On the resistanceless statement of the two-dimensional Neumann–Kelvin problem for a surface-piercing tandem

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A new set of supplementary conditions is proposed for the two-dimensional Neumann–Kelvin problem describing the steady-state forward motion of a surface-piercing tandem in an infinite depth fluid. This problem is shown to be uniquely solvable for almost every value of the forward speed U. The velocity potential solving the problem corresponds to a flow about the tandem providing no resistance (wave and spray resistance vanish simultaneously). On the other hand, for the exceptional values of U examples of non-uniqueness (trapped modes) are constructed using the inverse procedure recently applied by McIver (1996) to the problem of time-harmonic water waves. For the proposed statement of the Neumann–Kelvin problem the inverse method involves the investigation of streamlines generated by two dipoles placed in the free surface. The spacing of dipoles delivering trapped modes depends on U.

1. Introduction

The present paper continues the investigation of waveless statement of the Neumann-Kelvin problem started in Kuznetsov & Motygin (1997) (which will be cited as (I) below), where the case of one surface-piercing cylinder has been considered. Here we treat the case of a tandem of two-dimensional bodies moving forward with constant velocity U in the free surface of an inviscid, incompressible fluid under gravity. We consider the resulting fluid motion in the framework of the linearized water-wave theory, and the corresponding boundary value problem is usually referred to as the Neumann-Kelvin problem. An account of known results on this problem is given in (I).

Since the pioneering paper by Ursell (1981), it is well-known that the Neumann-Kelvin problem for surface-piercing bodies requires supplementary conditions. In the case of a tandem four conditions must be added to the boundary value problem. In the present work we use the "resistanceless" set of supplementary conditions which provides vanishing the total resistance (a sum of wave resistance and spray resistance) for a surface-piercing tandem. The existence of solution to this statement is proved for tandems of arbitrary geometry, but uniqueness is established only for tandems which are symmetric with respect to the y-axis. The solvability and uniqueness hold for all values of U, except possibly a sequence tending to zero (a similar result was established for a single body in (I)).

The question of unique solvability for *all* values of U was open for the Neumann–Kelvin problem augmented by any set of supplementary conditions. Different types of such conditions are surveyed in (I). Here we show that for the waveless statement uniqueness does not take place for all values of U even for symmetric about the y-axis tandems. This is demonstrated with the help of the so-called inverse method which was recently applied by McIver (1996) for construction of a non-uniqueness example in the two-dimensional problem of time-harmonic water waves. The idea of inverse procedure is as follows (we describe it for our problem). A potential with special properties is obtained by placing two singularities in the free surface of fluid. The distance between them is chosen depending on U in order to cancel waves at infinity downstream. Investigation of potential's streamlines shows that there is a pair of them with the both ends in the free surface and enclosing the singularities inside. These streamlines are interpreted as rigid contours delivering a geometry of tandem for which the homogeneous problem has a nontrivial solution. However, singularities applied here are dipoles because sources, which were successfully used by McIver (1996), do not allow to satisfy supplementary conditions imposed in the present statement of the Neumann-Kelvin problem.

In fact, our example delivers non-uniqueness to two statements of the Neumann-Kelvin problem. Along with the resistanceless statement introduced in the present paper the same example works in the case of the least singular statement proposed by Ursell (1981).

We formulate the Neumann-Kelvin problem and some auxiliary results in § 2. The resistanceless potential is defined in § 3, where the source method is also applied to prove existence of this potential. In § 4 the uniqueness theorem is proved for symmetric tandems. The non-uniqueness example constructed in § 5 shows that exceptional value admitted in the uniqueness theorem does exist at least for some special geometries. Concluding remarks are given in § 6. Appendix A contains necessary properties of Green's function, and Appendix B treats the auxiliary problems used for proving the uniqueness theorem.

2. The Neumann-Kelvin problem and some auxiliary results

By D_+ and D_- we denote cross-sections of surface-piercing cylinders moving forward in the direction of the x-axis, and $W = \mathbb{R}^2_- \setminus (\overline{D_+} \cup \overline{D_-})$ is the cross-section of the fluid domain (see fig. 1). Three components of the free surface are F_+, F_0 and F_-, and S_+ (S_-) denotes the wetted contour of the front (back) cylinder.

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[b]

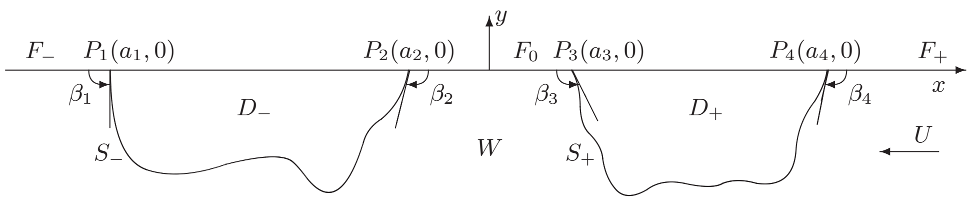

Figure 1. A definition sketch of the geometry and the corresponding notations.

We assume $S = S_+ \cup S_-$ to be a C^2-curve, and the unilateral tangents to S to form angles $\beta_i \neq 0, \pi$ with $F = F_+ \cup F_0 \cup F_-$ at the points P_i, $i = 1, 2, 3, 4$.

We consider the steady-state wave pattern due to the tandem in the framework of the linear surface wave theory (see Wehausen & Laitone 1960), and use the coordinate system attached to bodies so that $-a_2 = a_3 = a$ (see fig. 1). Then the velocity field is described by means of potential u satisfying the following boundary value problem (it is usually referred to as the Neumann–Kelvin problem):

$$\nabla^2 u = 0 \quad \text{in} \quad W, \tag{2.1}$$

$$u_{xx} + \nu u_y = 0 \quad \text{on} \quad F, \tag{2.2}$$

$$\partial u/\partial n = U \cos(n, x) \quad \text{on} \quad \text{int}\, S = S \setminus \{P_1, P_2, P_3, P_4\}, \tag{2.3}$$

$$\lim_{x \to +\infty} |\nabla u| = 0, \tag{2.4}$$

$$\sup\{|\nabla u| : (x, y) \in W \setminus E\} < \infty, \tag{2.5}$$

$$\int_{W \cap E} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}y < \infty. \tag{2.6}$$

Here $\nu = gU^{-2}$, g is the acceleration due to gravity, U is the forward velocity of the tandem. By $\mathbf{n}$ we denote the unit normal to S directed into W, and E is an arbitrary compact set in $\overline{\mathbb{R}^2_-}$ such that $\overline{D_+} \cup \overline{D_-} \subset E$ and $F_\pm \cap E \neq \emptyset$, $F_0 \cap E \neq \emptyset$. It is obvious, that u is defined up to an arbitrary constant term.

The condition (2.6) means that the kinetic energy is locally finite, which implies that there are no strong singularities at $P_1, \ldots, P_4$. This follows from the asymptotics of u in a vicinity of a corner point derived by Kuznetsov & Maz'ya (1989) (see also I). It shows that ∇u could be singular at $P_1, \ldots, P_4$, but the finite limits

$$u_x(P_i) \stackrel{\text{def}}{=} \lim_{x \to a_i - 0} u_x(x, 0), \text{ for } i = 1, 3 \quad \text{and} \quad u_x(P_i) \stackrel{\text{def}}{=} \lim_{x \to a_i + 0} u_x(x, 0) \text{ for } i = 2, 4$$

do exist.

Any solution of (2.1)–(2.6) (one can admit an arbitrary right-hand side term in (2.3)) has the following asymptotics as $|z| \to \infty$ ($z = x + iy$) :

$$u(x, y) = C + Q \log(\nu |z|) + H(-x) \mathrm{e}^{\nu y} (\mathcal{A} \sin \nu x + \mathcal{B} \cos \nu x) + \psi(x, y). \tag{2.7}$$

Here C is an arbitrary constant, H denotes the Heaviside function, and the estimates $\psi = O(|z|^{-1})$, $|\nabla\psi| = O(|z|^{-2})$ hold for the remainder term. To determine the constants Q, $\mathcal{A}$, $\mathcal{B}$ we have the following equalities:

$$\pi\nu Q + \sum_{i=1}^{4}(-1)^i u_x(P_i) = \nu \int_S \partial u/\partial n \, \mathrm{d}s. \tag{2.8}$$

$$\mathcal{A} = -2\Big\{ \int_S \Big[u\frac{\partial}{\partial n}(\mathrm{e}^{\nu y}\cos\nu x) - \frac{\partial u}{\partial n}\mathrm{e}^{\nu y}\cos\nu x \Big] \mathrm{d}s + \sum_{i=1}^{4}(-1)^i \big[\nu^{-1}u_x(P_i)\cos\nu a_i + u(P_i)\sin\nu a_i\big]\Big\}, \tag{2.9}$$

$$\mathcal{B} = 2\Big\{ \int_S \Big[u\frac{\partial}{\partial n}(\mathrm{e}^{\nu y}\sin\nu x) - \frac{\partial u}{\partial n}\mathrm{e}^{\nu y}\sin\nu x \Big] \mathrm{d}s + \sum_{i=1}^{4}(-1)^i \big[\nu^{-1}u_x(P_i)\sin\nu a_i - u(P_i)\cos\nu a_i\big]\Big\}, \tag{2.10}$$

The constant Q is equal (up to a certain factor) to the additional flux of fluid produced by tandem at infinity downstream; $\mathcal{A}$ and $\mathcal{B}$ are proportional to the amplitudes of sine and cosine waves behind the tandem. Since $\int_S \cos(n, x)\,\mathrm{d}s = 0$, then by (2.3) the right-hand side term vanishes in (2.8).

3. The resistanceless velocity potential and its existence

First we remind the formula from (I), § 5 expressing the total resistance to the forward motion of a tandem:

$$R = -\frac{\rho\nu}{4}\big(\mathcal{A}^2 + \mathcal{B}^2\big) - \frac{\rho}{2\nu}\sum_{i=1}^{4}(-1)^i \left[u_x(P_i)\right]^2. \tag{3.1}$$

Here ρ denotes the density of fluid. Since the first term in the right-hand side depends on wave amplitudes, it should be referred to as the wave resistance. This term coincides with the formula for the wave resistance of a totally submerged cylinder (see Kostyukov 1968). The coefficients $\mathcal{A}$ and $\mathcal{B}$ in the latter case do not contain the out of integral terms in (2.9) and (2.10). The second term in (3.1) is known as the spray resistance because it depends on the fluid behaviour at the bow and stern points $P_1, \dots, P_4$, which through (2.8) is related to the additional flux generated by bodies.

In the present work we consider the following statement, which obviously provides a *resistanceless* potential solving (2.1)–(2.6).

Definition 3.1. We say that u is a resistanceless potential if it satisfies (2.1)–(2.6), and the following supplementary conditions

$$u_x(P_1) - u_x(P_2) = 0, \qquad u_x(P_3) - u_x(P_4) = 0, \tag{3.2}$$

$$\mathcal{A} = 0, \qquad \mathcal{B} = 0 \tag{3.3}$$

hold.

In what follows the problem satisfied by the resistanceless potential will be referred to as Problem (R). Its solvability will be shown in the present section for all $\nu > 0$ except possibly for a sequence of values tending to infinity. The proof does not use the specific form of the right-hand side term in (2.3), and we replace it by the general Neumann condition

$$\partial u/\partial n = f \quad \text{on} \quad \operatorname{int} S \tag{3.4}$$

with arbitrary f from the Hölder space $C^{0,\alpha}(S),\ 0 < \alpha < 1$. The same is applicable to the supplementary conditions. In particular, any real values (generally depending on ν) can be prescribed for the functionals $\mathcal{A}$ and $\mathcal{B}$. However, we restrict ourselves by the homogeneous conditions (3.2) and (3.3) to make calculations simpler.

As in Kuznetsov & Maz'ya (1989) we apply the source method reducing Problem (R) to an integro-algebraic system for which the Fredholm alternative holds in a suitable Banach space. Then we show that the system has unique solution for sufficiently small ν. This allow to apply the theorem by Trofimov (1968) on the invertibility of an operator-function in Banach space depending analytically on a parameter (ν in our case) for proving the unique solvability of the integro-algebraic system (for all $\nu > 0$ except possibly for a sequence of values tending to infinity). The solvability (but not uniqueness) of Problem (R) follows from the solvability of the integro-algebraic system.

We seek a solution in the form

$$u(z) = (\mathcal{U}\mu)(z) + \sum_{i=1}^{4} \mu_i G(z, a_i). \tag{3.5}$$

Here G is Green's function (see Appendix A), μ_i are unknown real numbers and the single layer potential

$$(\mathcal{U}\mu)(z) = \int_S \mu(\zeta) G(z,\zeta)\, \mathrm{d}s_\zeta, \quad z \in \overline{\mathbb{R}^2_-}, \quad \zeta = \xi + i\eta$$

has an unknown real density μ belonging to the class $C^{0,\alpha}(\operatorname{int} S)$, $0 < \alpha < 1$, and to a Banach space $C_\kappa(S)$. The latter consists of continuous on $\operatorname{int} S$ functions, and

$$\|\mu\|_\kappa = \sup\{|y|^{1-\kappa}|\mu(z)| : \ z \in \operatorname{int} S\}, \quad 0 < \kappa < 1$$

defines the norm in this space.

For any unknowns the properties of G guarantee that the function (3.5) satisfies all relations of Problem (R) except for (3.4), (3.3) and (3.2).

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As in (I) substituting (3.5) into (3.4) one obtains the following integro-algebraic equation

$$-\mu(z) + (T\mu)(z) + 2\sum_{i=1}^{4} \mu_i (\partial G/\partial n_z)(z, a_i) = 2f(z), \quad z \in \operatorname{int} S, \tag{3.6}$$

where

$$(T\mu)(z) = 2\int_S \mu(\zeta)(\partial G/\partial n_z)(z,\zeta)\,\mathrm{d}s_\zeta$$

is not a compact operator in $C_\kappa(S)$. However, if κ satisfies the inequality

$$\kappa < \min_{i=1,2,3,4} \left[1 + \left|1 - 2\beta_i/\pi\right|\right]^{-1}, \tag{3.7}$$

then $|T| < 1$, which is sufficient for validity of the Fredholm theorems for $I - T$ in $C_\kappa(S)$. Here $|T|$ is the essential norm of T and I is the identity operator (see (I) for the details).

Using (3.2) and (3.3) we complement the equation (3.6) by an algebraic system for μ_i containing integral functionals of μ. By (A.5) the asymptotics of (3.5) at infinity has the following wave term:

$$-2\mathrm{e}^{\nu y}\left(\int_S \mu(\zeta)\mathrm{e}^{\nu\eta}\sin\nu(x-\xi)\,\mathrm{d}s_\zeta + \sum_{i=1}^{4}\mu_i \sin\nu(x-a_i)\right).$$

Comparing this with (2.7) and taking into account (3.3) we obtain:

$$\sum_{i=1}^{4}\mu_i\cos\nu a_i = -\int_S \mu(\zeta)\mathrm{e}^{\nu\eta}\cos\nu\xi\,\mathrm{d}s_\zeta, \quad \sum_{i=1}^{4}\mu_i\sin\nu a_i = -\int_S \mu(\zeta)\mathrm{e}^{\nu\eta}\sin\nu\xi\,\mathrm{d}s_\zeta. \tag{3.8}$$

From (3.2) we get

$$\sum_{i=1}^{4}\mu_i\left[G_x(a_{2\pm1}, a_i) - G_x(a_{3\pm1}, a_i)\right] = \int_S \mu(\zeta)\left[G_x(a_{3\pm1},\zeta) - G_x(a_{2\pm1},\zeta)\right]\mathrm{d}s_\zeta, \tag{3.9}$$

where $G_x(a_i, a_i)$ should be calculated with the help of (A.4). The equations (3.6), (3.8) and (3.9) constitute an integro-algebraic system for the unknown vector $X = (\mu, \mu_1, \ldots, \mu_4)^t$.

To prove the solvability theorem for this system (see Theorem 3.2 below) it is necessary to show that the Fredholm alternative holds for it. The suitable Banach space is $C_\kappa(S) \times \mathbb{R}^4$ supplied with norm $\max\{\|\mu\|_\kappa, |\mu_1|, \ldots, |\mu_4|\}$, and the system can be written as follows:

$$(-\mathcal{I} + \mathcal{K})\,X = V. \tag{3.10}$$

Here $V = (2f, 0, 0, 0, 0)^t$, $\mathcal{I}$ is the identity matrix operator, and $\mathcal{K}$ is the operator

$$\begin{bmatrix} T & N \\ L & A \end{bmatrix},$$

where $N = (N_1, N_2, N_3, N_4)$, $L = (L_1, L_2, L_3, L_4)^t$, $A = \{a_{ij}\}_{i,j=1}^4$. By N_i the operator of multiplication by $2(\partial G/\partial n_z)(z, a_i)$ is denoted, and

$$L_1\mu = \int_S \mu(\zeta)\mathrm{e}^{\nu\eta}\cos\nu\xi\,\mathrm{d}s, \quad L_2\mu = \int_S \mu(\zeta)\mathrm{e}^{\nu\eta}\sin\nu\xi\,\mathrm{d}s,$$
$$L_{(7\pm1)/2}\mu = \int_S \mu(\zeta)\big[G_x(a_{2\pm1},\zeta) - G_x(a_{3\pm1},\zeta)\big]\,\mathrm{d}s.$$

Also, $a_{ij} = f_{ij} + \delta_{ij}$, where δ_{ij} is the Kronecker delta and f_{ij} are defined as follows:

$$f_{1,j} = \cos\nu a_j, \quad f_{2,j} = \sin\nu a_j, \quad f_{(7\pm1)/2,j} = G_x(a_{2\pm1}, a_j) - G_x(a_{3\pm1}, a_j).$$

Now, to prove that under the condition (3.7) the Fredholm alternative holds for (3.10) in $C_\kappa(S)\times\mathbb{R}^4$ one has to apply the scheme of proof of Theorem 3.1 in (I), and we leave this to the reader.

As in (I) the system (3.6), (3.8) and (3.9) can be investigated in detail for small $\nu > 0$. We put $l_\pm = a_{3\pm1} - a_{2\pm1}$ (that is, l_+ (l_-) is the length of the front (back) body along the free surface), and begin with the following

LEMMA 3.1. For any $a > 0$, l_+, l_- and sufficiently small values $\nu > 0$ the equations (3.8), (3.9) are solvable with respect to μ_i, $i = 1, 2, 3, 4$.

Proof. Using (A.3) we obtain after some algebra the following asymptotic representation for the determinant of (3.8), (3.9):

$$\Delta = \nu^4(l_+ + l_-)\,\lambda(l_+, l_-, a) + O(\nu^5\log\nu) \quad \text{as} \quad \nu\to 0.$$

Here

$$\lambda(l_+, l_-, a) = -\,2a\log(2a) - (2a + l_+ + l_-)\log(2a + l_+ + l_-)$$
$$+\,(2a + l_+)\log(2a + l_+) + (2a + l_-)\log(2a + l_-).$$

It is obvious that $\lambda(l_+, 0, a) = 0$, and for $l_+ > 0$

$$\frac{\partial\lambda}{\partial l_-} = \log(2a + l_-) - \log(2a + l_+ + l_-) < 0.$$

Thus, $\lambda(l_+, l_-, a) < 0$ for $l_+, l_- > 0$, and hence, $\Delta \neq 0$ for sufficiently small ν, which completes the proof.

The solution of (3.8), (3.9) can be written in the form

$$\mu_i = \Delta^{-1}\int_S \mu(\zeta)\Delta_i(\zeta)\,\mathrm{d}s_\zeta \quad i = 1, 2, 3, 4,$$

where Δ_i is the determinant having the i-th column of the system (3.8), (3.9) matrix replaced by the following vector:

$$\big(-\mathrm{e}^{\nu\eta}\cos\nu\xi,\, -\mathrm{e}^{\nu\eta}\sin\nu\xi,\, G_x(a_1,\zeta) - G_x(a_2,\zeta),\, G_x(a_3,\zeta) - G_x(a_4,\zeta)\big)^t.$$

Substituting the formulae for μ_i into (3.6) we arrive at

LEMMA 3.2. For sufficiently small $\nu > 0$ (3.10) is equivalent to the equation

$$-\mu(z) + (T_\nu \mu)(z) = 2f, \quad z \in \operatorname{int} S, \tag{3.11}$$

where the operator T_ν is defined by

$$(T_\nu \mu)(z) = (T\mu)(z) + \frac{2}{\Delta} \int_S \mu(\zeta) \sum_{i=1}^{4} \Delta_i(\zeta) \frac{\partial G}{\partial n_z}(z, a_i)\, \mathrm{d}s_\zeta.$$

REMARK 3.1. By (A.1) and (A.2) the difference $T_\nu - T$ is a finite-dimensional operator in the space $C_\kappa(S)$. Therefore, the essential norms of T_ν and of T are equal, which implies that the Fredholm alternative holds for (3.11) if κ satisfies (3.7).

THEOREM 3.1. If κ satisfies (3.7), then for sufficiently small $\nu > 0$ equation (3.11) is uniquely solvable in $C_\kappa(S)$.

Proof. Let us consider the equation

$$-\mu(z) + (T_0 \mu)(z) = 2f, \quad z \in \operatorname{int} S, \tag{3.12}$$

where

$$(T_0 \mu)(z) = \frac{1}{\pi} \int_S \mu(\zeta) \frac{\partial}{\partial n_z} \big(\log|z - \overline{\zeta}| - \log|z - \zeta|\big)\, \mathrm{d}s_\zeta.$$

Extending μ and f to $S' = \{z : \ \overline{z} \in S\}$ as odd functions with respect to y, we note that (3.12) is the integral equation of the Neumann problem in the exterior domain outside $S \cup S'$. The unique solvability of (3.12) in $C_\kappa(S)$ was shown by Carleman (1916) under the condition (3.7). Thus, it suffices to demonstrate that the norm of $T_\nu - T_0$ is small in this space for positive ν close to zero.

From (A.1), (A.2) and Lemma 3.2 the kernel of $T_\nu - T_0$ has the form

$$\frac{\partial g}{\partial n_z}(z, \zeta) + \Delta^{-1} \sum_{i=1}^{4} \Delta_i(\zeta) \frac{\partial g}{\partial n_z}(z, a_i). \tag{3.13}$$

By (A.2) we have for all $\zeta \in S$:

$$(\partial g / \partial n_z)(z, \zeta) = -2 \cos(n_z, y) \nu \log \nu + O(\nu) \quad \text{as } \nu \to 0.$$

Then, (3.13) can be written for small ν as follows:

$$\big[2\pi^{-1} \cos(n_z, y) \nu \log \nu + O(\nu)\big] \Big[1 + \Delta^{-1} \sum_{i=1}^{4} \Delta_i(\zeta)\Big]. \tag{3.14}$$

Straightforward but rather lengthy analysis leads to the following asymptotic representation:

$$\sum_{i=1}^{4} \Delta_i(\zeta) = -\nu^4 (l_+ + l_-)\, \lambda(l_+, l_-, a) + O(\nu^5 \log \nu) \quad \text{as} \quad \nu \to 0.$$

Taking into account the asymptotics of Δ obtained in the proof of Lemma 3.1 we find that (3.13) is $O(\nu^2 \log^2 \nu)$ as $\nu \to 0$. This proves the theorem.

By Lemma 3.2 the system (3.6), (3.8) and (3.9) is equivalent to (3.11) for small ν. Then, Theorem 3.1 yields the following

Corollary 3.1. If κ satisfies (3.7), then for sufficiently small $\nu > 0$ the system (3.6), (3.8) and (3.9) is uniquely solvable in $C_\kappa(S) \times \mathbb{R}^4$.

Now, the general results on solvability can be proved in the same way as the similar theorem and corollary in (I). Thus, we only give the corresponding formulations and mention that the crucial points are application of Trofimov's (1968) theorem on the invertibility of operator-function (which uses the Fredholm alternative), and verifying that $\mu \in C^{0,\alpha}(S)$ based on results from Colton & Kress (1983).

Theorem 3.2. For all $\nu > 0$, except possibly for a sequence tending to infinity, (3.10) is uniquely solvable in $C_\kappa(S) \times \mathbb{R}^4$, where κ satisfies (3.7).

Corollary 3.2. For all values $\nu > 0$, except possibly for a sequence tending to infinity, Problem (R) has a solution for any $f \in C^{0,\alpha}(S)$.

4. On the uniqueness of the resistanceless potential

Here we prove that Problem (R) has no more than one solution for all $\nu > 0$ except for a certain discrete sequence of values. The method of proof was proposed by Kuznetsov & Maz'ya (1989), and is based on the fact that the solution to the original problem is unique if an "adjoint" problem is solvable. The supplementary conditions used by Kuznetsov & Maz'ya (1989) cancel all out of integral terms in Green's identity, which makes them well suited for defining a solvable adjoint problem by virtue of this identity. Since the supplementary conditions (3.3) are poorly adapted to removing the out of integral terms, we impose a geometrical restriction to make the same method applicable to Problem (R). Namely, we assume that S_+ and S_- are symmetric to each other about the y-axis. This allows to consider symmetric and antisymmetric solutions separately. In either case it is possible to define an appropriate "adjoint" Problem $(\mathrm{R}_1)/(\mathrm{R}_2)$, which is coupled with symmetric/antisymmetric solution by Green's formula containing only integral. The exact definition of these problems is given in Appendix B, where we also prove that they are solvable.

Let u be a solution to the homogeneous problem (2.1)–(2.6), (3.2) and (3.3). Since W is symmetric, u is a sum of even and odd functions with respect to x:

$$u(x,y) = u^{(s)}(x,y) + u^{(a)}(x,y),$$

where

$$2u^{(s)}(x,y) = u(x,y) + u(-x,y), \qquad 2u^{(a)}(x,y) = u(x,y) - u(-x,y).$$

It is obvious that $u^{(s)}$ and $u^{(a)}$ satisfy the homogeneous Problem (R). Moreover,

$$\begin{aligned} &u^{(s)}_x(P_3) + u^{(s)}_x(P_2) = 0, \quad u^{(s)}(P_4) - u^{(s)}(P_3) + u^{(s)}(P_2) - u^{(s)}(P_1) = 0, \\ &u^{(a)}_x(P_3) - u^{(a)}_x(P_2) = 0, \quad u^{(a)}(P_4) - u^{(a)}(P_3) - u^{(a)}(P_2) + u^{(a)}(P_1) = 0. \end{aligned} \tag{4.1}$$

According to (2.7), (3.2) and (3.3) we have:

$$u^{(s)}(x,y) = C + \psi^{(s)}(x,y), \quad u^{(a)}(x,y) = \psi^{(a)}(x,y) \quad \text{as} \quad |z| \to \infty. \tag{4.2}$$

Here $\psi^{(s,a)} = O(|z|^{-1})$, $|\nabla\psi^{(s,a)}| = O(|z|^{-2})$ and C is an arbitrary constant.

THEOREM 4.1. The problem (2.1)–(2.6), (3.2) and (3.3) for a symmetric tandem has at most one solution (up to a constant term) for all $\nu > 0$ except possibly a discrete sequence of values.

Proof. We have to show that $u^{(s)}$ and $u^{(a)}$ introduced above are constants (it is obvious that $u^{(a)} = 0$ in this case). According to Theorem B.1 (see Appendix B) Problem (R_1)/(R_2) is solvable for all $\nu > 0$, except possibly a sequence tending to infinity. Let ν be such a value of the parameter that either of these problems has a solution $u^{(1)}$ and $u^{(2)}$ for arbitrary Neumann data having the zero mean value on S.

Since (4.2) holds for $u^{(s)}$, we can use Lemma B.1 and get

$$\int_S u^{(s)} (\partial u^{(1)}/\partial n)\, \mathrm{d}s = 0,$$

because (4.1) and (B.1) imply that the right-hand side term vanishes in (B.5). The second factor in the integrand is an arbitrary function orthogonal to a constant, which yields that $u^{(s)} = \text{const}$ on S. Now, $u^{(s)} = \text{const}$ on W by the uniqueness theorem for the Cauchy problem for Laplace's equation.

Similar consideration leads to the conclusion that $u^{(a)} = 0$ on W, which completes the proof.

5. A non-trivial potential satisfying the homogeneous Problem (R)

In the previous section it was shown that if the tandem is symmetric, then the uniqueness theorem for Problem (R) is true for all $\nu > 0$ except possibly for a certain sequence. Here we demonstrate that the exceptional values of ν, admitting a non-trivial solution to the homogeneous Problem (R), do exist.

To construct examples we use the so-called inverse procedure, which replaces finding a solution to a given problem by determining physically reasonable fluid region for a given solution. We define the latter with the help of Green's function. Putting

$$u(z) = \pi\nu^{-1}\left[G_x(z;\pi/\nu) - G_x(z;-\pi/\nu)\right], \tag{5.1}$$

we obtain a solution to the homogeneous problem (2.1)–(2.6) for a surface-piercing tandem if at least one of the streamlines of the flow connects the x-axis on either side of one singular point and another streamline similarly surrounds the other singular point (we interpret these streamlines as rigid contours where the homogeneous Neumann condition holds). The streamlines are level lines of the streamfunction v, which is a harmonic conjugate to u and can be written as follows:

$$\begin{aligned} v(z) &= \int_0^\infty \frac{\cos k(x-\pi/\nu) - \cos k(x+\pi/\nu)}{k-\nu}\,\mathrm{e}^{ky}\,\mathrm{d}k \\ &= \mathrm{Re}\left\{\mathrm{e}^{-i\nu z}\left[\mathrm{Ei}\big(i\nu(z-\pi/\nu)\big) - \mathrm{Ei}\big(i\nu(z+\pi/\nu)\big)\right]\right\}. \end{aligned} \tag{5.2}$$

The second expression in terms of the exponential integral is a consequence of 8.212.5, Gradshteyn & Ryzhik (1980). The particular combination of dipoles (5.1) is chosen so as to cancel wave terms in the asymptotics of u, which is an immediate consequence of (A.5).

It follows from (5.2) that

$$v_y(x,y) - \nu v(x,y) = \frac{y}{y^2 + (x-\pi/\nu)^2} - \frac{y}{y^2 + (x+\pi/\nu)^2}. \tag{5.3}$$

Hence,

$$v_y(x,0) - \nu v(x,0) = 0, \quad \text{when} \quad x \neq \pm\pi/\nu, \tag{5.4}$$

and the derivative $v_y = u_x$ has the same value at points of intersection of a streamline enclosing one of the dipoles with the x-axis. Thus, taking such a contour as S_+ and a similar contour as S_-, we see that u delivers a solution to Problem (R) in the corresponding fluid domain W. The asymptotic behaviour of Ei implies that $v(z) \sim \pm \log|z \pm \pi/\nu|$ as $z \to \mp\pi/\nu$, and so u behaves like a vortex near singlar points as has been pointed out by Ursell (1981). Hence, the streamlines enclosing the dipoles do exist for sufficiently large values of v. These lines are close to semicircles which are the level lines of $\log|z \pm \pi/\nu|$.

Now, we shall investigate the family of streamlines in more details, and, in particular, prove that any streamline of positive level encloses one of the singular points as shown on fig. 2. Since $v(x, y)$ is an odd function with respect to x, we shall restrict ourselves only with positive values of x. Properties of harmonic functions yield that streamlines can have end-points either on the free surface or at infinity (see, for example, McIver 1996). It follows from the asymptotics of (5.1), that the nodal lines, that is, the loci where $v = 0$ are the only streamlines going to infinity. These lines divide the fluid domain into subdomains, each containing a family of contours with the same properties. Thus, we have to investigate the behaviour of the function $v(x, 0)$ and of the nodal lines of $v(x, y)$ in the quadrant $\{x > 0, y < 0\}$.

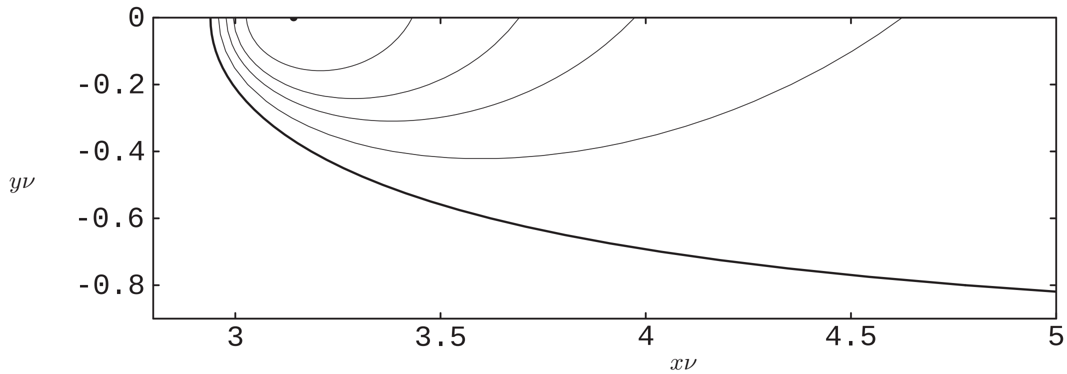

Figure 2. The streamlines $v = 0$ (bold line), 0.2, 0.4, 0.6, 1.0.

LEMMA 5.1. On the half-axis $x > 0$ the function $v(x, 0)$ has only one zero $\xi_0 \in (2\pi/3\nu, \pi/\nu)$.

Proof. Let us demonstrate that $v(x, 0) < 0$ for $x \in (0, 2\pi/3\nu]$. From 3.354.2 and 3.722.7, Gradshteyn & Ryzhik (1980), we get

$$\int_0^{+\infty} \frac{\cos ak}{k-1}\, \mathrm{d}k = \int_0^{+\infty} \frac{k\mathrm{e}^{-ak}}{1+k^2}\, \mathrm{d}k - \pi \sin a, \quad a > 0. \tag{5.5}$$

Then, for $x \in (0, \pi/\nu)$ we have $v(x, 0) = I(x) - 2\pi \sin x\nu$, where

$$I(x) = \int_0^{+\infty} \frac{k\left[\mathrm{e}^{(x\nu-\pi)k} - \mathrm{e}^{-(x\nu+\pi)k}\right]}{1+k^2}\, \mathrm{d}k.$$

It is obvious that $I'(x) > 0$, and hence, $I(x)$ is a non-negative increasing function. Then,

$$I(x) \le I(2\pi/3\nu) \le \mathrm{e}^{-1} \int_0^{+\infty} \frac{\mathrm{e}^{(1-\pi/3)k}}{1+k^2}\, \mathrm{d}k < \mathrm{e}^{-1} \int_0^{+\infty} \frac{\mathrm{d}k}{1+k^2} = \frac{\pi}{2\mathrm{e}},$$

because $\mathrm{e}^{k-1} \ge k$ for $k \ge 0$.

We define

$$x^* = \nu^{-1} \arcsin(1/4\mathrm{e}), \tag{5.6}$$

so that $2\pi \sin x\nu \ge \pi/2\mathrm{e}$ for $x \in [x^*, 2\pi/3\nu]$. Hence, $v(x, 0) \le 0$ on this interval. Now, let us prove that $v(x, 0) < 0$ for $x \in (0, x^*]$. Assuming the contrary and taking into account that $v(0, 0) = 0$, we note that $v_x(\xi, 0) = 0$ for some $\xi \in (0, x^*)$. This is impossible since

$$v_x(x, 0) < 2\pi\nu \left\{[\pi^2 - (x^*\nu)^2]^{-1} - \cos x^*\nu\right\} < 0,$$

which follows from

$$v_x(x, 0) = 2\pi\nu \left\{[\pi^2 - (x\nu)^2]^{-1} - \cos x\nu\right\} - \nu \int_0^{+\infty} \frac{\mathrm{e}^{(x\nu-\pi)k} + \mathrm{e}^{-(x\nu+\pi)k}}{1+k^2}\, \mathrm{d}k. \tag{5.7}$$

It is obvious that $v(x,0) \to +\infty$ as $x \to \pi/\nu$. Then, $v(x,0)$ vanishes at some point $\xi_0 \in (2\pi/3\nu, \pi/\nu)$, because $v(2\pi/3\nu, 0) < 0$.

Let us prove that the zero of $v(x,0)$ is unique. From (5.5)

$$v(x,0) = \int_0^{+\infty} \frac{k\left(\mathrm{e}^{(\pi - x\nu)k} - \mathrm{e}^{-(\pi + x\nu)k}\right)}{1+k^2}\,\mathrm{d}k > 0 \quad \text{for} \quad x > \pi/\nu.$$

Moreover, when $x\nu \in [2\pi/3, \pi)$ the inequalities $-\cos x\nu \geq 1/2$,

$$\int_0^{+\infty} \frac{\mathrm{e}^{(x\nu - \pi)k} + \mathrm{e}^{-(x\nu+\pi)k}}{1+k^2}\,\mathrm{d}k \leq \pi$$

and the formula (5.7) imply that

$$v_x(x,0) \geq 2\pi\nu[\pi^2 - (x\nu)^2]^{-1} > 0.$$

This completes the proof.

Lemma 5.2. On the half-axis $x > 0$ the function $v_x(x,0)$ has only one zero $\xi_1 \in (x^*, 2\pi/3\nu)$, where x^* is defined in (5.6).

Proof. In the proof of Lemma 5.1 it was shown that $v_x(x,0) > 0$ when $x \in [2\pi/3\nu, \pi/\nu)$ and $v_x(x,0) < 0$ when $x \in (0, x^*] \cup (\pi/\nu, +\infty)$. Thus, there is at least one zero ξ_1 of $v_x(x,0)$ and $\xi_1 \in (x^*, 2\pi/3\nu)$. To prove the assertion it is sufficient to show that $v_{xx} \neq 0$ in this interval. Differentiating (5.7) and comparing the result with $v(x,0)$ we get

$$v_{xx} = -\nu^2 v(x,0) + \frac{4\pi x\nu^3}{[\pi^2 - (x\nu)^2]^2}.$$

Since $v(x,0) < 0$ in $(x^*, 2\pi/3\nu)$, the inequality

$$v_{xx} > \frac{4\pi x\nu^3}{[\pi^2 - (x\nu)^2]^2} > 0$$

holds in this interval, which completes the proof.

The following corollary is an immediate consequence of Lemmas 5.1 and 5.2.

Corollary 5.1. The inequality $\xi_1 < \xi_0$ holds. The function $v(x,0)$ is negative for $0 < x < \xi_0$ and positive for $x > \xi_0$ and $x \neq \pi/\nu$. It decreases for $0 < x < \xi_1$ and $x > \pi/\nu$ and increases between ξ_1 and π/ν (see fig. 3a for the graph).

Let us describe the nodal streamlines. We remind that $v(x,y)$ is an odd function with respect to x, and hence, the negative y-axis is a nodal line. By Lemma 5.1 there is only one nodal line in $\{x > 0, y < 0\}$, emanating from $(\xi_0, 0)$. Its equation can be found as follows. From (5.3), we get

$$v(x,y) = \mathrm{e}^{\nu y}\left[v(x,0) - \int_y^0 \left\{\frac{t}{t^2 + (x+\pi/\nu)^2} - \frac{t}{t^2 + (x - \pi/\nu)^2}\right\}\mathrm{e}^{-t}\,\mathrm{d}t\right].$$

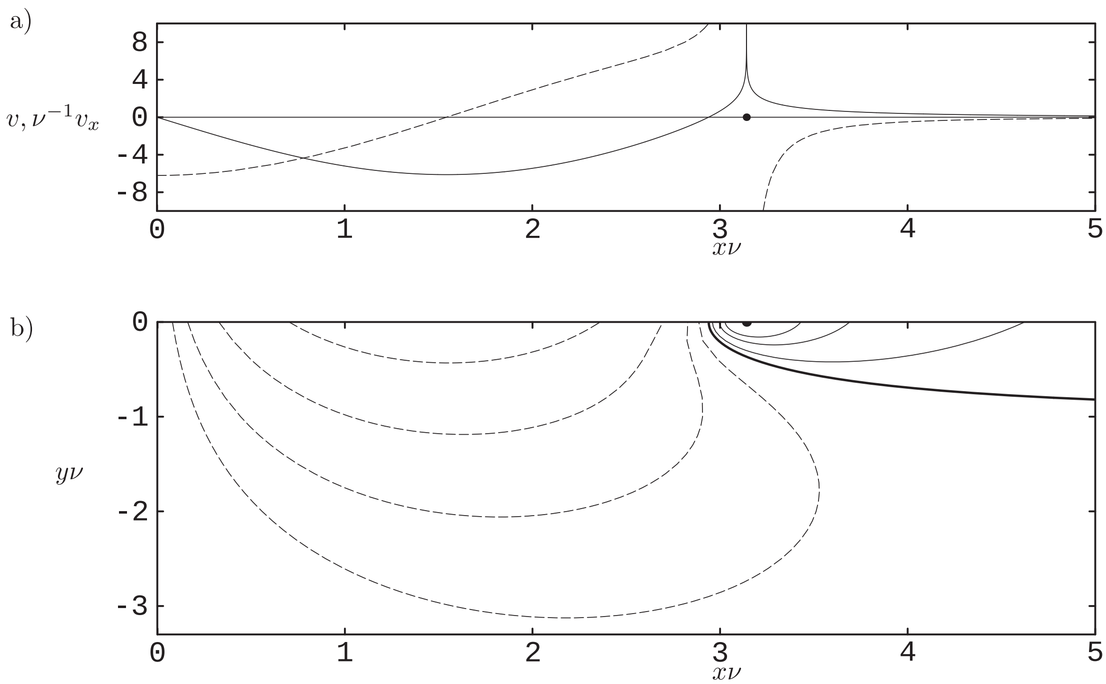

Figure 3. a) the values of v and $\nu^{-1}v_x$ (dashed line) on $y=0$; b) streamlines $v=-0.5$, -1.0, -2.0, -4.0 (dashed lines), $v=0$ (bold line) and $v=0.2$, 0.6, 1.0 (solid lines).

Hence,

$$v(x,0) = \int_y^0 \left\{ \frac{t}{t^2+(x+\pi/\nu)^2} - \frac{t}{t^2+(x-\pi/\nu)^2} \right\} \mathrm{e}^{-t}\,\mathrm{d}t$$

is the sought equation. The last integral is positive in the quadrant, and $v(x,0)>0$ for $x>\xi_0$. Thus, the nodal line emanating from $(\xi_0,0)$ lies below the interval $(\xi_0,+\infty)$ on the x-axis. It divides the quadrant in two domains each covered with a family of streamlines. By Corollary 5.1 the streamlines of one family, say Γ, correspond to positive values of v and have the left end-point in $(\xi_0,\pi/\nu)$ and the right end-point in $(\pi/\nu,+\infty)$ (solid lines in fig. 3b). The streamlines of negative level form the other family and have the right and left end-points in $(0,\xi_1)$ and (ξ_1,ξ_0) respectively (dashed lines in fig. 3b).

A streamline from Γ and a line from the family symmetric about the y-axis to Γ give a tandem, for which the potential (5.1) delivers an example of non-uniqueness to Problem (R).

REMARK 5.1. Let us consider an arbitrary streamline $v(x,y)=\mathrm{const}\neq 0$, having angles β_1 and β_2 with the free surface at the left (P_1) and the right (P_2) end-points respectively (cf fig. 1). Then,

$$\tan\beta_i = (-1)^i v_x(P_i)/v_y(P_i) = (-1)^i v_x(P_i)/\nu v(P_i), \quad i=1,2,$$

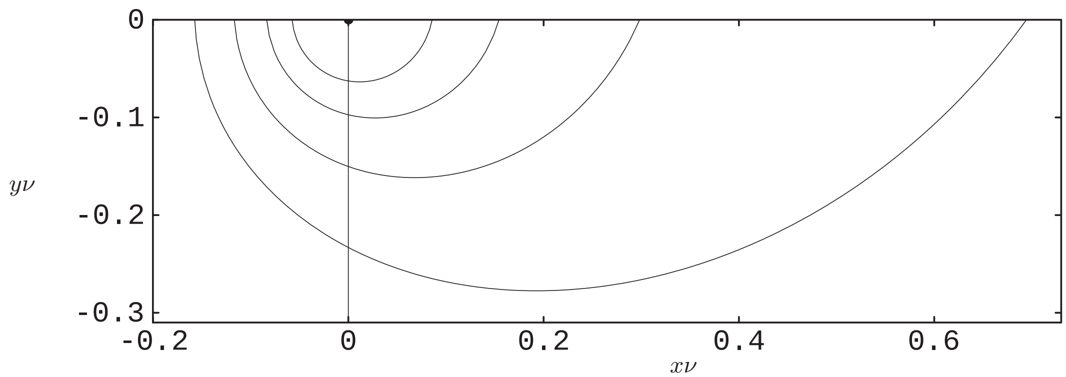

Figure 4. The streamlines $v^{(1)} = 0.5,\ 1.0,\ 1.5,\ 2.0$.

where the second equality is a consequence of (5.4). By Corollary 5.1 illustrated in fig. 3a the last expression is negative, and hence, $\beta_i > \pi/2$ (see fig. 3b).

It is known that the velocity field can generally be singular for a potential satisfying (2.1)–(2.6) in a flow domain having a non-acute corner between the free surface and rigid contour. This is a consequence of the corner local asymptotics (1.6) in Kuznetsov & Maz'ya (1989). To avoid such singularities Ursell (1981) and Kuznetsov & Maz'ya (1989) proposed the so-called least singular statement for a single surface-piercing body, but it can be easily generalized for any number of bodies with non-acute angles of intersection with the free surface. In Remark 5.1 we demonstrated that the streamlines defined by (5.2) have obtuse angles with the free surface. On the other hand, the potential (5.1) gives a non-singular velocity field everywhere outside the singular points. Thus, this potential delivers examples of non-uniqueness for the least singular statement when two, three or four bodies are presented by the streamlines corresponding to (5.2).

Remark 5.2. An example of non-uniqueness for the least singular statement does exist in the case of a single surface-piercing body. It is given by the following potential

$$u^{(1)}(z) = \pi\nu^{-1} G_x(z; 0).$$

The corresponding streamlines are plotted in fig. 4. Like (5.1) this potential satisfies a condition similar to (3.2), which implies that there is no spray resistance. However, in contrast to (5.1) the potential is not waveless.

6. Conclusion

The resistanceless velocity potential is shown to deliver a well-posed statement to the two-dimensional Neumann–Kelvin problem for a surface-piercing tandem when $\nu > 0$ does not belong to a certain dicrete sequence of possible exceptional values. Besides, a new type of non-uniqueness for the Neumann–Kelvin problem is described. The well-known kind of non-uniqueness follows from the fact that this boundary value problem is under-determined for surface-piercing bodies (see § 1), and occurs for all such bodies and all values of ν. The new type of non-uniqueness takes place only for special values of ν depending on the geometry. These values are point eigenvalues of the relevant pseudo-differential operator and are embedded in the continuous spectrum known to be $(0, +\infty)$. The corresponding modes of finite energy are usually referred to as trapped modes.

To construct trapped modes we use the Green's function x-derivatives, whereas McIver (1996) applies Green's function itself. The reason of using the derivatives is that they deliver an example to two statements (resitanceless and least singular) of the Neumann–Kelvin problem simultaneously. It can be shown that a potential involving two sources gives an example of non-uniqueness only for the least singular statement, but cannot satisfy the second pair of supplementary conditions in Problem (R).

There is no unique set of supplementary conditions vanishing the total resistance to the forward motion in the case of more than two surface-piercing bodies. At the same time, the least singular solution is naturally defined for any number of bodies. Some of the corresponding non-uniqueness examples are considered.

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Appendix A. The Green function of the Neumann-Kelvin problem

Here we summarize some known properties of the Green function (the two-dimensional Kelvin source), the corresponding references are given in (I). The best-known formula for Green's function is as follows:

$$G(z,\zeta) = -\,(2\pi)^{-1}\big[\log(\nu|z-\zeta|) + \log(\nu|z-\overline{\zeta}|) \\ + 2\int_0^\infty \frac{\cos\mu(x-\xi)}{\mu-\nu}\mathrm{e}^{\mu(y+\eta)}\,\mathrm{d}\mu + 2\pi\mathrm{e}^{\nu(y+\eta)}\sin\nu(x-\xi)\big],$$

where the integral is understood as the Cauchy principal value. Another useful representation is as follows:

$$G(z,\zeta) = -(2\pi)^{-1}\{\log|z-\zeta| - \log|z-\overline{\zeta}| + g(z,\zeta)\}, \tag{A.1}$$

where

$$g(z,\zeta) = -\,2\,\mathrm{Re}\Big\{\log(\nu|z-\overline{\zeta}|)\sum_{m=1}^{\infty}\frac{[-i\nu(z-\overline{\zeta})]^m}{m!} \\ + \exp\{-i\nu(z-\overline{\zeta})\}\Big(\gamma - \frac{\pi}{2}i + \sum_{m=1}^{\infty}\frac{[i\nu(z-\overline{\zeta})]^m}{m!m}\Big)\Big\}. \tag{A.2}$$

Here $\gamma = 0,5772\ldots$ is Euler's constant. From here it follows (cf Ursell 1981):

$$G_x(x,y;\xi,0) = \\ \frac{\nu}{\pi}\sum_{m=0}^{\infty}\frac{(-\nu r)^m}{m!}\left\{\left(\varphi-\frac{\pi}{2}\right)\cos m\left(\varphi+\frac{\pi}{2}\right) + \left[\log(\nu r) - \frac{\Gamma'(m+1)}{\Gamma(m+1)}\right]\sin m\left(\varphi+\frac{\pi}{2}\right)\right\}, \tag{A.3}$$

$$G_y(x,y;\xi,0) = \\ \frac{\nu}{\pi}\sum_{m=0}^{\infty}\frac{(-\nu r)^m}{m!}\left\{\left(\frac{\pi}{2}-\varphi\right)\sin m\left(\varphi+\frac{\pi}{2}\right) + \left[\log(\nu r) - \frac{\Gamma'(m+1)}{\Gamma(m+1)}\right]\cos m\left(\varphi+\frac{\pi}{2}\right)\right\},$$

where $re^{i\varphi} = (x-\xi)+iy$, $\varphi\in[-\pi,0]$. From the first formula (A.3) one immediately obtains that

$$\lim_{x\to\xi\pm 0} G_x(x,0;\xi,0) = \nu(-1\pm 1/2). \tag{A.4}$$

For $|\zeta| < C < \infty$ the following asymptotic formula holds:

$$G(z,\zeta) = -\pi^{-1}\log(\nu|z|) - H(-x)2\mathrm{e}^{\nu(y+\eta)}\sin\nu(x-\xi) + \psi(x,y), \tag{A.5}$$

where $\psi = O(|z|^{-1}),\ |\nabla\psi| = O(|z|^{-2})$ as $|z|\to\infty$.

Appendix B. Auxiliary Problems (R_1) and (R_2)

The proof of uniqueness theorem in § 4 follows that of Kuznetsov & Maz'ya (1989) and needs two auxiliary problems, which are formulated and investigated here.

Definition B.1. We say that $u^{(i)}$ $(i = 1, 2)$ is a solution to Problem (R_i), if it satisfies (2.1), (2.2), (3.4), (2.4)–(2.6), (3.2), and the following supplementary conditions hold:

$$u_x^{(i)}(P_3) + (-1)^i u_x^{(i)}(P_2) = 0,\ \ u^{(i)}(P_4) - u^{(i)}(P_3) + (-1)^i \left[u^{(i)}(P_2) - u^{(i)}(P_1)\right] = 0. \qquad \text{(B.1)}$$

The method used in § 3 for proof of the solvability of Problem (R) is also applicable for Problems (R_1) and (R_2). Let us seek a solution to Problem (R_i) in the form (3.5), where by $(\mu^{(i)}, \mu_1^{(i)}, \dots, \mu_4^{(i)})^t$ we denote the unknown vector. Substituting (3.5) into (B.1) we obtain the algebraic equations:

$$\begin{aligned} &\sum_{j=1}^{4} \mu_j^{(i)}[G_x(a, a_j) + (-1)^i G_x(-a, a_j)] + \int_S \mu^{(i)}(\zeta)[G_x(a, \zeta) + (-1)^i G_x(-a, \zeta)]\, \mathrm{d}s = 0, \\ &\sum_{j=1}^{4} \mu_j^{(i)}[G(a_4, a_j) - G(a_3, a_j) + (-1)^i G(a_2, a_j) - (-1)^i G(a_1, a_j)] \\ &+ \int_S \mu^{(i)}(\zeta)[G(a_4, \zeta) - G(a_3, \zeta) + (-1)^i G(a_2, \zeta) - (-1)^i G(a_1, \zeta)]\, \mathrm{d}s = 0. \end{aligned} \qquad \text{(B.2)}$$

Thus, the integro-algebraic system for Problem (R_i) consists of equations (3.6), (3.9) and (B.2).

In what follows we assume that $l_+ = l_- = l$, because this is the only case used in the proof of Theorem 4.1. In the same way as in the proof of Lemma 3.1 one finds that the algebraic part of the obtained system is solvable for any a, l and sufficiently small $\nu > 0$. Using (A.1)–(A.4) one finds that the asymptotics of the determinant $\Delta^{(i)}$ of (3.9), (B.2) has the form:

$$\Delta^{(1)} = \nu^6\big[-4al^2 + O(\nu \log \nu)\big],\ \ \Delta^{(2)} = \nu^5\big[8l\pi^{-1}\sigma(a, l) + O(\nu \log \nu)\big] \text{ as } \nu \to 0,$$

where

$$\sigma(a, l) = a \log(2a) - (2a + l) \log(2a + l) + (a + l) \log(2a + 2l).$$

It is obvious that $\Delta^{(1)} \neq 0$ for sufficiently small ν. Let us show that the same is true for $\Delta^{(2)}$. Since $\sigma(a, 0) = 0$ and

$$\partial\sigma/\partial l = \log(2a + 2l) - \log(2a + l) > 0,$$

$\sigma(a, l) > 0$ for $a, l > 0$, which completes the proof.

The solution of (3.9) and (B.2) has the form

$$\mu_j^{(i)} = \frac{1}{\Delta^{(i)}} \int_S \mu^{(i)}(\zeta) \Delta_j^{(i)}(\zeta)\, \mathrm{d}s_\zeta, \quad j = 1, 2, 3, 4$$

where $\Delta_j^{(i)}(\zeta)$ are the determinants arising when solving the system (3.9), (B.2). Substituting $\mu_j^{(i)}$ into the integral equation (3.6), we get

$$-\mu^{(i)}(z) + (T_\nu^{(i)} \mu^{(i)})(z) = 2f(z),\ \ z \in \operatorname{int} S,$$

$$(T_\nu^{(i)}\mu^{(i)})(z) = (T\mu)(z) + \frac{2}{\Delta^{(i)}}\int_S \mu^{(i)}(\zeta)\sum_{j=1}^{4}\Delta_j^{(i)}(\zeta)\frac{\partial G}{\partial n_z}(z, a_j)\,\mathrm{d}s_\zeta.$$

As in § 3 the crucial point in the proof of solvability of Problems (R_1) and (R_2) is to show that the norm of $T_\nu^{(i)} - T_0$ is small when ν is close to zero (cf the proof of Theorem 3.1). Again we restrict ourselves with consideration of the case $l_+ = l_- = l$. Similarly to (3.13) and (3.14) the norm of $T_\nu^{(i)} - T_0$ is small if

$$\frac{1}{\Delta^{(i)}}\sum_{j=1}^{4}\Delta_j^{(i)}(\zeta) \tag{B.3}$$

is bounded as $\nu \to 0$. Direct calculation shows that as $\nu \to 0$:

$$\sum_{j=1}^{4}\Delta_j^{(1)} = \nu^6\Big\{4l^2\pi^{-1}\Big[a\sum_{i=1}^{4}(-1)^i\varphi_i(\zeta) + l(\varphi_2(\zeta) - \varphi_3(\zeta))\Big] + O(\nu\log\nu)\Big\},$$

$$\sum_{j=1}^{4}\Delta_j^{(2)} = \nu^5\big[4l\pi^{-2}\sigma(a,l)(3\pi + \varphi_1(\zeta) + \varphi_4(\zeta)) + O(\nu\log\nu)\big],$$

where $\varphi_i(\zeta) = \arg(\zeta - a_i) \in [-\pi, 0]$. These formulae and the asymptotics for $\Delta^{(i)}$ demonstrate that (B.3) is bounded for small ν. In the same way as in § 3 we obtain

THEOREM B.1. For all values $\nu > 0$, except possibly for a sequence tending to infinity, Problems (R_1) and (R_2) have solutions for any $f \in C^{0,\alpha}(S)$.

Now we prove a lemma used in § 4.

LEMMA B.1. Let u be a resistanceless potential such that

$$u(x,y) = C + \psi(x,y) \quad \text{as } |z| \to \infty, \tag{B.4}$$

where ψ decays at infinity like the remainder term in (2.7). If $u^{(i)}$ satisfies Problem (R_i) with f having zero mean value over S, then

$$\int_S \left(u\frac{\partial u^{(i)}}{\partial n} - u^{(i)}\frac{\partial u}{\partial n}\right)\mathrm{d}s = \nu^{-1}\sum_{j=1}^{4}(-1)^j\big[u(P_j)u_x^{(i)}(P_j) - u^{(i)}(P_j)u_x(P_j)\big]. \tag{B.5}$$

Proof. The asymptotics (2.7) is true for the functions $u^{(i)}(x,y)$. Let $Q^{(i)}$, $\mathcal{A}^{(i)}$, $\mathcal{B}^{(i)}$ be the coefficients in this asymptotics. From (2.8) $Q^{(i)} = 0$, because of (3.2) and the orthogonality of f to constant. Without loss of generality we can cancel the constant term in the asymptotics of $u^{(i)}$:

$$u^{(i)}(x,y) = H(-x)\mathrm{e}^{\nu y}(\mathcal{A}^{(i)}\sin\nu x + \mathcal{B}^{(i)}\cos\nu x) + \psi^{(i)}(x,y) \quad \text{as } |z| \to \infty, \tag{B.6}$$

where $\psi^{(i)}$ has the same behaviour at infinity as ψ in (B.4).

Let $R_d = \{|x| < d, -d < y < 0\}$ be a rectangle containing $\overline{D}_+ \cup \overline{D}_-$. By p_0, p_d and $q_{\pm d}$ we denote the upper, lower, right and left sides of R_d respectively, $W_d = R_d \setminus (\overline{D}_+ \cup \overline{D}_-)$. By Green's formula

$$0 = \int_{W_d} (u^{(i)} \nabla^2 u - u \nabla^2 u^{(i)})\, \mathrm{d}x\mathrm{d}y = \int_{\partial W_d} \left(u \frac{\partial u^{(i)}}{\partial n} - u^{(i)} \frac{\partial u}{\partial n} \right) \mathrm{d}s,$$

where $\mathbf{n}$ is directed into W_d. Thus,

$$\int_S \left(u \frac{\partial u^{(i)}}{\partial n} - u^{(i)} \frac{\partial u}{\partial n} \right) \mathrm{d}s = \int_{\partial W_d \setminus (\overline{D}_+ \cup \overline{D}_-)} \left(u^{(i)} \frac{\partial u}{\partial n} - u \frac{\partial u^{(i)}}{\partial n} \right) \mathrm{d}s,$$

and we have to consider the integrals along straight segments whose sum gives the latter integral.

Substituting (B.4) and (B.6) into the integral over p_d we find after simple calculation that this integral is $O(d^{-1})$ as $d \to \infty$. Similarly, we establish that the same is true for the integral over q_d, and that the integral over q_{-d} is equal to

$$-C(\mathcal{A}^{(i)} \cos \nu d + \mathcal{B}^{(i)} \sin \nu d) + O(d^{-1}).$$

The boundary condition (2.2) yields that

$$\int_{p_0 \cap \partial W_d} \left(u^{(i)} \frac{\partial u}{\partial n} - u \frac{\partial u^{(i)}}{\partial n} \right) \mathrm{d}s = \nu^{-1} \left(\int_{-d}^{a_1} + \int_{a_2}^{a_3} + \int_{a_4}^{d} \right) \left[u^{(i)} u_{xx} - u u^{(i)}_{xx} \right]_{y=0} \mathrm{d}x.$$

Integrating by parts we find that the sum of integrals is equal to

$$\left[u^{(i)}(x,0) u_x(x,0) - u(x,0) u^{(i)}_x(x,0) \right]_{x=-d}^{x=d} - \sum_{j=1}^{4} (-1)^j \left[u(P_j) u^{(i)}_x(P_j) - u^{(i)}(P_j) u_x(P_j) \right].$$

According to the asymptotics of u and $u^{(i)}$ at infinity the first term in the last expression is equal to

$$C(\mathcal{A}^{(i)} \cos \nu d + \mathcal{B}^{(i)} \sin \nu d) + O(d^{-1}).$$

Summing up the asymptotics of the segment integrals and tending d to infinity we arrive at (B.5).

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