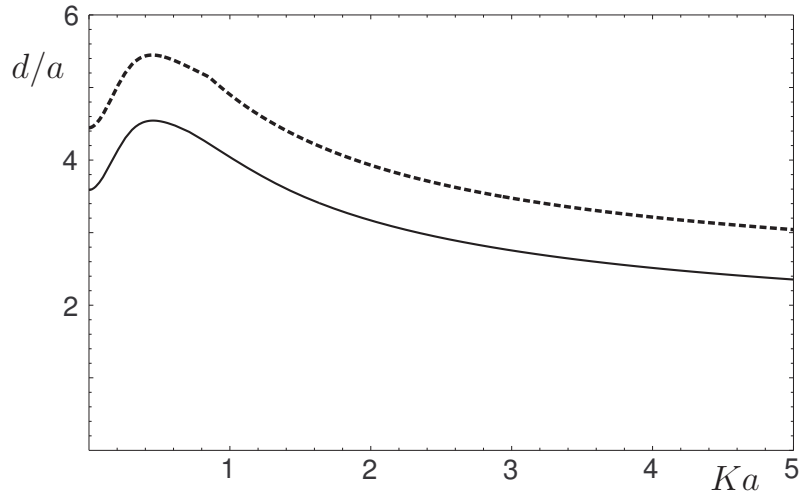


Figure 3: Bounds given by the formulae (3.4) (solid line) and (3.5) (dashed line) for two spheres of radius a and whose centres are located at a depth d . The uniqueness domain is located above the curves.

The asymptotic behaviour for small and large Ka of the curves in figures 2 and 3 can be obtained as follows. Consider first of all the two-dimensional case shown in fig. 2. From (3.3), as $Ka \rightarrow 0$

$$g(K, y) \sim \frac{2}{\pi|y|}$$

and therefore

$$\int_{S(Y)} g(K, y) ds_{(\mathbf{x}, y)} \sim \frac{4a}{\pi} \int_0^{2\pi} \frac{d\theta}{d - a \sin \theta} = \frac{8}{((d/a)^2 - 1)^{1/2}}$$

so that (3.4) holds provided $d/a \rightarrow \sqrt{65}$ as $Ka \rightarrow 0$. In the same limit

$$\sum_{i=1}^2 g(K, Y_i) |S_i| \sim \frac{8}{d/a - 1}$$

and, thus, $d/a \rightarrow 9$ as $Ka \rightarrow 0$ for (3.5). It may be noted that since the estimates (3.4), (3.5) in terms of the functions (3.2) and (3.3) are based on the expansion (A.6) of the exponential integral and this expansion is most suited for large values of the argument (see Appendices), we cannot expect the estimates (3.4) and (3.5) to be effective for small values of K , and indeed d/a has a finite limit as $Ka \rightarrow 0$. However, the theorem of Simon & Ursell (1984) guarantees (in the two-dimensional case) uniqueness for any submergence when Ka is sufficiently small or large.

Further, as $Ka \rightarrow \infty$, (3.3) gives

$$g(K, y) \sim \frac{2\sqrt{2} + 1}{\pi K y^2}$$

and therefore

$$\int_{S(Y)} g(K, y) \, ds_{(x,y)} \sim \frac{2(2\sqrt{2}+1)a}{\pi K} \int_0^{2\pi} \frac{d\theta}{(d-a\sin\theta)^2} = \frac{4(2\sqrt{2}+1)d/a}{Ka((d/a)^2-1)^{3/2}}.$$

For the last expression to be non-zero in the limit $Ka \rightarrow \infty$, it is required that $d/a \rightarrow 1$ and hence

$$\frac{4(2\sqrt{2}+1)d/a}{Ka((d/a)^2-1)^{3/2}} \sim \frac{4(1+1/(2\sqrt{2}))}{Ka(d/a-1)^{3/2}},$$

so that (3.4) holds provided

$$\frac{d}{a} \sim 1 + \left(\frac{4(1+1/(2\sqrt{2}))}{Ka} \right)^{2/3} \quad \text{as } Ka \rightarrow \infty.$$

As $Ka \rightarrow \infty$

$$\sum_{i=1}^2 g(K, Y_i) |S_i| \sim \frac{4(2\sqrt{2}+1)}{Ka(d/a-1)^2}$$

and (3.5) holds provided

$$\frac{d}{a} \sim 1 + 2 \left(\frac{2\sqrt{2}+1}{Ka} \right)^{1/2} \quad \text{as } Ka \rightarrow \infty.$$

Similar calculations may be made for the asymptotics of the three-dimensional case shown in fig. 3 and in view of (3.2) the results are as follows. As $Ka \rightarrow 0$, (3.4) and (3.5) give

$$\frac{d}{a} \rightarrow \left[1 + 8\sqrt{2} \left(1 + \frac{1}{\pi^2} \right)^{1/2} \right]^{1/2}, \quad \frac{d}{a} \rightarrow 1 + 4 \left[2^{-1} \left(1 + \frac{1}{\pi^2} \right) \right]^{1/4},$$

respectively. As $Ka \rightarrow \infty$ (3.4) and (3.5) hold provided

$$\frac{d}{a} \sim 1 + \sqrt{2} \left(\frac{5}{Ka} \right)^{1/2} \left(1 + \frac{1}{\pi^2} \right)^{1/4}, \quad \frac{d}{a} \sim 1 + 2 \left[\frac{5}{Ka} \left(1 + \frac{1}{\pi^2} \right)^{1/2} \right]^{1/3},$$

respectively. Similar asymptotic analysis can be done for bodies of arbitrary shape.

4. Modification of the uniqueness criterion

In the section we shall show how the uniqueness result of Theorem 2.1 can be improved. Suppose we can find potentials Φ_i defined and bounded on W and satisfying the equations (1.1a) and (1.1b). Then, by Green's identity (provided

that the potentials and their derivatives have the behaviour at infinity that is required for the application of the identity) we find

$$0 = \int_S u(P) \partial_n \Phi_i(P) \, ds,$$

Combining the latter with (2.5) we get

$$u(Q) = \int_S u(P) \left[\partial_{n(P)} G(P, Q) + \sum_i c_i \partial_n \Phi_i(P, Q) \right] \, ds_{(P)}, \quad Q \in W \cup F, \quad (4.1)$$

where c_i are arbitrary constants and we indicate explicitly that Φ_i can also contain dependence on Q . We fix the functions $c_i = c_i(Q)$ as solutions to the minimization problem

$$\min_{c_1, c_2, \dots} \int_S \left| \partial_{n(P)} G(P, Q) + \sum_i c_i \partial_n \Phi_i(P, Q) \right| \, ds_{(P)}, \quad Q \in F, \quad (4.2)$$

and from (4.1) arrive at the inequality

$$\sup_F |u| \leq \max_S |u| \sup_{Q \in F} \left\{ \int_S \left| \partial_{n(P)} G(P, Q) + \sum_i c_i(Q) \partial_n \Phi_i(P, Q) \right| \, ds_{(P)} \right\}.$$

which, along with the maximum principles, proves the assertion

THEOREM 4.1: *If for a given geometry S and a value of the parameter K the inequality*

$$\mathcal{M}(F, S, \{\Phi_i\}) \equiv \sup_{Q \in F} \left\{ \int_S \left| \partial_{n(P)} G(P, Q) + \sum_i c_i(Q) \partial_n \Phi_i(P, Q) \right| \, ds_{(P)} \right\} \leq 1 \quad (4.3)$$

holds, where the functions $c_i(Q)$ are defined by (4.2), then the problem (1.1) has only the trivial solution.

The minimization problem (4.2) is a problem of linear programming and can be effectively solved numerically by the simplex method. More details will be given below in the section 6.

The supremum in the formula (4.3) is sought in the infinite set F , which is inappropriate for numerical application of the criterion. Therefore, it is important to note that the computation can be reduced to a finite set because a finite part of the free surface F_0 can be found such that the value of the expression $\mathcal{M}(F \setminus F_0, S, \{\Phi_i\})$ over the remainder of the free surface is guaranteed to be smaller than one.

Consider the expression (4.2) in the two-dimensional case for a point in the free surface $Q = (x_0, 0)$. Let the point Q be located to the right of the bodies, so that $x \leq x_0$ for all $P = (x, y) \in S$. Then, by (A.8) we have

$$\begin{aligned} \partial_x G(P, Q) &= K e^{Ky} \cos K(x - x_0) - K \pi^{-1} \operatorname{Im} \{R_1(iK\zeta)\}, \\ \partial_y G(P, Q) &= K e^{Ky} \sin K(x - x_0) - K \pi^{-1} \operatorname{Re} \{R_1(iK\zeta)\}, \end{aligned}$$

where $\zeta = x - x_0 + iy$ and R_1 is given in (A.7).

Then, one can find by the particular choice $c_1 = -1$, $\Phi_1 = e^{Ky} \sin K(x - x_0)$ and $c_i = 0$, $i \neq 1$, in (4.3) that

$$\begin{aligned} & \min_{c_1, c_2, \dots} \int_S \left| \partial_{n(P)} G(P, Q) + \sum_i c_i \partial_{n(P)} \Phi_i(P, Q) \right| ds_{(P)} \\ & \leq \int_S \left| \partial_{n(P)} \left(G(P, Q) - e^{Ky} \sin K(x - x_0) \right) \right| ds_{(P)} \\ & \leq \int_S R_G(|x - x_0|, y) ds_{(x, y)}, \end{aligned} \quad (4.4)$$

where the function R_G is defined in (A.9). The estimate (4.4) can be obtained analogously for the case when the point Q is located to the left of the bodies.

Let the inequality

$$\int_S R_G(|x - x_0|, y) ds_{(x, y)} < 1, \quad (4.5)$$

hold for $x_0 = a_+^*$ and $x_0 = a_-^*$, where

$$a_+^* \geq \max\{x : (x, y) \in S\}, \quad a_-^* \leq \min\{x : (x, y) \in S\}.$$

Since the function $R_G(|x - x_0|, y)$ decays monotonically in $|x - x_0|$, the inequalities (4.5) hold for all $x_0 \in F$, such that $\pm x_0 > \pm a_\pm^*$. Then, the inequalities (4.4) and (4.5) guarantees that

$$\mathcal{M}(F \setminus F_0, S, \{\Phi_i\}) < 1, \quad F_0 = \{(x, 0) : a_-^* \leq x \leq a_+^*\}$$

and to establish uniqueness using the criterion of Theorem 4.1 we have only to check the corresponding inequality over the finite part of the free surface F_0 .

In the three-dimensional case it is much more obvious that the criterion can be evaluated in a finite part of the free surface because the Green function and its gradient decay at infinity (see e.g. Kuznetsov et al., 2002, part I) as $O(|\mathbf{x} - \mathbf{x}_0|^{-1/2})$. A better result can be obtained by extracting simple waves in the way it has been done above for the two-dimensional problem, but such analysis is beyond the scope of this work.

5. Auxiliary potentials

In this section we suggest a set of auxiliary potentials Φ_i . Consider first the two-dimensional problem. The simplest auxiliary potentials are the wave solutions, which have been used in the previous section,

$$\tilde{\Phi}^{(+)}(P, Q) = e^{K(y-y_0)} \cos K(x - x_0), \quad \tilde{\Phi}^{(-)}(P, Q) = e^{K(y-y_0)} \sin K(x - x_0). \quad (5.1)$$

Introduce two polar coordinate systems (r, θ) and (r', θ') , where r and r' are given by (2.2) and θ and θ' are such that

$$x - x_0 = r \sin \theta = r' \sin \theta', \quad y = r \cos \theta + y_0 = -r' \cos \theta' - y_0.$$

Using the notation we write the following multipole solutions, satisfying (1.1a) and (1.1b),

$$\Phi_n(P, Q) = \frac{e^{-in\theta}}{r^n} + \frac{(-1)^n}{(n-1)!} \int_0^\infty \frac{\mu + K}{\mu - K} \mu^{n-1} e^{\mu(y+y_0)} e^{i\mu(x-x_0)} d\mu, \\ n = 1, 2, \dots, \quad \Phi_0(P, Q) = G(P, Q). \quad (5.2)$$

Let Q_j , $j = 0, 1, 2, \dots$, be a system of points located inside the contours S . Then, obviously, the real and imaginary parts of $\Phi_n(P, Q_j)$ can be used as auxiliary potentials.

However, numerical computation of the potentials (5.2) is a difficult task due to the integration of oscillatory functions with singularities, which should be performed over the infinite interval. Hence, much more convenient objects for numerical computation are the auxiliary potentials based on the wave-free potentials

$$\Psi_i^{(+)}(P) = \operatorname{Re}\{\Psi_{n_i+1}(P, Q_{j_i})\}, \quad \Psi_i^{(-)}(P) = \operatorname{Im}\{\Psi_{n_i+1}(P, Q_{j_i})\}, \quad (5.3)$$

where $n_i, j_i \in \{0, 1, \dots\}$ and for $n = 1, 2, 3, \dots$

$$\Psi_n(P, Q) = K \frac{e^{-in\theta}}{r^n} - K \frac{e^{-in\theta'}}{r'^n} - n \frac{e^{-i(n+1)\theta}}{r^{n+1}} - n \frac{e^{-i(n+1)\theta'}}{r'^{n+1}}$$

(see e.g. [Eatock Taylor & Hu, 1991](#)).

The expansion theorem of [Ursell \(1950\)](#) states that any wave potential can be written at great distances as the sum of a wave source, a horizontal wave dipole, regular waves and wave-free potentials and, thus, at least formally, substantiates choosing auxiliary potentials from the set $\{\tilde{\Phi}^\pm, \Phi_0, \operatorname{Im} \Phi_1, \Psi_1^\pm, \Psi_2^\pm, \dots\}$.

Now consider the three-dimensional problem. We introduce cylindrical coordinate system with its origin at the point $Q(x_0, y_0, z_0)$, where the radius R and the angular coordinate α for the point $P(x, y, z)$ are defined by

$$R = |\mathbf{x} - \mathbf{x}_0|, \quad x - x_0 = R \cos \alpha, \quad z - z_0 = R \sin \alpha. \quad (5.4)$$

Then, it is easy to check that the functions

$$\tilde{\Phi}_m(P, Q) = e^{K(y-y_0)} J_m(KR) \begin{Bmatrix} \cos m\alpha \\ \sin m\alpha \end{Bmatrix}, \quad m = 0, 1, 2, \dots$$

satisfy the Laplace equation (1.1a) and the condition on the free surface (1.1b), thus they can be applied in the criterion (4.3).

Similar to the two-dimensional case, more potentials Φ for application in (4.3) could be constructed using multipole singular solutions of Laplace's equation (for their definition see e.g., [Wehausen & Laitone, 1960](#)) with the singular points located inside the surfaces S . However, for effective computation the following system is preferable (the points Q_j are located inside the surfaces S)

$$\Phi_i(P) = \Psi_{\ell_i}^{m_i}(P, Q_{j_i}), \quad j_i, \ell_i, m_i = 0, 1, 2, \dots$$

which is based on the following wave-free potentials (see [Wehausen & Laitone, 1960](#), 13.21)

$$\Psi_\ell^m(P, Q) = \left[\frac{P_{\ell+1}^m(-\cos \theta)}{r^{\ell+2}} - \frac{K}{\ell - m + 1} \frac{P_\ell^m(-\cos \theta)}{r^{\ell+1}} + (-1)^m \frac{P_{\ell+1}^m(\cos \theta')}{r'^{\ell+2}} + \frac{(-1)^m K}{\ell - m + 1} \frac{P_\ell^m(\cos \theta')}{r'^{\ell+1}} \right] \begin{Bmatrix} \cos m\alpha \\ \sin m\alpha \end{Bmatrix},$$

where $\ell, m = 0, 1, 2, \dots$, P_ℓ^m are associated Legendre functions (see e.g. 8.752(1) in [Gradsteyn & Ryzhik \(1994\)](#), 8.6.6 in [Abramowitz & Stegun \(1965\)](#)).

6. Numerical results

In this section we shall give examples of numerical evaluation of the criterion (4.3). Numerical investigation was performed for the criterion (4.3) in the two-dimensional case with the minimisation (4.2) implemented using the NAG (Numerical Algorithms Group) routine E04CCF which is based on the simplex method (see the NAG web site at www.nag.co.uk for details of this routine). The calculations were carried out on a Sun Sparc Ultra 5 and to produce each data point in the figures below took 3–4 minutes of CPU time. As in § 3 we consider a system consisting of two circles of radius a whose centres are located at a distance l from the line $x = 0$ and at a depth d . Results of computation for the criterion (4.3) are presented in fig. 4 and 5 for $d/a = 2$ and $d/a = 3$ respectively. The notations \circ , \times and \square are used for the points where non-existence of trapped modes is shown using (4.3). The mark \circ means that no auxiliary potentials are needed, at the points \times the non-existence of trapped modes is established using the auxiliary simple-wave potentials (5.1) and \square means that both simple waves and four waveless potentials (5.3) are involved to improve the criterion. The points marked by the symbols \blacksquare indicate where the criterion (4.3) could not be satisfied using any of the above-mentioned auxiliary potentials; calculations with an increased number of auxiliary potentials failed due to difficulties in finding a minimum in (4.2).

In contrast to fig. 5, which seems to suggest that for the whole considered region of parameters the uniqueness takes place, fig. 4, where d/a is smaller, contains an area, where the method cannot prove uniqueness. This observation of a relation between depth of submergence and uniqueness is factual for the simple bounds studied in § 3. For any geometry the bounds guarantee uniqueness for depths of submergence exceeding some value (see fig. 2).

In fig. 5 the results of our computations are compared with results of [Kuznetsov & Simon \(1999\)](#) which are only applicable to symmetric geometries (that work is unable to make predictions about the case shown in fig. 4). The grey areas in fig. 5 correspond to the domain where uniqueness is guaranteed for both symmetric and antisymmetric modes by the criterion of [Kuznetsov & Simon \(1999\)](#).

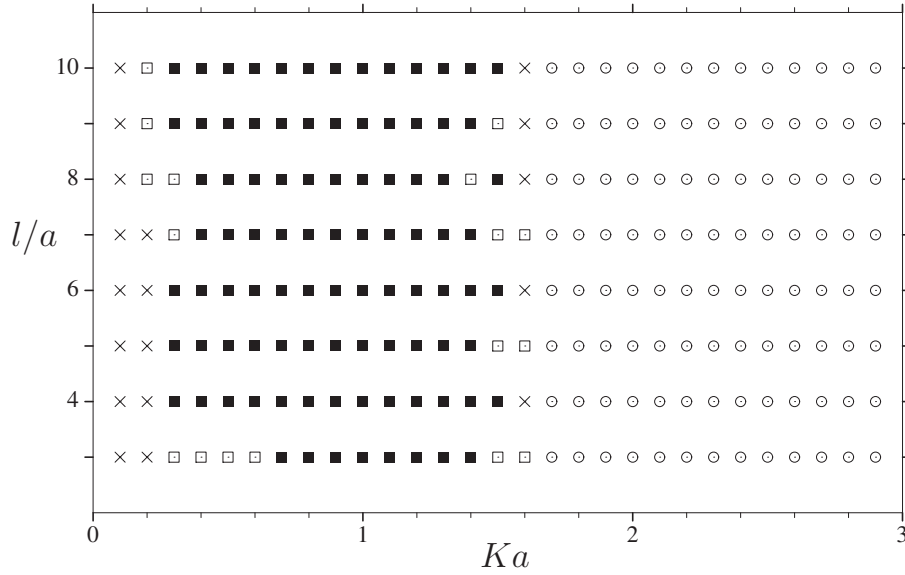


Figure 4: Results of numerical investigation of criterion (4.3) for two cylinders with $d/a = 2$.

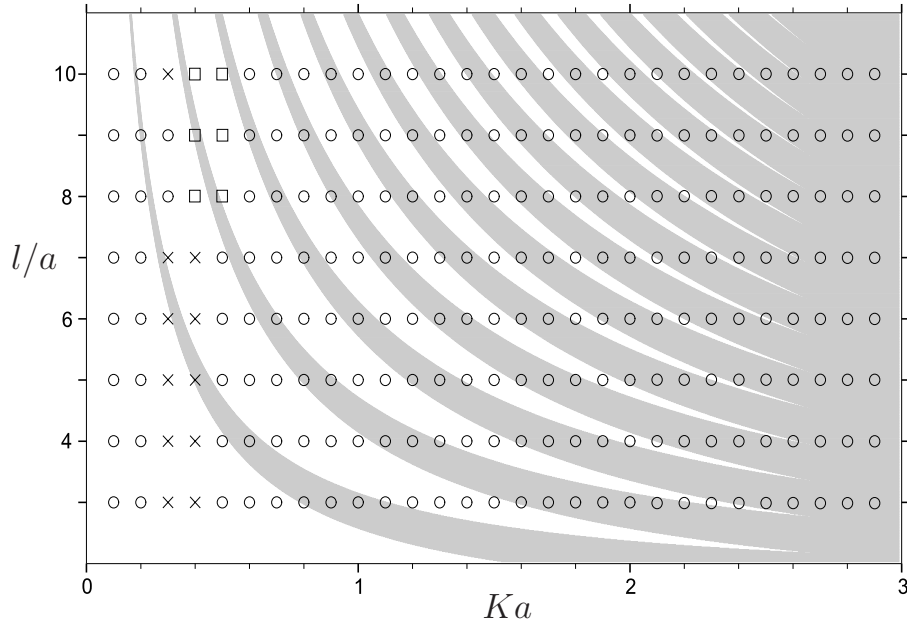


Figure 5: Results of numerical investigation of criterion (4.3) for two cylinders with $d/a = 3$. In the grey area uniqueness is given by criterion of Kuznetsov & Simon (1999)

7. Conclusion

In this paper the question of uniqueness for the two and three-dimensional linearized boundary-value problems of wave-body interaction is considered in the case when the fluid has an unbounded free surface and the bodies are totally submerged. A new criterion for uniqueness is proposed, which is based on Green's integral identity and maximum principles for elliptic differential equations. The criterion is given in the form of an inequality that involves integrals of Green's function on the wetted part of submerged bodies (in this way including the frequency of oscillations and the geometry). The integrals can

also contain auxiliary potentials and a set of such potentials is suggested for improvement of the result. The criterion can be checked numerically for fixed geometrical configurations and frequencies and if it is satisfied (with the accuracy of the computation taken into account) this proves uniqueness for the particular geometry and frequency. Examples of such computations are given. In addition, simple bounds are also derived from the criterion, which deliver uniqueness sets in the space of parameters defined by submergence of the system of bodies and the frequency of oscillation. It is noteworthy that these bounds do not depend on the horizontal spacing between the bodies (for some bounds dependence on the geometry is reduced to include only the area of the wetted surface of the bodies) and, thus, for the case when the bodies are widely separated, the bounds have a significant advantage when compared with those of [Simon & Ursell \(1984\)](#) or [Kuznetsov & Simon \(1999\)](#) (the latter bounds are for symmetric structures). The suggested criterion is applicable to any number of bodies of arbitrary shape, subject to the external sphere condition, and can also be generalised for more complicated elliptic problems. It seems very likely that an improvement of the criterion can be achieved for symmetric structures; in particular, for the two-dimensional problem symmetric and antisymmetric in x modes can be considered separately. Perhaps the simplest further applications of the suggested criterion could be to the problem in finite-depth fluid and to the two-dimensional problem describing oblique incidence of waves to cylindrical structures where in the set of equations (1.1) the field equation is replaced by $\Delta u - k^2 u = 0$. It would also be of interest to compare the results by the criterion with the recently obtained non-uniqueness examples by [Porter \(2002\)](#) to study how close are the points of uniqueness by the criterion and the bounds to the parameters corresponding to the trapped modes.

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Appendix A. On derivatives of the two-dimensional Green function

In the appendix we shall derive estimates for derivatives of the function $G(P, Q)$ in the case when the source point Q belongs to the free surface F . From the

representation (2.4) we have

$$G(x, y, x_0, 0) = \frac{1}{\pi} \int_0^\infty \frac{e^{\mu y} \cos \mu(x - x_0)}{\mu - K} d\mu.$$

Then, we find

$$\begin{aligned} \partial_x G(x, y, x_0, 0) &= -\frac{1}{\pi} \int_0^\infty e^{\mu y} \sin \mu(x - x_0) d\mu - \frac{K}{\pi} \int_0^\infty \frac{e^{\mu y} \sin \mu(x - x_0)}{\mu - K} d\mu \\ &= -\frac{x - x_0}{\pi[(x - x_0)^2 + y^2]} - \frac{K}{\pi} \operatorname{Im}\{e^{-iK\zeta} \operatorname{Ei}(iK\zeta)\}, \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} \partial_y G(x, y, x_0, 0) &= -\frac{y}{\pi[(x - x_0)^2 + y^2]} + \frac{K}{\pi} \int_0^\infty \frac{e^{\mu y} \cos \mu(x - x_0)}{\mu - K} d\mu \\ &= -\frac{y}{\pi[(x - x_0)^2 + y^2]} - \frac{K}{\pi} \operatorname{Re}\{e^{-iK\zeta} \operatorname{Ei}(iK\zeta)\}, \end{aligned} \quad (\text{A.2})$$

where $\zeta = x - x_0 + iy$ and Ei is the integral exponent (see 8.212.5 in [Gradsteyn & Ryzhik \(1994\)](#))

$$\operatorname{Ei}(z) = -e^z \int_0^\infty \frac{e^{-zt} dt}{t - 1}, \quad \operatorname{Re}\{z\} > 0.$$

It will be convenient to use the following form of the formulae (A.1) and (A.2)

$$\partial_x G - i\partial_y G = \frac{iK}{\pi} \left[-\frac{1}{iK\zeta} + e^{-iK\zeta} \operatorname{Ei}(iK\zeta) \right]. \quad (\text{A.3})$$

Now we derive an expansion of the function Ei . We write

$$\int_0^\infty \frac{e^{-zt} dt}{t - 1} = \oint_0^\infty \frac{e^{-zt} dt}{t - 1} + \pi i e^{-z},$$

where \oint means that the line of integration passes above the pole. Moving the contour of integration we have

$$\oint_0^\infty \frac{e^{-zt} dt}{t - 1} - \pi i e^{-z} = \int_{\ell^+} \frac{e^{-zt} dt}{t - 1},$$

where ℓ^+ is the ray emanating from the origin and going to infinity through the point $-(iz)^{-1}$. Then, by change of variable $t = \tau/iz$ we obtain

$$\int_{\ell^+} \frac{e^{-zt} dt}{t - 1} = \int_0^{-\infty} \frac{e^{i\tau} d\tau}{\tau - iz} = \int_0^{+\infty} \frac{e^{i\tau} d\tau}{\tau - iz} - 2\pi i e^{-z}.$$

Let $\operatorname{Im}\{z\} > 0$. Integrating by parts N times we arrive at

$$\int_0^{+\infty} \frac{e^{i\tau} d\tau}{\tau - iz} = -\sum_{k=1}^N (k-1)! z^{-k} + \frac{N!}{i^N} \int_0^\infty \frac{e^{i\tau} d\tau}{(\tau - iz)^{N+1}}. \quad (\text{A.4})$$

Further we use the inequality $n^{-1} \left| \sum_{i=1}^n a_i \right| \leq \left(n^{-1} \sum_{i=1}^n a_i^2 \right)^{1/2}$ (see 11.115 in [Gradsteyn & Ryzhik \(1994\)](#)) to write

$$|\tau - iz| = \left[(\tau - \operatorname{Re}\{iz\})^2 + (\operatorname{Im}\{iz\})^2 \right]^{1/2} \geq 2^{-1/2} (\tau - \operatorname{Re}\{iz\} + \operatorname{Im}\{iz\})$$

where we take into account signs of τ , $\operatorname{Re}\{z\}$ and $\operatorname{Im}\{z\}$. Then we have

$$\begin{aligned} \left| \int_0^\infty \frac{e^{i\tau} d\tau}{(\tau - iz)^{N+1}} \right| &\leq \int_0^\infty \frac{d\tau}{|\tau - iz|^{N+1}} \leq 2^{(N+1)/2} \int_0^\infty \frac{d\tau}{(\tau + \operatorname{Re}\{z\} + \operatorname{Im}\{z\})^{N+1}} \\ &= 2^{(N+1)/2} N^{-1} (\operatorname{Re}\{z\} + \operatorname{Im}\{z\})^{-N} \leq 2^{(N+1)/2} N^{-1} |z|^{-N}. \end{aligned} \quad (\text{A.5})$$

The case $\operatorname{Im}\{z\} < 0$ is analogous, in this case we use the formula

$$\int_0^\infty \frac{e^{-zt} dt}{t-1} - \pi i e^{-z} = - \int_{-\infty}^0 \frac{e^{i\tau} d\tau}{\tau - iz}.$$

Finally, combining (A.4) and (A.5) we find

$$\operatorname{Ei}(z) = \pi i \operatorname{sign}(\operatorname{Im}\{z\}) + e^z \left\{ \sum_{n=1}^N (n-1)! z^{-n} + R_N(z) \right\}, \quad (\text{A.6})$$

where

$$\begin{aligned} R_N(z) &= -\frac{N!}{i^N} \int_0^\infty \frac{e^{i\tau} d\tau}{(\tau - iz)^{N+1}}, \\ |R_N| &\leq 2^{(N+1)/2} (N-1)! |z|^{-N}, \quad |R_N| \leq (2^{N/2+1} + 1) N! |z|^{-(N+1)}. \end{aligned} \quad (\text{A.7})$$

Thus, by (A.3) and (A.6) with $N = 1$ we arrive at

$$\partial_x G - i \partial_y G = -K \operatorname{sign}(x - x_0) e^{Ky} e^{-iK(x-x_0)} + \frac{iK}{\pi} R_1(iK\zeta), \quad (\text{A.8})$$

where the non-wave term is estimated as follows:

$$\left| \frac{iK}{\pi} R_1(iK\zeta) \right| \leq R_G(|x - x_0|, y) \equiv \frac{1}{\pi} \min \left\{ \frac{2}{r}, \frac{K(2\sqrt{2} + 1)}{(Kr)^2} \right\}. \quad (\text{A.9})$$

Appendix B. On derivatives of the three-dimensional Green function

In the appendix we shall derive estimates for derivatives of the three-dimensional Green's function. By formula 6.611.1 from ([Gradsteyn & Ryzhik, 1994](#)) and in view of the representation (2.3) we have for $Q \in F$

$$G(\mathbf{x}, y; \mathbf{x}_0, 0) = \frac{1}{2\pi r} + \frac{K}{2\pi} I(R, y), \quad I(R, y) = \int_0^\infty \frac{e^{\mu y} J_0(\mu R)}{\mu - K} d\mu, \quad (\text{B.1})$$

where r and R are given in (2.2) and (5.4) respectively. By using formula 6.611.1 from (Gradsteyn & Ryzhik, 1994) again, we find

$$\partial_y I(R, y) = \frac{1}{r} + K I(R, y). \quad (\text{B.2})$$

The term $J_0(\mu R)$ can be written as follows

$$J_0(\mu R) = \frac{1}{\pi} \int_{-1}^1 (1 - t^2)^{-1/2} \cos(\mu R t) dt$$

(see formula 8.411.8 in Gradsteyn & Ryzhik (1994)). Substituting the latter representation to the definition of I (B.1), changing order of integration and using the function Ei (cf. (A.2)) we get

$$\begin{aligned} I(R, y) &= \frac{1}{\pi} \int_{-1}^1 (1 - t^2)^{-1/2} dt \int_0^\infty \frac{e^{\mu y} \cos(\mu R t)}{\mu - K} d\mu \\ &= -\frac{1}{\pi} \int_{-1}^1 (1 - t^2)^{-1/2} \text{Re} \{ e^{-iK \zeta(t)} \text{Ei}(iK \zeta(t)) \} dt \\ &= -\frac{1}{\pi} \int_{-1}^1 (1 - t^2)^{-1/2} \text{Re} \left\{ \pi i \text{sign}(t) e^{-iK \zeta(t)} + \frac{1}{iK \zeta(t)} \right. \\ &\quad \left. - \frac{1}{K^2 [\zeta(t)]^2} + R_2(iK \zeta(t)) \right\} dt, \end{aligned} \quad (\text{B.3})$$

where $\zeta(t) = Rt + iy$ and the expansion (A.6) with $N = 2$ is used. By formula 8.551.1 from (Gradsteyn & Ryzhik, 1994) we have

$$\begin{aligned} &\int_{-1}^1 (1 - t^2)^{-1/2} \text{Re} \{ \pi i \text{sign}(t) e^{-iK \zeta(t)} \} dt \\ &= 2\pi e^{Ky} \int_0^1 (1 - t^2)^{-1/2} \sin(KRt) dt = \pi^2 e^{Ky} \mathbf{H}_0(KR), \end{aligned} \quad (\text{B.4})$$

where \mathbf{H}_n is the Struve function (Gradsteyn & Ryzhik, 1994, sect. 8.55). We also find

$$\int_{-1}^1 (1 - t^2)^{-1/2} \text{Re} \left\{ \frac{1}{iK \zeta(t)} \right\} dt = -\frac{y}{K} \int_{-1}^1 (1 - t^2)^{-1/2} \frac{dt}{(Rt)^2 + y^2} = \frac{\pi}{Kr}, \quad (\text{B.5})$$

$$\begin{aligned} \int_{-1}^1 (1 - t^2)^{-1/2} \text{Re} \left\{ \frac{1}{K^2 [\zeta(t)]^2} \right\} dt &= \frac{1}{K^2} \int_{-1}^1 \frac{(1 - t^2)^{-1/2} (R^2 t^2 - y^2) dt}{(R^2 t^2 - y^2)^2 + 4R^2 t^2 y^2} \\ &= \frac{\pi y}{K^2 r^3} = -\frac{\pi}{K^2} \partial_y (r^{-1}). \end{aligned} \quad (\text{B.6})$$

Thus, by (B.1)–(B.6) we arrive at

$$\partial_y G(\mathbf{x}, y; \mathbf{x}_0, 0) = -\frac{K^2}{2} e^{Ky} \mathbf{H}_0(KR) + \sigma_1(R, y), \quad (\text{B.7})$$

where

$$\sigma_1(R, y) = -\frac{K^2}{2\pi^2} \int_{-1}^1 (1-t^2)^{-1/2} \operatorname{Re}\{R_2(iK \zeta(t))\} dt.$$

Taking into account (A.7) and the equality $\int_{-1}^1 (1-t^2)^{-1/2} dt = \pi$ we find

$$\begin{aligned} |\sigma_1(R, y)| &\leq \frac{K^2}{2\pi^2} \max_{t \in [-1, 1]} |R_2(iK \zeta(t))| \int_{-1}^1 (1-t^2)^{-1/2} dt \\ &\leq \frac{K^2}{2\pi} \min \left\{ \max_{t \in [-1, 1]} \frac{2\sqrt{2}}{|K \zeta(t)|^2}, \max_{t \in [-1, 1]} \frac{10}{|K \zeta(t)|^3} \right\} \\ &= \frac{K^2}{2\pi} \min \left\{ \frac{2\sqrt{2}}{K^2 y^2}, \frac{10}{K^3 |y|^3} \right\}. \end{aligned} \quad (\text{B.8})$$

Consider now $\partial_x G$ and $\partial_z G$. We note that

$$\partial_x G = \cos \alpha \partial_R G, \quad \partial_z G = \sin \alpha \partial_R G \quad (\text{B.9})$$

(see (5.4) for definition of α) and, thus, $\partial_R G$ should only be treated. By using formulae 8.473.4 and 6.611.1 in (Gradsteyn & Ryzhik, 1994), we find

$$\partial_R I(R, y) = -\frac{r+y}{Rr} - K I_1(R, y), \quad I_1(R, y) = \int_0^\infty \frac{e^{\mu y} J_1(\mu R)}{\mu - K} d\mu. \quad (\text{B.10})$$

By using 8.411.8 from (Gradsteyn & Ryzhik, 1994) and integrating by parts we write

$$\begin{aligned} I_1(R, y) &= \frac{R}{\pi} \int_{-1}^1 (1-t^2)^{1/2} dt \int_0^\infty \frac{\mu e^{\mu y} \cos(\mu R t)}{\mu - K} d\mu \\ &= \frac{1}{\pi} \int_{-1}^1 t(1-t^2)^{-1/2} dt \int_0^\infty \frac{e^{\mu y} \sin(\mu R t)}{\mu - K} d\mu. \end{aligned}$$

The principal value integral can be written in terms of \exp and Ei (see (A.1)) and, then, in view of (A.6) we have

$$\begin{aligned} I_1(R, y) &= \frac{1}{\pi} \int_{-1}^1 t(1-t^2)^{-1/2} \operatorname{Im} \left\{ \pi i \operatorname{sign}(t) e^{-iK \zeta(t)} + \frac{1}{iK \zeta(t)} \right. \\ &\quad \left. - \frac{1}{K^2 [\zeta(t)]^2} + R_2(iK \zeta(t)) \right\} dt. \end{aligned} \quad (\text{B.11})$$

By 8.551.1 in (Gradsteyn & Ryzhik, 1994),

$$\begin{aligned} &\int_{-1}^1 t(1-t^2)^{-1/2} \operatorname{Im} \{ \pi i \operatorname{sign}(t) e^{-iK \zeta(t)} \} dt \\ &= 2\pi e^{Ky} \int_0^1 t(1-t^2)^{-1/2} \cos(KRt) dt = -\pi^2 e^{Ky} \mathbf{H}_1(KR) + 2\pi e^{Ky}. \end{aligned} \quad (\text{B.12})$$

Analogously to (B.5) and (B.6) we obtain

$$\begin{aligned} \int_{-1}^1 t(1-t^2)^{-1/2} \operatorname{Im} \left\{ \frac{1}{iK \zeta(t)} \right\} dt \\ = -\frac{R}{K} \int_{-1}^1 t^2(1-t^2)^{-1/2} \frac{dt}{(Rt)^2 + y^2} = -\frac{\pi(r+y)}{KRr}, \end{aligned} \quad (\text{B.13})$$

and

$$\begin{aligned} \int_{-1}^1 t(1-t^2)^{-1/2} \operatorname{Im} \left\{ \frac{1}{K^2[\zeta(t)]^2} \right\} dt \\ = -\frac{2Ry}{K^2} \int_{-1}^1 \frac{t^2(1-t^2)^{-1/2} dt}{(R^2t^2 - y^2)^2 + 4R^2t^2y^2} = \frac{\pi R}{K^2r^3} = -\frac{\pi}{K^2} \partial_R(r^{-1}). \end{aligned} \quad (\text{B.14})$$

Finally, by (B.1) and (B.9)–(B.14) we arrive at

$$\partial_{\left\{ \begin{smallmatrix} x \\ z \end{smallmatrix} \right\}} G(\mathbf{x}, y; \mathbf{x}_0, 0) = \frac{K^2}{2\pi} \left\{ \begin{smallmatrix} \cos \alpha \\ \sin \alpha \end{smallmatrix} \right\} [-\pi e^{Ky} \mathbf{H}_1(KR) + 2e^{Ky} + \sigma_2(R, y)], \quad (\text{B.15})$$

where

$$\sigma_2(R, y) = \frac{1}{\pi} \int_{-1}^1 t(1-t^2)^{-1/2} \operatorname{Im} \{ R_2(iK \zeta(t)) \} dt,$$

and in view of (A.7) and the equality $\int_0^1 t(1-t^2)^{-1/2} dt = 1$,

$$|\sigma_2(R, y)| \leq \frac{1}{\pi} \min \left\{ \frac{2\sqrt{2}}{K^2y^2}, \frac{10}{|Ky|^3} \right\}. \quad (\text{B.16})$$

We summarize the results given in (B.7), (B.8), (B.15) and (B.16) and arrive at the sought estimate

$$\begin{aligned} |\nabla G(\mathbf{x}, y; \mathbf{x}_0, 0)| \leq K^2 \left[\left(2^{-1}e^{Ky} + \pi^{-1} \min \left\{ \frac{\sqrt{2}}{(Ky)^2}, \frac{5}{(K|y|)^3} \right\} \right)^2 \right. \\ \left. + \left(\pi^{-1}e^{Ky} + \pi^{-2} \min \left\{ \frac{\sqrt{2}}{(Ky)^2}, \frac{5}{(K|y|)^3} \right\} \right)^2 \right]^{1/2}. \end{aligned} \quad (\text{B.17})$$

Here we also used the inequalities $|\mathbf{H}_0(KR)| \leq 1$, $|\pi \mathbf{H}_1(KR) - 2| \leq 2$, which obviously follows from (B.4) and (B.12).