

Justification of the Kirchhoff hypotheses and error estimation for two-dimensional models of anisotropic and inhomogeneous plates, including laminated plates

BY O. V. MOTYGIN, S. A. NAZAROV

*Laboratory for Mathematical Modelling of Wave Phenomena,
Institute of Mechanical Engineering Problems, Russian Academy of Sciences,
V.O., Bol'shoy pr., 61, St. Petersburg 199178, Russia*

Asymptotic analysis of the problem describing deformation of a thin cylindric plate with clamped lateral side is performed. The problem is considered in the most general statement with the plate being laminated and consisting of arbitrary number of nonhomogeneous and anisotropic (21 elastic moduli) layers. Explicit integral representations of the differential operators which form the two-dimensional model of the plate are derived. In the case when the elastic moduli of each of the layers are constant, these integral representations turn into algebraic ones. The asymptotic procedure is justified with the help of a weighted inequality of Korn's type. The error estimates obtained give a rigorous mathematical proof of both Kirchhoff's hypotheses (kinematic and static) and shed a light to the well-known intrinsic inconsistency of the couple of the hypotheses.

Introduction

Numerous investigations related to the theory of thin plates are conventionally divided into two groups. As of the first group we classify the papers oriented to development of concrete mechanical applications of the theory. Within this group, based on the classical Kirchhoff hypotheses, the theory itself becomes heuristically evident and, therefore, does not need any justification at all. Being, for a long time, the only computational tool in engineering, such mechanical approaches have resulted in a plethora of practical formulae and numerical results. At the same time, there has been appearing a series of paradoxes originating from the well-known intrinsic inconsistency of Kirchhoff's couple of the kinematic and static hypotheses.

In contrast to the first group presented by papers whose number increases by hundreds annually, the second group still consists of a few publications devoted to estimation of the approximation accuracy of two-dimensional models of plates (see [Morgenstern 1959](#), [Shoikhet 1973, 1976](#), [Ciarlet & Kesavan 1981](#), [Destuynder 1981](#), [Leora et al. 1986](#), [Destuynder & Gruias 1995](#)) and asymptotic analysis of elasticity problems in thin domains (see [Gol'denveizer 1962](#), [Gol'denveizer & Kolos 1965](#), [Ciarlet & Destuynder 1979](#), [Nazarov 1982](#), [Caillerie 1984](#), [Sánchez-Palencia 1990](#), [Zorin 1987](#), [Nazarov 1995](#)).

Besides, we mention here the boundary layer phenomenon which, in mechanics, is known as “abnormal behaviour of stress-strain state near plate edges” since it cannot be described by the Kirchhoff hypotheses (see [Friedrichs & Dressler 1961](#), [Gol'denveizer & Kolos 1965](#), [Nazarov 1982](#), [Iosif'jan et al. 1982](#), [Gregory & Wan 1984](#), [Zorin & Nazarov 1989](#), [Nazarov 1996](#), [Dauge & Gruais 1995](#), [Dauge et al. 1998](#), [Mazja et al. 1991](#), Ch. 15,16 for the case of smooth boundary and [Nazarov 1991a](#), [1991b](#), [1992a](#), [1993](#) for plates with corners).

By virtue of estimation of the asymptotic accuracy the cited above papers confirm the Kirchhoff hypotheses and the classical two-dimensional model of homogeneous isotropic plates. Nevertheless, a rigorous justification of numerous engineering solutions for anisotropic and laminated plates is still an open question. In this respect there are the papers ([Caillerie 1984](#), [Panasenکو & Reztsov 1987](#), [Nazarov 1995](#)) where homogenization of plates with periodic elastic properties was performed. Since those results were not presented by explicit formulae, it is difficult to find correct mechanical analogies for them.

In the present work we undertake the asymptotic analysis of the elasticity problem in arbitrary anisotropic and inhomogeneous plates under the only condition that the elastic properties vary smoothly in longitudinal directions. We derive and justify two-dimensional models for such plates including laminated ones. By appealing to the obtained error estimates, we prove the Kirchhoff hypotheses and, hence, establish mathematical basis for the applied investigations mentioned above. We also use the asymptotic analysis to reveal origins of the Kirchhoff hypotheses inconsistency and formulate a rule which can help to avoid typical mistakes in employments of the hypotheses.

The paper is organized as follows. In Sect. 2 we describe a general procedure intended to construct asymptotics of solutions to the elasticity problem and based on polynomial property ([Nazarov 1995](#)) of the elasticity system (see also [Nazarov & Plamenevsky 1994](#), Ch. 5 and [Nazarov 1997a](#), which make use of this property to implement the theory of boundary layer for thin plates and rods). The procedure results in a system of three partial differential equations, the Dirichlet problem for which forms a two-dimensional model of plates. In Sect. 3 we check up the asymptotic accuracy of the introduced models and obtain estimates for remainders in asymptotic formulae for three-dimensional stress and displacement fields in the plate. Based on the weighted Korn's inequality (see [Shoikhet 1973](#), [Kondrat'ev & Oleinik 1988](#), [Nazarov 1992b](#), [Cioranescu et al. 1989](#), [Nazarov 1997c](#) and others) these estimates turn out to be asymptotically sharp. Finally, in Sect. 4 we prove the Kirchhoff hypotheses for arbitrary anisotropic and inhomogeneous plate and derive an integral representation for matrix coefficients of the differential operator in the resultant system defining the two-dimensional model. For a laminated plate this representation becomes explicit. The proof due to its generality mathematically substantiates many results of the mechanical approach (see, e.g. [Zorin 1987](#), where formulae equivalent to those in Sect. 4 were obtained).

We also mention the papers ([Reissner 1986](#), [Arnold & Falk 1996](#)) devoted to corrected classical models of plates and the papers ([Babuška 1992](#), [Babuška & Li 1991](#), [Schwab](#)

1996) dealing with hierarchic modeling of plates. Those approaches are mostly related to numerical methods for the elasticity problem on thin plates and lie far enough from the asymptotic analysis applied in the present paper. We note that the asymptotic precision of the models mentioned above coincides with the precision of the Kirchhoff model (see Theorem 3.3 and Corollary 3.4). The fact that it is impossible to find a correction which improves the usual accuracy in the framework of the two-dimensional models, has natural origin in the boundary layer phenomenon or the edge effects, implying change of type of the stress-strain state near the lateral side of a plate. Namely, the plane strain state replaces the plane stress state with simultaneous change of orientation. We emphasize that construction of the boundary layer may allow to obtain pointwise estimates of errors in calculation of stresses and displacements as was shown in (Nazarov 1997b) for thin rods.

1. Formulation of the problem

1. Domains and equations. Let Ω_h be the cylinder $\omega \times (-hH_-, hH_+)$, where ω is a domain in \mathbb{R}^2 and $\partial\omega$ is a simple, smooth, and closed contour. The cylinder Ω_h consists of N superposed layers

$$\Omega_h^n = \{x \in \Omega_h : hH_{n-1} < z < hH_n\},$$

where $x = (y, z)$, $y = (y_1, y_2)$, and

$$-H_- = H_0 < H_1 < \dots < H_{N-1} < H_N = H_+.$$

We assume that the elastic material of Ω_h has different properties in the layers Ω_h^n and we denote by \mathbf{a}^n corresponding Hooke's tensors of rank 4. Let \mathbf{a}^n depend on the "slow" variables $y \in \omega$ and the "rapid" variable $\zeta \in \Upsilon = (-H_-, H_+)$. The Hooke's tensor \mathbf{a} of the whole plate, where $\mathbf{a}(x) = \mathbf{a}^n(x)$ when $x \in \Omega_h^n$, is a piecewise smooth tensor having jumps on the interfaces $\Gamma_h^n = \omega \times \{hH_n\}$.

Let $\boldsymbol{\sigma} = (\sigma_{ij})$ and $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$, resp., be the stress and strain tensors related by the Hooke's law:

$$\sigma_{ij} = \sum_{k,\ell=1}^3 \mathbf{a}_{ijkl} \varepsilon_{k\ell}.$$

It is convenient here to use matrix/column notation. We introduce the strain and stress columns of height 6,

$$\begin{aligned} \boldsymbol{\varepsilon} &= (\varepsilon_{11}, \varepsilon_{22}, \alpha^{-1} \varepsilon_{12}, \alpha^{-1} \varepsilon_{13}, \alpha^{-1} \varepsilon_{23}, \varepsilon_{33})^t, \\ \boldsymbol{\sigma} &= (\sigma_{11}, \sigma_{22}, \alpha^{-1} \sigma_{12}, \alpha^{-1} \sigma_{13}, \alpha^{-1} \sigma_{23}, \sigma_{33})^t, \end{aligned} \tag{1.1}$$

where t means transposition and the factors $\alpha = 2^{-\frac{1}{2}}$ are introduced to equalize the natural norms of stress/strain columns and tensors. Then, the Hooke's law takes the form

$$\boldsymbol{\sigma} = A\boldsymbol{\varepsilon}.$$

Here A is a symmetric and positive definite matrix of size 6×6 . To connect A and \mathbf{a} we introduce two sets of indices such that $\bar{p} = 1,1;2,2;3,3$ corresponds to $p = 1;2;6$ and $\bar{q} = 1,2;1,3;2,3$ corresponds to $q = 3;4;5$. Then one can directly check the following presentations for components of the matrix A :

$$A_{pp} = \mathbf{a}_{\bar{p}\bar{p}}, \quad A_{qp} = \alpha^{-1} \mathbf{a}_{\bar{q}\bar{p}}, \quad A_{qq} = \alpha^{-2} \mathbf{a}_{\bar{q}\bar{q}} \quad (p, q = 1, \dots, 6).$$

We introduce a 3×6 -matrix of differential operators,

$$D(\nabla_x) = \begin{pmatrix} \partial_1 & 0 & \alpha \partial_2 & \alpha \partial_3 & 0 & 0 \\ 0 & \partial_2 & \alpha \partial_1 & 0 & \alpha \partial_3 & 0 \\ 0 & 0 & 0 & \alpha \partial_1 & \alpha \partial_2 & \partial_3 \end{pmatrix}, \quad \nabla_x = (\partial_1, \partial_2, \partial_3)^t, \quad \partial_j = \frac{\partial}{\partial x_j}.$$

Let us interpret a displacement vector u as the column $(u_1, u_2, u_3)^t$ in \mathbb{R}^3 . Taking into account the definition $\varepsilon_{jk}(u) = 2^{-1}(\partial_j u_k + \partial_k u_j)$ of cartesian components of the strain tensor ε , we get the formula for the strain column

$$\varepsilon(u) = D(\nabla_x)^t u. \quad (1.2)$$

Using the above notation, the elasticity problem in the plate Ω_h can be written as follows:

$$D(-\nabla_x)A(y, h^{-1}z)D(\nabla_x)^t u(x) = f(x), \quad x \in \Omega_h^1 \cup \dots \cup \Omega_h^N, \quad (1.3)$$

$$D(-e^3)A^1(y, -H_-)D(\nabla_x)^t u(x) = g^-(y), \quad x \in \Gamma_h^0, \quad (1.4)$$

$$D(e^3)A^N(y, H_+)D(\nabla_x)^t u(x) = g^+(y), \quad x \in \Gamma_h^N, \quad (1.5)$$

$$u(x) = 0, \quad x \in \Upsilon_h. \quad (1.6)$$

Here f stands for volume forces, g^\pm for tractions on the faces, whilst the condition (1.6) corresponds to the clamped lateral side $\Upsilon_h = \partial\omega \times (-hH_-, hH_+)$ of the plate. Furthermore, we complete the problem (1.3)–(1.6) with the intrinsic transmission conditions implying the displacement and normal stress vectors to be continuous on Γ_h^n ,

$$u^n(y, hH_n - 0) = u^{n+1}(y, hH_n + 0), \quad (1.7)$$

$$\begin{aligned} D(e^3)A^n(y, H_n - 0)D(\nabla_x)^t u^n(y, hH_n - 0) \\ = D(e^3)A^{n+1}(y, H_n + 0)D(\nabla_x)^t u^{n+1}(y, hH_n + 0), \quad y \in \omega. \end{aligned} \quad (1.8)$$

Here $n = 1, \dots, N-1$; u^n is the restriction of u on Ω_h^n and $v(H \pm 0)$ denotes one-sided limit of the function $z \mapsto v(z)$ as $z \rightarrow H \pm 0$.

Let us introduce a small parameter $\delta > 0$ and smooth the function $z \mapsto A(y, h^{-1}z)$ in the δ -neighbourhood of $z = hH_n$. From now on we consider the smoothed Hooke's matrix A_δ

which allows us to omit the conditions (1.7), (1.8) during intermediate calculations while in final formulae we perform the limit passage $\delta \rightarrow +0$. As verified, this yields correct formulae for a laminated plate. Nevertheless, we shall be forced to return back to piecewise smooth matrix A for a justification of asymptotic analysis performed in the formal way.

2. Formal asymptotics

1. The limit problem. Let introduce the rapid variable $\zeta = h^{-1}z$. Obviously, for a differentiable function $x \mapsto U(y, h^{-1}z)$, we have

$$D(\nabla_x)^t U(y, h^{-1}z) = (\mathbb{D}_y^t U(y, h^{-1}z) + h^{-1} \mathbb{D}_\zeta^t U(y, \zeta)) \Big|_{\zeta=h^{-1}z}, \quad (2.1)$$

where

$$\mathbb{D}_\zeta = D(0, 0, \partial_\zeta) = \begin{pmatrix} 0 & 0 & 0 & \alpha \partial_\zeta & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha \partial_\zeta & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_\zeta \end{pmatrix}, \quad (2.2)$$

$$\mathbb{D}_y = D(\partial_1, \partial_2, 0) = \begin{pmatrix} \partial_1 & 0 & \alpha \partial_2 & 0 & 0 & 0 \\ 0 & \partial_2 & \alpha \partial_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \partial_1 & \alpha \partial_2 & 0 \end{pmatrix}. \quad (2.3)$$

We denote by $\mathcal{L}(h, x, \partial_x)$, $\mathcal{B}^-(h, x, \partial_x)$ and $\mathcal{B}^+(h, x, \partial_x)$ the differential operators in the left-hand side of (1.3), (1.4) and (1.5), resp. Then, in view of (2.1) the operators can be represented as follows:

$$\begin{aligned} \mathcal{L}(h, x, \nabla_x) &= h^{-2} \mathcal{L}^0(y, \zeta, \partial_\zeta) + h^{-1} \mathcal{L}^1(y, \zeta, \nabla_y, \partial_\zeta) + h^0 \mathcal{L}^2(y, \zeta, \nabla_y), \\ \mathcal{B}^\pm(h, x, \nabla_x) &= h^{-1} \mathcal{B}^{0\pm}(y, \zeta, \partial_\zeta) + h^0 \mathcal{B}^{1\pm}(y, \zeta, \nabla_y). \end{aligned} \quad (2.4)$$

Here

$$\begin{aligned} \mathcal{L}^0 &= -\mathbb{D}_\zeta A \mathbb{D}_\zeta^t, \quad \mathcal{L}^1 = -\mathbb{D}_\zeta A \mathbb{D}_y^t - \mathbb{D}_y A \mathbb{D}_\zeta^t, \quad \mathcal{L}^2 = -\mathbb{D}_y A \mathbb{D}_y^t, \\ \mathcal{B}^{0\pm} &= \pm \mathbb{D}_1 A^\pm \mathbb{D}_\zeta^t, \quad \mathcal{B}^{1\pm} = \pm \mathbb{D}_1 A^\pm \mathbb{D}_y^t, \end{aligned} \quad (2.5)$$

and

$$\mathbb{D}_1 = D(0, 0, 1), \quad A^\pm = A(y, \pm H_\pm).$$

The principal (with respect to h) parts of the differential operators (2.4) form the limit problem

$$\mathcal{L}^0(y, \zeta, \partial_\zeta) \mathbf{U}(\zeta) = \mathbf{F}(\zeta), \quad \zeta \in (-H_-, H_+), \quad (2.6)$$

$$\mathcal{B}_\pm^0(y, \zeta, \partial_\zeta) \mathbf{U}(\zeta) = \mathbf{G}^\pm, \quad \zeta = \pm H_\pm. \quad (2.7)$$

This problem is but the Neumann problem for a system of ordinary differential equations in ζ with the parameter $y \in \omega$. The matrix \mathbb{D}_ζ is algebraic complete (see Nečas 1967) and, therefore, \mathcal{L}^0 is a formal positive operator which possesses the polynomial property (see Nazarov & Plamenevsky 1994, Nazarov 1995). This leads to the following conclusions. First, the problem (2.6), (2.7) has a solution if and only if there holds the compatibility condition

$$\int_{-H_-}^{H_+} \mathbf{F}(\zeta) d\zeta + \mathbf{G}^+ + \mathbf{G}^- = 0. \quad (2.8)$$

Moreover, its solution is defined up to an additive constant vector and, under the normalization condition

$$\int_{-H_-}^{H_+} \mathbf{U} d\zeta = 0, \quad (2.9)$$

the solution \mathbf{U} becomes unique and inherits the smoothness properties in y from the right-hand sides \mathbf{F} and \mathbf{G}^\pm .

2. Asymptotic ansatz. We assume that the functions in the right-hand side of (1.3)–(1.5) take the form

$$\begin{aligned} f(y, \zeta) &= h^{-1} f^0(y, \zeta) + h^0 \bar{f}^0(y) + \tilde{f}(y, z), \\ g^\pm(y) &= h^0 g^{0\pm}(y) + \tilde{g}^\pm(y), \end{aligned} \quad (2.10)$$

where the detached terms f^0 , $g^{0\pm}$ and \bar{f}^0 satisfy the conditions

$$\begin{aligned} \int_{-H_-}^{H_+} f_3^0(y, \zeta) d\zeta + g_3^{0+}(y) + g_3^{0-}(y) &= 0, \quad y \in \bar{\omega}, \\ \bar{f}_1^0(y) = \bar{f}_2^0(y) &= 0, \quad y \in \omega. \end{aligned} \quad (2.11)$$

As we prove by further considerations, the following asymptotic ansatz for a solution to the problem (1.3)–(1.5) is consistent with the decomposition (2.10) provided the conditions (2.11) and certain estimates for the remainders \tilde{f} and \tilde{g}^\pm hold true:

$$u(h, x) \sim \mathcal{U}(h, y, \zeta) = h^{-2} \mathcal{U}^{-2}(y) + h^{-1} \mathcal{U}^{-1}(y, \zeta) + h^0 \mathcal{U}^0(y, \zeta) + h^1 \mathcal{U}^1(y, \zeta). \quad (2.12)$$

Moreover, appealing to various arguments presented either in (Shoikhet 1973, Ciarlet & Destuynder 1979, Destuynder 1981, Leora et al. 1986, Sánchez-Palencia 1990, Nazarov

1995), or in other works for homogeneous plates, we keep the usual forms for the first couple of terms in (2.12), namely,

$$\mathcal{U}^{-2}(y) = w_3(y)e^3, \quad (2.13)$$

and

$$\mathcal{U}^{-1}(y, \zeta) = \sum_{i=1}^2 e^i (w_i(y) - \zeta \partial_i w_3(y)) \quad (2.14)$$

where $h^{-2}w_3(y)$ and $h^{-1}w_1(y)$, $h^{-1}w_2(y)$ imply unknown mean values of the deflection and longitudinal displacements in the point $y \in \omega$ of the middle cross-section of the plate.

REMARK 2.1. Since the problem (1.3)–(1.8) is linear, one can easily achieve the conditions (2.10) and (2.11) while multiplying the right-hand sides f and g^\pm by a normalization factor h^m . Estimates for the remainders \tilde{f} and \tilde{g}^\pm in (2.10) needed to justify the asymptotic representation (2.12), will be formulated in Sect. 4. At the same time, the equalities (2.11) prescribe that the longitudinal and transversal forces applied to the plate are of the orders h^{-1} and h^0 , resp. This assumption can be confirmed by everyday observation that it is easier to bend a plate, than to stretch it. Those primitive observations are reflected by the multipliers h^{-2} at w_3 and h^{-1} at w_1, w_2 as well.

3. Constructing the asymptotic terms. We substitute the expressions (2.4), (2.10) and (2.12) into the equations (1.3)–(1.5) and collect coefficients at the same powers of h . As a result, we obtain a recursive sequence of problems to define \mathcal{U}^m , the first of which is given by (2.6), (2.7).

The function (2.13) does not depend on ζ and, hence, satisfies the homogeneous problem (2.6), (2.7). The problem for \mathcal{U}^{-1} has the form

$$\begin{aligned} \mathcal{L}^0 \mathcal{U}^{-1} &= -\mathcal{L}^1 \mathcal{U}^{-2} \equiv \mathbb{D}_\zeta A \mathbb{D}_y^t \mathcal{U}^{-2}, \quad \zeta \in (-H_-, H_+), \\ \mathcal{B}_\pm^0 \mathcal{U}^{-1} &= -\mathcal{B}_\pm^1 \mathcal{U}^{-2} \equiv \mp \mathbb{D}_1 A^\pm \mathbb{D}_y^t \mathcal{U}^{-2}, \quad \zeta = \pm H_\pm. \end{aligned} \quad (2.15)$$

The function (2.14) satisfies (2.15) by virtue of the identity

$$\mathbb{D}_y^t e^3 = \sum_{i=1}^2 \mathbb{D}_\zeta^t \zeta e^i \frac{\partial}{\partial y_i} = \sum_{i=1}^2 \mathbb{D}_1^t e^i \frac{\partial}{\partial y_i} \quad (2.16)$$

which is of the permanent use throughout the paper.

According to (2.4), the function \mathcal{U}^0 is subject to the equations

$$\begin{aligned} \mathcal{L}^0 \mathcal{U}^0 &= -\mathcal{L}^1 \mathcal{U}^{-1} - \mathcal{L}^2 \mathcal{U}^{-2} \equiv \mathbb{D}_\zeta A \mathbb{D}_y^t \mathcal{U}^{-1} + \mathbb{D}_y A \{ \mathbb{D}_\zeta^t \mathcal{U}^{-1} + \mathbb{D}_y^t \mathcal{U}^{-2} \}, \quad \zeta \in (-H_-, H_+), \\ \mathcal{B}_\pm^0 \mathcal{U}^0 &= -\mathcal{B}_\pm^1 \mathcal{U}^{-1} \equiv \mp \mathbb{D}_1 A^\pm \mathbb{D}_y^t \mathcal{U}^{-1}, \quad \zeta = \pm H_\pm. \end{aligned} \quad (2.17)$$

By (2.16), (2.13) and (2.14), the sum in the curly brackets vanishes. Obviously,

$$\int_{-H_-}^{H_+} \mathbb{D}_\zeta \Psi(\zeta) d\zeta = \mathbb{D}_1 \int_{-H_-}^{H_+} \partial_\zeta \Psi(\zeta) d\zeta = \mathbb{D}_1 (\Psi(H_+) - \Psi(-H_-)). \quad (2.18)$$

When $\Psi = A\mathbb{D}_y^t \mathcal{U}^{-1}$, the latter identity yields the compatibility condition (2.8) for the problem (2.17).

Based on (2.14) simple algebraic transformations lead to the formula

$$\mathbb{D}_y^t \mathcal{U}^{-1} = \mathcal{Y}(\zeta) \mathcal{D}(\nabla_y)^t w, \quad (2.19)$$

where \mathcal{D} and \mathcal{Y} are 3×6- and 6×6-matrices defined as follows:

$$\mathcal{D}(\nabla_y) = \begin{pmatrix} \partial_1 & 0 & \alpha \partial_2 & 0 & 0 & 0 \\ 0 & \partial_2 & \alpha \partial_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \partial_1^2 & \alpha \partial_2^2 & \partial_1 \partial_2 \end{pmatrix}, \quad (2.20)$$

$$\mathcal{Y}(\zeta) = \begin{pmatrix} \mathbb{I} & -\alpha^{-1} \zeta \mathbb{I} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}, \quad (2.21)$$

while \mathbb{I} and \mathbb{O} , resp., denote the unit and zero matrices of size 3×3. Employing the above definitions we find the representation

$$\mathcal{U}^0(y, \zeta) = \mathcal{V}(y, \zeta) \mathcal{D}(\nabla_y)^t w(y), \quad (2.22)$$

where the 3×6-matrix $\mathcal{V} = (\mathcal{V}^1, \dots, \mathcal{V}^6)$ satisfies the problem

$$\begin{aligned} -\mathbb{D}_\zeta A \mathbb{D}_\zeta^t \mathcal{V} &= \mathbb{D}_\zeta A \mathcal{Y}, \quad \zeta \in (-H_-, H_+), \\ \pm \mathbb{D}_1 A^\pm \mathbb{D}_\zeta^t \mathcal{V} &= \mp \mathbb{D}_1 A^\pm \mathcal{Y}, \quad \zeta = \pm H_\pm. \end{aligned} \quad (2.23)$$

In other words, the columns $\mathcal{V}^1, \dots, \mathcal{V}^6$ solve the problem (2.6), (2.7) with special right-hand sides. We subject $\mathcal{V}^1, \dots, \mathcal{V}^6$ to the orthogonality conditions (2.9) and, hence, we make the matrix-function \mathcal{V} to be smooth in $y \in \bar{\omega}$. We also introduce the matrix

$$\mathcal{Z}(y, \zeta) = \mathbb{D}_\zeta^t \mathcal{V}(y, \zeta) + \mathcal{Y}(\zeta) \quad (2.24)$$

and mention that (2.23) is equivalent to

$$\begin{aligned} -\mathbb{D}_\zeta A(y, \zeta) \mathcal{Z}(y, \zeta) &= 0, \quad \zeta \in (-H_-, H_+), \\ \pm \mathbb{D}_1 A(y, \pm H_\pm) \mathcal{Z}(y, \pm H_\pm) &= 0. \end{aligned} \quad (2.25)$$

We consider the next problem in the sequence which is intended to find \mathcal{U}^1 , i.e.,

$$\begin{aligned}\mathcal{L}^0 \mathcal{U}^1 &= -\mathcal{L}^1 \mathcal{U}^0 - \mathcal{L}^2 \mathcal{U}^{-1} + f^0 \equiv \mathbb{D}_\zeta A \mathbb{D}_y^t \mathcal{U}^0 + \mathbb{D}_y A \mathcal{Z} \mathcal{D}^t w + f^0, \quad \zeta \in (-H_-, H_+), \\ \mathcal{B}_\pm^0 \mathcal{U}^1 &= -\mathcal{B}_\pm^1 \mathcal{U}^0 + g_\pm^0 \equiv \mp \mathbb{D}_1 A^\pm \mathbb{D}_y^t \mathcal{U}^0 + g_\pm^0, \quad \zeta = \pm H_\pm,\end{aligned}\quad (2.26)$$

Due to the identity (2.18) with $\Psi = A \mathbb{D}_y^t \mathcal{U}^0$ the compatibility conditions (2.8) for (2.26) takes the form

$$\int_{-H_-}^{H_+} (e^i)^t \mathbb{D}_y A(y, \zeta) \mathcal{Z}(y, \zeta) d\zeta \mathcal{D}^t w + \mathcal{F}_i(y) = 0 \quad (2.27)$$

where

$$\mathcal{F}_i(y) = \int_{-H_-}^{H_+} f_i^0(y, \zeta) d\zeta + g_i^{0+}(y) + g_i^{0-}(y). \quad (2.28)$$

It is worth noting that $\mathcal{F}_3(y) = 0$ by virtue of (2.11). Moreover, by employing (2.16) we conclude that in view of (2.25)

$$\begin{aligned}\int_{-H_-}^{H_+} (e^3)^t \mathbb{D}_y A(y, \zeta) \mathcal{Z}(y, \zeta) d\zeta &= \sum_{i=1}^2 \frac{\partial}{\partial y_i} \int_{-H_-}^{H_+} (e^i)^t \mathbb{D}_\zeta \zeta A(y, \zeta) \mathcal{Z}(y, \zeta) d\zeta \\ &= - \sum_{i=1}^2 \frac{\partial}{\partial y_i} \left\{ \int_{-H_-}^{H_+} (e^i)^t \zeta \mathbb{D}_\zeta A(y, \zeta) \mathcal{Z}(y, \zeta) d\zeta + (e^i)^t \zeta \mathbb{D}_\zeta A(y, \zeta) \mathcal{Z}(y, \zeta) \Big|_{\zeta=-H_-}^{\zeta=H_+} \right\} = 0.\end{aligned}$$

Thus, the equality (2.27) is satisfied identically for $i = 3$. Therefore, the relationships (2.27) give only two differential equations for the unknown vector-function w . The third equation is to be derived by examining the discrepancy of the asymptotic solution \mathcal{U} in the equations (1.3)–(1.5). By (2.4), (2.10) we arrive at

$$\begin{aligned}\mathcal{L}\mathcal{U} - f &= h^{-4} \mathcal{L}^0 \mathcal{U}^{-2} + h^{-3} (\mathcal{L}^0 \mathcal{U}^{-1} + \mathcal{L}^1 \mathcal{U}^{-2}) + h^{-2} (\mathcal{L}^0 \mathcal{U}^0 + \mathcal{L}^1 \mathcal{U}^{-1} + \mathcal{L}^2 \mathcal{U}^{-2}) \\ &\quad + h^{-1} (\mathcal{L}^0 \mathcal{U}^1 + \mathcal{L}^1 \mathcal{U}^0 + \mathcal{L}^2 \mathcal{U}^{-1} - f^0) + h^0 (\mathcal{L}^1 \mathcal{U}^1 + \mathcal{L}^2 \mathcal{U}^0 - \bar{f}^0) + h^1 \mathcal{L}^2 \mathcal{U}^1 - \tilde{f}.\end{aligned}$$

The coefficients of h^{-4}, \dots, h^{-1} are zero due to the definitions (2.13), (2.15), (2.17) and (2.26) of the functions $\mathcal{U}^{-2}, \dots, \mathcal{U}^1$. Thus, we have

$$\mathcal{L}\mathcal{U} - f = -h^0 f^1 + h^1 \mathcal{L}^2 \mathcal{U}^1 - \tilde{f}, \quad \mathcal{B}_\pm \mathcal{U} - g^\pm = -h^0 g^{1\pm} - \tilde{g}^\pm,$$

where

$$f^1 = \mathbb{D}_\zeta A \mathbb{D}_y^t \mathcal{U}^1 + \mathbb{D}_y A (\mathbb{D}_\zeta^t \mathcal{U}^1 + \mathbb{D}_y^t \mathcal{U}^0) + \bar{f}^0, \quad g^{1\pm} = \mp \mathbb{D}_1 A^\pm \mathbb{D}_y^t \mathcal{U}^1. \quad (2.29)$$

We assume the orthogonality condition

$$\int_{-H_-}^{H_+} f_3^1(y, \zeta) d\zeta + g_3^{1+}(y) + g_3^{1-}(y) = 0 \quad (2.30)$$

which is analogous to the first condition in (2.11). This, in principal, allows next term of the ansatz (2.12) to be constructed (see Mazja et al. 1991, Ch. 16). However, since we avoid here a discussion of junior terms of asymptotic expansion, we emphasize that (2.30) is crucial for justification of the asymptotic procedure in Sect. 4.

By (2.18) with $\Psi = A\mathbb{D}_y^t \mathcal{U}^1$ and (2.29), the condition (2.30) becomes

$$\int_{-H_-}^{H_+} (e^3)^t \mathbb{D}_y A (\mathbb{D}_\zeta^t \mathcal{U}^1 + \mathbb{D}_y^t \mathcal{U}^0) d\zeta + \bar{f}_3^0 H = 0 \quad (2.31)$$

where $H = H_+ + H_-$. Taking (2.16) and (2.26) into account we treat the left-hand side of (2.31) in the following way:

$$\begin{aligned} \int_{-H_-}^{H_+} (e^3)^t \mathbb{D}_y A (\mathbb{D}_\zeta^t \mathcal{U}^1 + \mathbb{D}_y^t \mathcal{U}^0) d\zeta &= \sum_{i=1}^2 \frac{\partial}{\partial y_i} \int_{-H_-}^{H_+} (e^i)^t \mathbb{D}_\zeta \zeta A (\mathbb{D}_\zeta^t \mathcal{U}^1 + \mathbb{D}_y^t \mathcal{U}^0) d\zeta \\ &= \sum_{i=1}^2 \frac{\partial}{\partial y_i} \left\{ - \int_{-H_-}^{H_+} (e^i)^t \zeta \mathbb{D}_\zeta A (\mathbb{D}_\zeta^t \mathcal{U}^1 + \mathbb{D}_y^t \mathcal{U}^0) d\zeta + (e^i)^t \zeta \mathbb{D}_\zeta A (\mathbb{D}_\zeta^t \mathcal{U}^1 + \mathbb{D}_y^t \mathcal{U}^0) \Big|_{\zeta=-H_-}^{\zeta=H_+} \right\} \\ &= \sum_{i=1}^2 \frac{\partial}{\partial y_i} \left\{ - \int_{-H_-}^{H_+} (e^i)^t \zeta \mathbb{D}_y A \mathfrak{Z} d\zeta \mathcal{D}^t w + \int_{-H_-}^{H_+} \zeta f_3^0(y, \zeta) d\zeta + (H_+ g_3^{0+} + H_- g_3^{0-}) \right\}. \end{aligned}$$

Finally, the condition (2.30) is equivalent to the equality

$$\int_{-H_-}^{H_+} \zeta \sum_{i=1}^2 \frac{\partial}{\partial y_i} (e^i)^t \mathbb{D}_y A \mathfrak{Z} d\zeta \mathcal{D}^t w + \mathcal{F}_3 = 0 \quad (2.32)$$

where

$$\mathcal{F}_3(y) = \bar{f}_3^0 H + \sum_{i=1}^2 \frac{\partial}{\partial y_i} \left\{ \int_{-H_-}^{H_+} \zeta f_3^0(y, \zeta) d\zeta + (H_+ g_3^{0+} - H_- g_3^{0-}) \right\}. \quad (2.33)$$

The equation (2.32) completes the resultant system (2.27) for w .

4. The resultant problem. Let us rewrite the system (2.27), (2.32) in a more convenient form. With this aim, we note that the matrix composed from lines

$$-(e^1)^t \mathbb{D}_y, \quad -(e^2)^t \mathbb{D}_y, \quad -\zeta \left((e^1)^t \frac{\partial}{\partial y_1} + (e^2)^t \frac{\partial}{\partial y_2} \right) \mathbb{D}_y$$

is equal to $\mathcal{D}(-\nabla_y) \mathcal{Y}(\zeta)^t$. Thus, the system in question takes the form

$$\mathcal{D}(-\nabla_y) \mathcal{M}(y) \mathcal{D}(\nabla_y)^t w(y) = \mathcal{F}(y), \quad y \in \omega, \quad (2.34)$$

where components of the vector $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)^t$ are given by (2.28) with $i = 1, 2$, and (2.33), \mathcal{M} is a matrix of size 6×6 ,

$$\mathcal{M}(y) = \int_{-H_-}^{H_+} \mathcal{Y}(\zeta)^t A(y, \zeta) \mathfrak{Z}(z, \zeta) d\zeta. \quad (2.35)$$

LEMMA 2.1. *For any $y \in \omega$ the matrix $\mathcal{M}(y)$ is symmetric and positive definite.*

Proof. By (2.35) and (2.25), we find

$$\mathcal{M}(y) = \int_{-H_-}^{H_+} \mathfrak{Z}(y, \zeta)^t A(y, \zeta) \mathfrak{Z}(y, \zeta) d\zeta,$$

i.e., \mathcal{M} is a Gram's matrix composed from the columns $\mathfrak{Z}^1, \dots, \mathfrak{Z}^6$ of (2.24) with the help of the inner product

$$\int_{-H_-}^{H_+} V(\zeta)^t A(y, \zeta) v(\zeta) d\zeta, \quad v, V \in L_2(-H_-, H_+)^6.$$

Since the left 3×3-block of \mathbb{D}_ζ and the lower 3×3-blocks in \mathcal{Y} vanish (see (2.2), (2.3) and (2.21)), linear independence of $\mathfrak{Z}^1, \dots, \mathfrak{Z}^6 \in L_2(-H_-, H_+)^6$ follows from the same property of the columns $\mathcal{Y}^1, \dots, \mathcal{Y}^6$. Thus, appealing to general properties of Gram's matrices completes the proof. ■

Based on the conditions (1.6), we supply the system (2.34) with the Dirichlet conditions

$$w(y) = 0, \quad \partial_\nu w_3(y) = 0, \quad y \in \partial\omega, \quad (2.36)$$

where ∂_ν means derivation along the inward normal ν .

The matrix (2.20) preserves the algebraic completeness (Nečas 1967) that makes the differential operator $\mathcal{D}(-\nabla_y)^t \mathcal{M}(y) \mathcal{D}(\nabla_y)$ to be formal positive (Nečas 1967) and to possess the polynomial property (Nazarov 1995, Nazarov & Plamenevsky 1994). Thus, we conclude

THEOREM 2.1. *For any right-hand side $\mathcal{F} \in H^{\ell-1}(\omega)^2 \times H^{\ell-2}(\omega)$ with $\ell \in \{1, 2, \dots\}$, the problem (2.34), (2.36) has the unique solution $w \in H^{\ell+1}(\omega)^2 \times H^{\ell+2}(\omega)$. There holds the estimate*

$$\|w; H^{\ell+1}(\omega)^2 \times H^{\ell+2}(\omega)\| \leq c \|\mathcal{F}; H^{\ell-1}(\omega)^2 \times H^{\ell-2}(\omega)\| \quad (2.37)$$

where c is independent of \mathcal{F} and w .

In the paper we use the notation $\|b; B\|$ for the norm of an element b of a Banach space B . Also, $H^\ell(\omega)$ with $\ell \in \{0, 1, \dots\}$ denotes the usual Sobolev space while $H^{-1}(\omega) = \mathring{H}^1(\omega)^*$ consists of functions Y of the form

$$Y = Y_0 - \nabla_y \cdot Y'; \quad Y' = (Y_1, Y_2) \in L_2(\omega)^2, \quad Y_0 \in L_2(\omega). \quad (2.38)$$

The norm in $H^{-1}(\omega)$ is equal to

$$\inf \{ \|Y_0; L_2(\omega)\| + \|Y'; L_2(\omega)\| \}$$

where the infimum is taken over all the representations (2.38). In what follows, the case $\ell = 1$ is of the most use.

3. Justification of the asymptotic procedure

1. Inequalities. We start with the formulation of a weighted inequality of Korn's type proved in (Nazarov 1992b) (see also Shoikhet 1973). In this section c and C denote positive constants independent of both the function u and the parameter $h \in (0, 1]$.

LEMMA 3.1. *Let $u \in H^1(\Omega_h)^3$ satisfy the Dirichlet conditions (1.6). There holds the Korn's inequality*

$$E(u, u; \Omega_h) \geq c |u|^2 \quad (3.1)$$

where $2^{-1}E(u, u; \Xi)$ means the elastic energy stored by the body Ξ ,

$$E(u, v; \Xi) = (AD(\nabla_x)^t u, D(\nabla_x)^t v)_{\Xi},$$

the weighted norm $|\cdot|$ in $H^1(\Omega_h)^3$ is defined by

$$\begin{aligned} |u|^2 = \int_{\Omega_h} \left\{ \sum_{i=1}^2 (|\nabla_y u_i|^2 + h^2 \rho_h^{-2} |\nabla_y u_3|^2 + h^2 \rho_h^{-2} |\partial_z u_i|^2 + \rho_h^{-2} |u_i|^2) \right. \\ \left. + |\partial_z u_3|^2 + h^2 \rho_h^{-4} |u_3|^2 \right\} dx, \quad \rho_h(y) = h + \text{dist}(y, \partial\omega). \end{aligned} \quad (3.2)$$

Following the standard way to prove inequalities for traces of functions on hyperplanes (see, e.g. Ladyzhenskaya 1973), from (3.1) and (3.2) we derive

LEMMA 3.2. *If $u \in H^1(\Omega_h)^3$, then*

$$h \sum_{n=0}^N \left\{ h^2 \|\rho_h^{-2} u_3; L_2(\Gamma_h^n)\|^2 + \sum_{i=1}^2 \|\rho_h^{-1} u_i; L_2(\Gamma_h^n)\|^2 \right\} \leq c |u|^2. \quad (3.3)$$

2. Smoothness assumptions. We suppose that in the representation (2.10)

$$f^0 \in L_2(\Omega_1)^3, \quad g^{0\pm} \in L_2(\omega)^3, \quad \bar{f}^0 = (0, 0, \bar{f}_3^0) \in H^{-1}(\omega)^3, \quad (3.4)$$

$$\bar{f}_3^0 = f_3^0 - \nabla_y \cdot f_3, \quad f_3^0 \in L_2(\omega), \quad f_3 = (f_3^1, f_3^2)^t \in L_2(\omega)^2. \quad (3.5)$$

In accordance with (3.2), (3.3) and (3.1) we put

$$\begin{aligned} \sum_{i=1}^2 \left\{ \|\rho_h \tilde{f}_i; L_2(\Omega_h)\| + h^{1/2} \sum_{\pm} \|\rho_h \tilde{g}_i^{\pm}; L_2(\omega)\| \right\} \\ + h^{-1} \left\{ \|\rho_h^2 \tilde{f}_3; L_2(\Omega_h)\| + h^{1/2} \sum_{\pm} \|\rho_h^2 \tilde{g}_3^{\pm}; L_2(\omega)\| \right\} = h^0 \tilde{\mathcal{N}}. \end{aligned} \quad (3.6)$$

By the assumption that $\tilde{\mathcal{N}}$ is of order h^0 , we express the smallness of the remainders in (3.6) with respect to the detached terms $h^{-1}f^0$, $h^0\tilde{f}^0$ and $h^0g^{0\pm}$. The inclusions (3.4) provide the

components f^0 , \tilde{f}^0 and $g^{0\pm}$ with the necessary differential properties and in what follows we use the notation

$$\mathcal{N} = \|f^0; L_2(\Omega_1)^3\| + \|g^{0\pm}; L_2(\omega)^3\| + \|\tilde{f}^0; H^{-1}(\omega)\|.$$

By (2.28) and (2.33), we have

$$\begin{aligned} \mathcal{F}_1, \mathcal{F}_2 &\in L_2(\omega), \quad \mathcal{F}_3 \in H^{-1}(\omega). \\ \|\mathcal{F}; L_2(\omega)^2 \times H^{-1}(\omega)\|^2 &\leq c \mathcal{N}. \end{aligned} \quad (3.7)$$

The estimates (3.7) and (2.37) in Theorem 2.1 furnish the relations

$$w_1, w_2 \in H^2(\omega), \quad w_3 \in H^3(\omega), \quad (3.8)$$

$$\|w; H^2(\omega)^2 \times H^3(\omega)\| \leq c \|\mathcal{F}; L_2(\omega)^2 \times H^{-1}(\omega)\| \leq C \mathcal{N}. \quad (3.9)$$

Let us relinquish the above convention on smoothing the matrix A . As a result, the terms \mathcal{U}^0 and \mathcal{U}^1 in (2.12) gain jumps of their derivatives at the points $z = H_n$; $n = 0, \dots, N-1$. Thus, in view of (2.13), (2.14) and (2.22) the formula (3.8) leads to the inclusions

$$\mathcal{U}^{-2} \in H^3(\omega \rightarrow H^2(\Delta))^3, \quad \mathcal{U}^{-1} \in H^2(\omega \rightarrow H^2(\Delta))^3, \quad \mathcal{U}^0 \in H^1(\omega \rightarrow \mathcal{H})^3, \quad (3.10)$$

where $H^s(\omega \rightarrow \mathcal{B})$ means the Sobolev space of abstract functions with the norm

$$\|v(y, \cdot); H^s(\omega \rightarrow \mathcal{B})\| = \left(\int_{\omega} \sum_{k=0}^s \|\nabla_y^k v(y, \cdot); \mathcal{B}\|^2 dy \right)^{\frac{1}{2}} \quad (3.11)$$

and

$$\begin{aligned} \mathcal{H} &= \{Y \in H^1(\Delta) : Y \in H^2(\Delta_n), n = 0, \dots, N-1\}, \\ \|Y; \mathcal{H}\| &= \left(\|Y; H^1(\Delta)\|^2 + \sum_{n=0}^{N-1} \|Y; H^2(\Delta_n)\|^2 \right)^{1/2}, \\ \Delta &= (-H_-, H_+), \quad \Delta_n = (H_n, H_{n-1}), \quad \overline{\Delta} = \overline{\Delta_0} \cup \dots \cup \overline{\Delta_{N-1}}. \end{aligned}$$

By (2.14) and (2.22), $\mathbb{D}_y^t \mathcal{U}^0, \mathbb{D}_y A (\mathbb{D}_\zeta^t \mathcal{U}^0 + \mathbb{D}_y^t \mathcal{U}^{-1}) \in L_2(\Omega_1)$ and, due to (2.26), there holds the inclusion

$$\mathcal{U}^1 \in L_2(\omega \rightarrow \mathcal{H})^3. \quad (3.12)$$

We emphasize that the norm of the vectors (3.10) and (3.12) are majorized by $\text{const } \mathcal{N}$.

3. The approximation solution and its discrepancy. The function \mathcal{U}^1 does not possess sufficient smoothness in y while \mathcal{U}^0 does not satisfy the condition (1.6) on the lateral side $\Upsilon_h = \partial\omega \times (-hH_-, hH_+)$ of the plate. Hence, instead of \mathcal{U} in (2.12) we should introduce the asymptotic solution

$$\mathcal{U}'(h, x) = h^{-2}\mathcal{U}^{-2}(y) + h^{-1}\mathcal{U}^{-1}(y, \zeta) + h^0\chi(y)\mathcal{U}^0(y, \zeta) \quad (3.13)$$

which does not contain the term \mathcal{U}^1 . Here we use standard cut-off function $\chi_0 \in C^\infty[0, \infty)$, where $\chi_0(t) = 1$ for $t \leq \frac{1}{2}$, $\chi_0(t) = 0$ for $t \geq 1$ and $0 \leq \chi_0(t) \leq 1$ for $t \in (\frac{1}{2}, 1)$, and we put

$$\chi(y) = 1 - \chi_0(h^{-1} \text{dist}\{y, \partial\omega\}), \quad \chi \in C_0^\infty(\omega). \quad (3.14)$$

The presence of χ in (3.13) and the conditions (2.36) guarantee that \mathcal{U}' satisfies the homogeneous Dirichlet conditions (1.6) on the lateral side mentioned above. Thus, according to (3.10),

$$\mathcal{U}' \in \mathring{H}^1(\Omega_h, \Upsilon_h)^3 = \{u \in H^1(\Omega_h)^3 : u = 0 \text{ on } \Upsilon_h\}.$$

We consider the discrepancy which the solution \mathcal{U}' leaves in the equations (1.3)–(1.5). Using (2.4), (2.10) and (3.13) we get

$$\begin{aligned} \mathcal{L}\mathcal{U}' - f &= h^{-4}\mathcal{L}^0\mathcal{U}^{-2} + h^{-3}(\mathcal{L}^0\mathcal{U}^{-1} + \mathcal{L}^1\mathcal{U}^{-2}) + h^{-2}(\mathcal{L}^0\chi\mathcal{U}^0 + \mathcal{L}^1\mathcal{U}^{-1} + \mathcal{L}^2\mathcal{U}^{-2}) \\ &\quad + h^{-1}(\mathcal{L}^1\chi\mathcal{U}^0 + \mathcal{L}^2\mathcal{U}^{-1} - f^0) + h^0(\mathcal{L}^2\chi\mathcal{U}^0 - \bar{f}^0) + \tilde{f}. \end{aligned}$$

The first two terms in the right-hand side of the last equality vanish due to (2.8) and (2.13). Thus, taking into account (2.26), we arrive at

$$\begin{aligned} \mathcal{L}\mathcal{U}' - f &= h^{-2}\mathcal{L}^0(\chi - 1)\mathcal{U}^0 + h^{-1}\mathcal{L}^1(\chi - 1)\mathcal{U}^0 + h^0\mathcal{L}^2(\chi - 1)\mathcal{U}^0 \\ &\quad - h^{-1}\mathcal{L}^0\mathcal{U}^1 + h^0(\mathcal{L}^2\mathcal{U}^0 - \bar{f}^0) + \tilde{f} \\ &= \mathcal{L}(\chi - 1)\mathcal{U}^0 - h^{-1}\mathcal{L}^0\mathcal{U}^1 + h^0(\mathcal{L}^2\mathcal{U}^0 - \bar{f}^0) + \tilde{f}. \end{aligned} \quad (3.15)$$

Similar consideration leads to the following presentation of the discrepancy of \mathcal{U}' in the boundary conditions (1.4) and (1.5):

$$\mathcal{B}^\pm\mathcal{U}' - g^\pm = \mathcal{B}^\pm(\chi - 1)\mathcal{U}^0 - h^0\mathcal{B}^{0\pm}\mathcal{U}^1 + \tilde{g}^\pm. \quad (3.16)$$

Analogously, the transmission conditions (1.8) on Γ_h^n , $n = 1, \dots, N - 1$ turn into the following:

$$\begin{aligned} &\mathcal{B}^n\mathcal{U}'(y, hH_n - 0) - \mathcal{B}^{n+1}\mathcal{U}'(y, hH_n + 0) \\ &= \mathcal{B}^n(\chi - 1)\mathcal{U}^0(y, hH_n - 0) - \mathcal{B}^{n+1}(\chi - 1)\mathcal{U}^0(y, hH_n + 0) \\ &\quad - h^0\mathcal{B}^{0,n}\mathcal{U}^1(y, hH_n - 0) + h^0\mathcal{B}^{0,n+1}\mathcal{U}^1(y, hH_n + 0), \end{aligned} \quad (3.17)$$

where, in analogy with (2.5),

$$\mathcal{B}^n = \mathbb{D}_1 A^n D(\nabla_x)^t, \quad \mathcal{B}^{0,n} = \mathbb{D}_1 A^n \mathbb{D}_\zeta^t.$$

We introduce the difference $\mathcal{R} = \mathcal{U}' - u$ and suppose for a while that all functions under consideration are smooth in $y \in \bar{\omega}$. This assumption does not bring a loss of generality. Indeed, for the final completion in $H^1(\Omega_h)^3$, the right-hand sides \bar{f}^0 , \bar{f}_3^0 , \tilde{f} , etc., can be approximated by smooth ones. The latter makes correct the integrations by parts, which we shall use in the sequel. We stress that in order to perform the completion mentioned above, we do not need to pay any attention to the small parameter h , because no estimation of that type is required.

Since \mathcal{R} vanishes on Υ_h , by (1.3)–(1.5) and (3.15)–(3.17) we arrive at the equality

$$E(\mathcal{R}, \mathcal{R}; \Omega_h) = (\mathcal{L}\mathcal{R}, \mathcal{R})_{\Omega_h} + \sum_{\pm} (\mathcal{B}^{\pm}\mathcal{R}, \mathcal{R})_{\Gamma_h^{\pm}} + \sum_{i=1}^{N-1} \left\{ (\mathcal{B}^i\mathcal{R}, \mathcal{R})_{\Gamma_h^{i-}} - (\mathcal{B}^{i+1}\mathcal{R}, \mathcal{R})_{\Gamma_h^{i+}} \right\},$$

where $\Gamma_h^{\pm} = \omega \times \{hH_i \pm 0\}$. Then, using (3.15), (3.16) and (3.17) we obtain

$$\begin{aligned} E(\mathcal{R}, \mathcal{R}; \Omega_h) &= \left(\mathcal{L}(\chi - 1)\mathcal{U}^0 - h^{-1}\mathcal{L}^0\mathcal{U}^1 + h^0\mathcal{L}^2\mathcal{U}^0 - \bar{f}^0 + \tilde{f}, \mathcal{R} \right)_{\Omega_h} \\ &+ \sum_{\pm} \left(\mathcal{B}^{\pm}(\chi - 1)\mathcal{U}^0 - h^0\mathcal{B}^{0\pm}\mathcal{U}^1 + \tilde{g}^{\pm}, \mathcal{R} \right)_{\Gamma_h^{\pm}} \\ &+ \sum_{i=1}^{N-1} \left\{ \left(\mathcal{B}^i(\chi - 1)\mathcal{U}^0 - h^0\mathcal{B}^{0,i}\mathcal{U}^1, \mathcal{R} \right)_{\Gamma_h^{i-}} - \left(\mathcal{B}^{i+1}(\chi - 1)\mathcal{U}^0 - h^0\mathcal{B}^{0,i+1}\mathcal{U}^1, \mathcal{R} \right)_{\Gamma_h^{i+}} \right\}. \end{aligned}$$

We rewrite the equality in the form

$$E(\mathcal{R}, \mathcal{R}; \Omega_h) = I_1 - I_2 + I_3 + \tilde{I},$$

where I_1 , I_2 , I_3 and \tilde{I} are defined as follows

$$\begin{aligned} I_1 &= (\mathcal{L}(\chi - 1)\mathcal{U}^0, \mathcal{R})_{\Omega_h} + \sum_{\pm} (\mathcal{B}^{\pm}(\chi - 1)\mathcal{U}^0, \mathcal{R})_{\Gamma_h^{\pm}} \\ &+ \sum_{i=1}^{N-1} \left\{ (\mathcal{B}^i(\chi - 1)\mathcal{U}^0, \mathcal{R})_{\Gamma_h^{i-}} - (\mathcal{B}^{i+1}(\chi - 1)\mathcal{U}^0, \mathcal{R})_{\Gamma_h^{i+}} \right\}, \\ I_2 &= h^{-1} (\mathcal{L}^0\mathcal{U}^1, \mathcal{R})_{\Omega_h} + \sum_{\pm} (\mathcal{B}^{0\pm}\mathcal{U}^1, \mathcal{R})_{\Gamma_h^{\pm}} + \sum_{i=1}^{N-1} \left\{ (\mathcal{B}^{0,i}\mathcal{U}^1, \mathcal{R})_{\Gamma_h^{i-}} - (\mathcal{B}^{0,i+1}\mathcal{U}^1, \mathcal{R})_{\Gamma_h^{i+}} \right\}, \\ I_3 &= (\mathcal{L}^2\mathcal{U}^0 - \bar{f}^0, \mathcal{R})_{\Omega_h}, \quad \tilde{I} = (\tilde{f}, \mathcal{R})_{\Omega_h} + \sum_{\pm} (\tilde{g}^{\pm}, \mathcal{R})_{\Gamma_h^{\pm}}. \end{aligned} \tag{3.18}$$

Integrating by parts cancels surface integrals in (3.18) and turns I_q into

$$\begin{aligned} I_1 &= (AD(\nabla_x)^t(\chi - 1)\mathcal{U}^0, D(\nabla_x)^t\mathcal{R})_{\Omega_h}, \quad I_2 = h^{-1} (A\mathbb{D}_\zeta^t\mathcal{U}^1, \mathbb{D}_\zeta^t\mathcal{R})_{\Omega_h} \\ I_3 &= (A\mathbb{D}_y^t\mathcal{U}^0, \mathbb{D}_y^t\mathcal{R})_{\Omega_h} - (\bar{f}_3^0, \mathcal{R}_3)_{\Omega_h}. \end{aligned}$$

Since \bar{f}^0 does not depend on ζ ,

$$(\bar{f}_3^0, \mathcal{R}_3)_{\Omega_h} = H (\bar{f}_3^0, R_3)_\omega, \quad \text{where } R_3(y) = \frac{1}{H} \int_{-H_-}^{H_+} \mathcal{R}_3(y, \zeta) d\zeta \quad \text{and } H = H_+ + H_-.$$

Hence, multiplying the orthogonality condition (2.31) by $R_3(y)$ and integrating by parts in ω we arrive at the equality

$$(A (\mathbb{D}_\zeta^t \mathcal{U}^1 + \mathbb{D}_y^t \mathcal{U}^0), \mathbb{D}_y^t e^3 R_3)_{\Omega_h} = (\bar{f}_3^0, \mathcal{R}_3)_{\Omega_h}.$$

Then, after some algebraic manipulations, we obtain

$$E(\mathcal{R}, \mathcal{R}; \Omega_h) = J_1 + J_2 + J_3 + \tilde{J}, \quad (3.19)$$

where $\tilde{J} = \tilde{I}$ and

$$\begin{aligned} J_1 &= (AD(\nabla_x)^t(\chi - 1)\mathcal{U}^0, D(\nabla_x)^t \mathcal{R})_{\Omega_h}, \\ J_2 &= -I_2 - (A\mathbb{D}_\zeta^t \mathcal{U}^1, \mathbb{D}_y^t \mathcal{R})_{\Omega_h} = - (A\mathbb{D}_\zeta^t \mathcal{U}^1, D(\nabla_x)^t \mathcal{R})_{\Omega_h}, \\ J_3 &= I_3 + (A\mathbb{D}_\zeta^t \mathcal{U}^1, \mathbb{D}_y^t \mathcal{R})_{\Omega_h} = (A (\mathbb{D}_\zeta^t \mathcal{U}^1 + \mathbb{D}_y^t \mathcal{U}^0), \mathbb{D}_y^t (\mathcal{R} - e^3 R_3))_{\Omega_h}. \end{aligned} \quad (3.20)$$

We stress that the inclusions $\mathcal{R} \in H^1(\Omega_h)^3$ and (3.10), (3.11) ensure that all multipliers in the scalar products (3.20) belong to $L_2(\Omega_h)$. Thus, by the completion in $H^1(\Omega_h)^3$ we can now remove the above assumption on the smoothness in y .

4. Estimation of the discrepancies. Let us evaluate the terms in the right-hand side of (3.19). Positive definiteness of the matrix A yields the inequality

$$E(\mathcal{R}, \mathcal{R}; \Omega_h) = (AD(\nabla_x)^t \mathcal{R}, D(\nabla_x)^t \mathcal{R}) \geq c \|D(\nabla_x)^t \mathcal{R}; L_2(\Omega_h)\|^2. \quad (3.21)$$

Now,

$$|J_1| \leq c E(\mathcal{R}, \mathcal{R}; \Omega_h)^{\frac{1}{2}} \|D(\nabla_x)^t(\chi - 1)\mathcal{U}^0; L_2(\Omega_h)\|. \quad (3.22)$$

Furthermore,

$$D(\nabla_x)^t(\chi - 1)\mathcal{U}^0 = (\mathbb{D}_y^t \chi)\mathcal{U}^0 + h^{-1}(\chi - 1)\mathbb{D}_\zeta^t \mathcal{U}^0 + (\chi - 1)\mathbb{D}_y^t \mathcal{U}^0. \quad (3.23)$$

By virtue of (3.10) the formula

$$\|(\chi - 1)\mathbb{D}_y^t \mathcal{U}^0; L_2(\Omega_h)\| \leq h^{1/2} \|(\chi - 1)\mathbb{D}_y^t \mathcal{U}^0; L_2(\Omega_1)\| \leq c h^{1/2} \mathcal{N}$$

is obvious where the term $h^{1/2}$ appears due to the small thickness h of the plate Ω_h . The first two terms in the right-hand side of (3.23) contain h^{-1} (see (3.14)). To estimate the terms, we use a variant of Hardy's inequality given by the following

LEMMA 3.3. For any $v \in H^1(\Omega_h)$, there holds the inequality

$$\int_{\Omega_1} (1 - \chi(y)) |v(y, \zeta)|^2 dy d\zeta \leq C h \int_{\Omega_1} (|\nabla_y v(y, \zeta)|^2 + |v(y, \zeta)|^2) dy d\zeta. \quad (3.24)$$

Proof. We denote by $\omega_\ell \subset \mathbb{R}^2$ a subdomain of ω formed by the lines $\text{dist}\{y, \partial\omega\} = \text{const} < \ell$. Obviously, it is sufficient to prove (3.24) with $h \leq \ell$. Let (s, ν) be a coordinate system given by these lines and the lines orthogonal to them so that $\partial\omega$ is represented by points $(s, 0)$. For a function $v(t) \in H^1(0, \ell)$ we have

$$\begin{aligned} h^{-1} \int_0^h |v(\nu)|^2 d\nu &\leq 2h^{-1} \int_0^h (|v(0)|^2 + |v(\nu) - v(0)|^2) d\nu \\ &\leq 2|v(0)|^2 + 2h \int_0^h \nu^{-2} |v(\nu) - v(0)|^2 d\nu. \end{aligned} \quad (3.25)$$

The inclusion $H^1(0, \ell) \subset C(0, \ell)$ yields the estimate

$$|v(0)| \leq c \|v; H^1(0, \ell)\|. \quad (3.26)$$

Using Hardy's inequality and the cut-off function χ_0 we obtain

$$\begin{aligned} \int_0^h \nu^{-2} |v(\nu) - v(0)|^2 d\nu &\leq \int_0^\ell |\chi_0(\ell^{-1}\nu)[v(\nu) - v(0)]|^2 \nu^{-2} d\nu \\ &\leq c \int_0^\ell |\partial_\nu \{\chi_0(\ell^{-1}\nu)[v(\nu) - v(0)]\}|^2 d\nu \leq C \|v; H^1(0, \ell)\|^2. \end{aligned} \quad (3.27)$$

Combining (3.25)–(3.27) we arrive at the inequality

$$h^{-1} \int_0^h |v(\nu)|^2 d\nu \leq c \|v; H^1(0, \ell)\|^2.$$

Since $|\partial_t v(y)|^2 \leq |\nabla_y v(y)|^2$, integrating in z and s completes the proof. \blacksquare

Applying the inequality (3.24) to estimate the first two terms in the right-hand side of (3.23) and taking into account the inclusion (3.10), we obtain

$$|J_1| \leq c E(\mathcal{R}, \mathcal{R}; \Omega_h)^{\frac{1}{2}} \left\| \mathcal{U}^0; H^1(\omega \rightarrow H^1(\Delta))^3 \right\| \leq C E(\mathcal{R}, \mathcal{R}; \Omega_h)^{\frac{1}{2}} \mathcal{N}. \quad (3.28)$$

Analogously, with (3.12) we gain

$$\begin{aligned} |J_2| &\leq \|D(\nabla_x)^t \mathcal{R}; L_2(\Omega_h)\| \times \|A \mathbb{D}_\zeta^t \mathcal{U}^1; L_2(\Omega_h)\| \\ &\leq c E(\mathcal{R}, \mathcal{R}; \Omega_h)^{\frac{1}{2}} h^{\frac{1}{2}} \left\| \mathcal{U}^1; L_2(\omega \rightarrow H^1(\Delta))^3 \right\| \leq C h^{\frac{1}{2}} E(\mathcal{R}, \mathcal{R}; \Omega_h)^{\frac{1}{2}} \mathcal{N}. \end{aligned} \quad (3.29)$$

In order to process the term J_3 in (3.20) we set $J_3 = \mathcal{J}_3 + J'_3$, where

$$\begin{aligned}\mathcal{J}_3 &= J_3(\mathcal{R}_3) = (A(\mathbb{D}_\zeta^t \mathcal{U}^1 + \mathbb{D}_y^t \mathcal{U}^0), \mathbb{D}_y^t e^3(\mathcal{R}_3 - R_3))_{\Omega_h}, \\ J'_3 &= J_3(\mathcal{R}_1, \mathcal{R}_2) = (A(\mathbb{D}_\zeta^t \mathcal{U}^1 + \mathbb{D}_y^t \mathcal{U}^0), \mathbb{D}_y^t (e^1 \mathcal{R}_1 + e^2 \mathcal{R}_2))_{\Omega_h}, \quad i = 1, 2.\end{aligned}$$

By virtue of (1.2) and (1.1) we have

$$|\mathbb{D}_y^t (e^1 \mathcal{R}_1 + e^2 \mathcal{R}_2)|^2 = \varepsilon_{11}(\mathcal{R})^2 + \varepsilon_{22}(\mathcal{R})^2 + 2\varepsilon_{12}(\mathcal{R})^2$$

and, owing to (3.21), we get

$$\|\mathbb{D}_y^t (e^1 \mathcal{R}_1 + e^2 \mathcal{R}_2); L_2(\Omega_h)\| \leq c E(\mathcal{R}, \mathcal{R}; \Omega_h)^{\frac{1}{2}}.$$

The latter and the inclusions (3.10), (3.12) furnish the formula

$$|J'_3| \leq c h^{\frac{1}{2}} E(\mathcal{R}, \mathcal{R}; \Omega_h)^{\frac{1}{2}} \mathcal{N}. \quad (3.30)$$

An estimation of \mathcal{J}_3 needs auxiliary inequalities.

LEMMA 3.4. *There holds the inequalities*

$$\|\partial_i(\mathcal{R}_3 - R_3); L_2(\Omega_h)\|^2 \leq c E(\mathcal{R}, \mathcal{R}; \Omega_h), \quad i = 1, 2 \quad (3.31)$$

with the constant c , independent of $\mathcal{R} \in \mathring{H}^1(\Omega_h, \Upsilon_h)$ and $h \in (0, 1]$.

Proof. We denote

$$\widehat{\mathcal{R}}(y, \zeta) = \mathcal{R}(y, \zeta) - \frac{1}{H} \int_{-H_-}^{H_+} \mathcal{R}(y, \zeta) d\zeta, \quad \widehat{\mathcal{R}}_3 = \mathcal{R}_3 - R_3. \quad (3.32)$$

Evidently,

$$\|\widehat{\mathcal{R}}; L_2(\Omega_1)\|^2 = \|\mathcal{R}; L_2(\Omega_1)\|^2 - \frac{1}{H} \left(\int_{-H_-}^{H_+} \mathcal{R}(y, \zeta) d\zeta \right)^2 \leq \|\mathcal{R}; L_2(\Omega_1)\|^2. \quad (3.33)$$

Since $\mathcal{R} = 0$ on Υ_h , the extension of \mathcal{R} by zero on the layer $\Pi_h = \{x : y \in \mathbb{R}^2, -H_- < h^{-1}z < H_+\}$ belongs to $H^1(\Pi_h)$. We cover the closure $\bar{\omega}$ by a family of squares Q_h^i of size $h \times h$ and consider one of the corresponding parallepipeds $C_h^i = Q_h^i \times (-hH_-, hH_+)$. On $C_h = C_h^i$ the vector-function \mathcal{R} can be represented in the form

$$\mathcal{R}(x) = \mathcal{R}^\perp(x) + d(x)\psi,$$

where

$$d(x) = \begin{pmatrix} 1 & 0 & -\alpha x_2 & \alpha x_3 & 0 & 0 \\ 0 & 1 & \alpha x_1 & 0 & -\alpha x_3 & 0 \\ 0 & 0 & 0 & -\alpha x_1 & \alpha x_2 & 1 \end{pmatrix}, \quad (3.34)$$

and the column $\psi \in \mathbb{R}^6$ is chosen such that

$$\int_{C_h} d(x)^t \mathcal{R}^\perp(x) \, dx = 0. \quad (3.35)$$

The latter leads to the algebraic system

$$d_C \psi = \int_{C_h} d(x)^t \mathcal{R}(x) \, dx,$$

where the 6×6-matrix

$$d_C = \int_{C_h} d(x)^t d(x) \, dx$$

is non-singular because

$$\det(d_C) = \frac{\alpha^6 h^{24}}{864} H^6 (1 + H^2)^2.$$

By virtue of (3.35) the Korn's inequality holds

$$E(\mathcal{R}^\perp, \mathcal{R}^\perp; C_h) \geq c \left\{ \|\nabla_x \mathcal{R}^\perp; L_2(C_h)\|^2 + h^{-2} \|\mathcal{R}^\perp; L_2(C_h)\|^2 \right\} \quad (3.36)$$

(see e.g. Kondrat'ev & Oleinik 1988, Nečas 1967 and Lemma 2.2 in Nazarov 1997c). Note the constant c does not depend on h because the change of variables $x \mapsto \xi = (h^{-1}[y - y^i], h^{-1}z)$, where y^i is the centre of Q_h^i , turns C_h into the standard parallepiped $(-1/2, 1/2)^2 \times (-H_-, H_+)$. Moreover, the factor h^{-2} appears in (3.36) due to the inverse change $\xi \mapsto x$.

Since $D(\nabla_x)d(x) = 0$, by using (3.33) we conclude that

$$E(\mathcal{R}, \mathcal{R}; C_h) = E(\mathcal{R}^\perp, \mathcal{R}^\perp; C_h) \geq c \|\partial_i \mathcal{R}^\perp; L_2(C_h)^3\|^2 \geq c \|\widehat{\partial_i \mathcal{R}_3^\perp}; L_2(C_h)^3\|^2, \quad i = 1, 2. \quad (3.37)$$

By definitions (3.32) and (3.34), one immediately observes that the matrix $\widehat{d}(x)$ does not depend on y . Hence,

$$\widehat{\partial_i \mathcal{R}_3^\perp} = \partial_i \widehat{\mathcal{R}_3}, \quad i = 1, 2,$$

and by summing (3.37) over all the cells C_h^i we obtain the inequality (3.31). ■

Lemma 3.4 and the inclusions (3.10), (3.12) yield the formula

$$|\mathcal{J}_3| \leq c h^{\frac{1}{2}} E(\mathcal{R}, \mathcal{R}; \Omega_h)^{\frac{1}{2}} \mathcal{N}, \quad (3.38)$$

needed to estimate the term J_3 in (3.19). It remains to mention that due to (3.1) and (3.3) the relationship (3.6) means that

$$|\widetilde{J}| \leq c h^0 E(\mathcal{R}, \mathcal{R}; \Omega_h)^{1/2} \widetilde{\mathcal{N}}. \quad (3.39)$$

5. Final theorems on approximation. Collecting the estimates (3.28), (3.29), (3.30), (3.38) and (3.39) for J_1 , J_2 , J_3 and \tilde{J} , we find the majorant $ch^0 E(\mathcal{R}, \mathcal{R}; \Omega_h)^{1/2} (\mathcal{N} + \tilde{\mathcal{N}})$ for the modulo of the right-hand side of (3.19). Thus, the inequalities (3.1) and (3.21) finish the proof of the following assertion.

THEOREM 3.2. *Let the right-hand sides of the problem (1.3)–(1.6) satisfy the conditions (2.10), (2.11), (3.4) and (3.6). Then the solution $u \in \mathring{H}^1(\Omega_h, \Upsilon_h)$ of the problem and the approximation solution (3.13) are in the relationship*

$$|u - \mathcal{U}'| + \|\varepsilon(u) - \varepsilon(\mathcal{U}'); L_2(\Omega_h)\| \leq ch^0(\mathcal{N} + \tilde{\mathcal{N}}) \quad (3.40)$$

where the constant c is independent of $h \in (0, 1)$, u , and the entries of the representation (2.10) of the right-hand sides.

In the following theorem we remove the cut-off function χ from the approximation of the displacement and strain fields u and $\varepsilon(u)$.

THEOREM 3.3. *Under the hypotheses of Theorem 3.2 there holds the inequality*

$$|u - h^{-2}\mathcal{U}^{-2} - h^{-1}\mathcal{U}^{-1} - h^0\mathcal{U}^0| + \|\varepsilon(u) - h^{-1}(\mathbb{D}_\zeta^t \mathcal{V} + \mathcal{Y})\mathcal{D}(\partial_y)^t w; L_2(\Omega_h)\| \leq ch^0(\mathcal{N} + \tilde{\mathcal{N}}) \quad (3.41)$$

where w is a solution of the resultant problem (2.34), (2.36); \mathcal{U}^j are indicated in (2.13), (2.14), (2.22); the matrices \mathbb{D}_ζ , \mathcal{D} and \mathcal{Y} , \mathcal{V} are defined in (2.2), (2.20) and (2.21), (2.23).

Proof. We start with the obvious inequality

$$|u - h^{-2}\mathcal{U}^{-2} - h^{-1}\mathcal{U}^{-1} - h^0\mathcal{U}^0| \leq |u - \mathcal{U}'| + |(1 - \chi)\mathcal{U}^0|. \quad (3.42)$$

Taking into account the inequality (3.1) we have

$$|(1 - \chi)\mathcal{U}^0|^2 \leq c E((1 - \chi)\mathcal{U}^0, (1 - \chi)\mathcal{U}^0; \Omega_h) \leq C \|D(\nabla_x)^t(1 - \chi)\mathcal{U}^0; L_2(\Omega_h)\|^2.$$

The last norm has been appeared in (3.22) and it was estimated in (3.28) as follows:

$$\|D(\nabla_x)^t(1 - \chi)\mathcal{U}^0; L_2(\Omega_h)\| \leq c \left\| \mathcal{U}^0, H^1(\omega \rightarrow H^1(\Delta))^3 \right\| \leq C \mathcal{N}.$$

Thus, combining (3.40) and (3.42), we arrive at the estimate

$$|u - h^{-2}\mathcal{U}^{-2} - h^{-1}\mathcal{U}^{-1} - h^0\mathcal{U}^0| \leq ch^0(\mathcal{N} + \tilde{\mathcal{N}}).$$

Let us estimate the second term in the left-hand side of (3.41). We have

$$\begin{aligned} \varepsilon(\mathcal{U}') &= D(\nabla_x)^t \mathcal{U}' = (\mathbb{D}_y^t + h^{-1}\mathbb{D}_\zeta^t) (h^{-2}\mathcal{U}^{-2} + h^{-1}\mathcal{U}^{-1} + h^0\chi\mathcal{U}^0) \\ &= h^{-3}\mathbb{D}_\zeta^t \mathcal{U}^{-2} + h^{-2}(\mathbb{D}_y^t \mathcal{U}^{-2} + \mathbb{D}_\zeta^t \mathcal{U}^{-1}) + h^{-1}(\mathbb{D}_y^t \mathcal{U}^{-1} + \chi\mathbb{D}_\zeta^t \mathcal{U}^0) + h^0\mathbb{D}_y^t \chi\mathcal{U}^0. \end{aligned} \quad (3.43)$$

The first two terms in the right-hand side of the last formula cancel by (2.13), (2.16) and (2.14). Taking into account (2.19) and (2.22), we obtain

$$\varepsilon(\mathcal{U}') = h^{-1} (\mathbb{D}_\zeta^t \mathcal{V} + \mathcal{Y}) \mathcal{D}(\nabla_y)^t w + h^{-1}(\chi - 1) \mathbb{D}_\zeta^t \mathcal{U}^0 + h^0 \mathbb{D}_y^t (\chi \mathcal{U}^0).$$

The last two terms in the right-hand side have been appeared in (3.23) and they were majorized by $ch^0\mathcal{N}$. Thus, in view of (3.40), we conclude the inequality (3.41). ■

COROLLARY 3.4. *Under the hypothesis of Theorem 3.2 the displacement field u satisfies the estimates*

$$h \|u_3 - h^{-2}w_3; L_2(\Omega_h)\| + \|u_i - h^{-1}(w_i - \zeta \partial_i w_3); L_2(\Omega_h)\| \leq ch^0(\mathcal{N} + \tilde{\mathcal{N}}). \quad (3.44)$$

Proof. It is sufficient to mention that, first, the weight factors ρ_h^{-2} and ρ_h^{-4} in (3.2) are larger than a positive constant and, secondly, the formulae (2.22) and (3.9) yield

$$\|\mathcal{U}^0; L_2(\Omega_h)\| \leq ch^{1/2} \|\mathcal{D}(\nabla_y)^t w; L_2(\omega)\| \leq c\mathcal{N}. \quad \blacksquare$$

4. The Kirchhoff hypotheses and explicit formulae for laminated plates

1. Discussion. Direct calculations show that each of the expressions

$$|h^{-2}\mathcal{U}^{-2}|, \quad |h^{-1}\mathcal{U}^{-1}|, \quad |h^0\mathcal{U}^0|, \quad \|h^{-1}(\mathbb{D}_\zeta^t \mathcal{V} + \mathcal{Y})\mathcal{D}(\partial_y)^t w; L_2(\Omega_h)\|$$

is of order $h^{-1/2}\|w; H^2(\omega)^2 \times H^3(\omega)\|$. The factor $h^{-1/2}$, indeed, confirms that all the asymptotic terms in (3.41) were detached correctly and the estimate (3.41), hence, justifies the asymptotic forms constructed. Moreover, the simplification (3.44) for the asymptotics of the displacement field holds true only in estimation of $L_2(\Omega_h)$ -norms while in the norm (3.2) which contains derivatives of u and is equivalent to $H^1(\Omega_h)$ -norm, the term \mathcal{U}^0 cannot be ignored. The asymptotics of strains figured in (3.41) is also calculated with the help of \mathcal{U}^0 (see (3.43)).

In the vicinity of the lateral side Υ_h of the plate Ω_h there appears the boundary layer phenomenon (see, e.g. Friedrichs & Dressler 1961, Gol'denveizer 1962, Gol'denveizer & Kolos 1965, Zorin & Nazarov 1989, Mazja et al. 1991, Ch. 16, Dauge & Gruais 1995, Dauge et al. 1998, etc.). Since the norms $|\cdot|$ and $\|\varepsilon(\cdot); L_2(\Omega_h)\|$ of the leading asymptotic term of the boundary layer type is of order h^0 (see, e.g., Zorin & Nazarov 1989 for an isotropic plate), the estimate (3.41) is asymptotically precise which means that the majorant of (3.41) cannot contain the factor $o(h^0)$ as $h \rightarrow 0$.

We emphasize that in the majorants of (3.40), (3.41) and (3.44), the values \mathcal{N} and $\tilde{\mathcal{N}}$ absorb completely the dependence on the right-hand sides of the initial problem (1.3)–(1.6) while the constants c depend only on the cross-section ω of the plate and the elastic

moduli introduced with the matrix $A(y, \zeta)$. Moreover, the obtained estimates are sharp with respect to the smoothness properties prescribed by (3.4) and (3.5). Namely, if for simplicity $g^\pm = 0$, $\tilde{f} = 0$, then a solution in $H^1(\Omega_h)^3$ of the problem (1.3)–(1.6) exists even in the case $f^0 \in H^{-1}(\Omega_h)^3$ while one can construct a counter-example such that the estimates (3.40) and (3.41) lose their validity due to internal boundary layers appearing near the subsets of the plate where the function f^0 is not smooth (see, e.g. Nazarov & Semenov 1981, Aldoshina & Nazarov 1998).

We also note that the difference $(1 - \chi)\mathcal{U}^0$ of the approximations employed in Theorems 3.2 and 3.3 has the norm $\|(1 - \chi)\mathcal{U}^0\|$ of the same order h^0 as the leading boundary layer term mentioned above. Thus, the boundary layer takes the role of the cut-off function χ in (3.13), i.e. compensation of the discrepancy generated by \mathcal{U}^0 in the Dirichlet conditions (1.6).

2. Justification of the hypotheses. A traditional derivation of the system (2.34), which is formal in the sense that it is not provided with estimates of precision, needs two hypotheses on the stress-strain state in a plate. Those are known as the Kirchhoff hypotheses and are naturally referred to as the kinematic and static ones. The kinematic hypothesis declares that, with the precision of the two-dimensional model, the components of the displacement field u can be approximated by the following functions, linear in z :

$$u_i(y, z) \sim \bar{u}_i(y) - z\partial_i\bar{u}_3(y), \quad i = 1, 2; \quad u_3(y, z) \sim \bar{u}_3(y). \quad (4.1)$$

Evidently, the inequality (3.44) justifies this hypothesis with

$$\bar{u}_i(y) = h^{-1}w_i(y), \quad \bar{u}_3(y) = h^{-2}w_3(y). \quad (4.2)$$

The static hypothesis predicts that the stresses

$$\sigma''(u) = (\sigma_{13}(u), \sigma_{23}(u), \sigma_{33}(u))^t$$

vanish in the plate Ω_h with the same precision. In order to prove this hypothesis, it is sufficient to note that, by virtue of the definition (1.1), (2.2) and the estimate (3.41),

$$(0, 0, 0, \sigma''(u)^t)^t = \mathbb{D}_1\sigma(u),$$

$$\|\mathbb{D}_1\sigma(u) - h^{-1}\mathbb{D}_1A(\mathbb{D}_\zeta^t\mathcal{V} + \mathcal{Y})\mathcal{D}(\nabla_y)^tw; L_2(\Omega_h)\| \leq ch^0(\mathcal{N} + \tilde{\mathcal{N}}).$$

Since, owing to the next lemma, the expression $\Sigma = \mathbb{D}_1A(\mathbb{D}_\zeta^t\mathcal{V} + \mathcal{Y})$ vanishes, the latter inequality is but a mathematical interpretation of the static Kirchhoff hypotheses.

LEMMA 4.1. *The equality $\Sigma(y, \zeta) = 0$ is valid with any $(y, \zeta) \in \Omega_1$.*

Proof. The equality follows from the fact that in terms of Σ the problem (2.23) takes the form

$$\partial_\zeta \Sigma = 0, \quad \zeta \in (-H_-, H_+); \quad \Sigma = 0, \quad \zeta = \pm H_\pm. \quad \blacksquare$$

The reason to call (4.1) a kinematic hypothesis is a well-known contradiction between the Kirchhoff hypotheses which results in the fact that the representation (4.1) is valid for manipulations with the displacements only. Indeed, if one calculates the stresses in accordance with (4.1) and (4.2), in view of (2.19) one obtains the relationship

$$\sigma(u; y, z) \sim A(y, z) \mathcal{Y}(\zeta) \mathcal{D}(\nabla_y)^t w(y) \quad (4.3)$$

which differs from the formula justified by (3.41)

$$\sigma(u; y, z) \sim A(y, \zeta) (\mathbb{D}_\zeta^t \mathcal{V}(y, \zeta) + \mathcal{Y}(y)) \mathcal{D}(\nabla_y)^t w(y). \quad (4.4)$$

The presence of the expression $\mathbb{D}_\zeta^t \mathcal{V}$ in (4.4) is stipulated by impossibility to ignore the term \mathcal{U}^0 in differentiation of the ansatz (2.12) as it was discussed in the previous subsection. It is to stress that some conclusions of the works which overlook the contradiction and use the “approximation” (4.3) happen to be wrong (see, e.g. Zorin & Nazarov 1989, where such mistakes are discussed at length).

3. Explicit formulae. In this section we use the following notations:

$$\begin{aligned} \sigma_{(1)} &= (\sigma_1, \sigma_2, \sigma_3)^t, \quad \varepsilon_{(1)} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^t, \quad \varepsilon_{(2)} = (\varepsilon_4, \varepsilon_5, \varepsilon_6)^t, \\ A &= \begin{pmatrix} A_{(11)} & A_{(12)} \\ A_{(21)} & A_{(22)} \end{pmatrix}, \quad \mathcal{Y}'(\zeta) = (\mathbb{I}, -\alpha^{-1} \zeta \mathbb{I}), \end{aligned} \quad (4.5)$$

where $A_{(ij)}$ stand for 3×3-blocks of the Hooke’s matrix A . Let

$$\tilde{\varepsilon} = h^{-1} (\mathbb{D}_\zeta^t \mathcal{V} + \mathcal{Y}) \mathcal{D}(\nabla_y)^t w, \quad \tilde{\sigma} = A \tilde{\varepsilon}$$

(see (3.41)). By Lemma 4.1 we conclude

$$\begin{aligned} \tilde{\sigma}_{(1)} &= A_{(11)} \tilde{\varepsilon}_{(1)} + A_{(12)} \tilde{\varepsilon}_{(2)}, \\ 0 &= A_{(21)} \tilde{\varepsilon}_{(1)} + A_{(22)} \tilde{\varepsilon}_{(2)}. \end{aligned}$$

Thus,

$$\tilde{\varepsilon}_{(2)} = -A_{(22)}^{-1} A_{(21)} \tilde{\varepsilon}_{(1)}. \quad (4.6)$$

Since the left 3×3-block of \mathbb{D}_1 vanishes, we obtain

$$\tilde{\varepsilon}_{(1)} = h^{-1} \mathcal{Y}' \mathcal{D}^t w. \quad (4.7)$$

Finally, combining (4.6) and (4.7) we arrive at

$$\tilde{\varepsilon} = h^{-1} \mathcal{E}(y, \zeta) \mathcal{D}(\nabla_y)^t w(y),$$

where

$$\begin{aligned} \mathcal{E}(y, \zeta) &= \left(\mathbb{I}, -A_{(22)}^{-1}(y, \zeta) A_{(21)}(y, \zeta) \right)^t \mathcal{Y}' = \\ &= \begin{pmatrix} \mathbb{I} & -\alpha^{-1} \zeta \mathbb{I} \\ -A_{(22)}^{-1}(y, \zeta) A_{(21)}(y, \zeta) & \alpha^{-1} \zeta A_{(22)}^{-1}(y, \zeta) A_{(21)}(y, \zeta) \end{pmatrix}. \end{aligned} \quad (4.8)$$

From the definition of $\tilde{\varepsilon}$ and the formulae (2.35), (2.24) we derive the representation

$$\mathcal{M} \mathcal{D}^t w = h \int_{-H_-}^{H_+} \mathcal{Y}^t A \tilde{\varepsilon} d\zeta = \int_{-H_-}^{H_+} \mathcal{Y}^t A \mathcal{E} d\zeta \mathcal{D}^t w. \quad (4.9)$$

By using (2.21), (4.5) and (4.8) we immediately obtain

$$\mathcal{Y}^t A \mathcal{E} = \begin{pmatrix} \mathbf{A} & -\alpha^{-1} \zeta \mathbf{A} \\ -\alpha^{-1} \zeta \mathbf{A} & \alpha^{-2} \zeta^2 \mathbf{A} \end{pmatrix}, \quad (4.10)$$

where \mathbf{A} is a matrix of size 3×3 ,

$$\mathbf{A}(y, \zeta) = A_{(11)}(y, \zeta) - A_{(12)}(y, \zeta) A_{(22)}^{-1}(y, \zeta) A_{(21)}(y, \zeta). \quad (4.11)$$

Finally, (4.9) and (4.10) give the formulae presentations

$$\begin{aligned} \mathcal{M}_{(11)} &= \int_{-H_-}^{H_+} \mathbf{A}(\zeta) d\zeta, \quad \mathcal{M}_{(12)} = \mathcal{M}_{(21)} = -\alpha^{-1} \int_{-H_-}^{H_+} \zeta \mathbf{A}(\zeta) d\zeta, \\ \mathcal{M}_{(22)} &= \alpha^{-2} \int_{-H_-}^{H_+} \zeta^2 \mathbf{A}(\zeta) d\zeta, \end{aligned} \quad (4.12)$$

for 3×3 -blocks of the matrix \mathcal{M} ,

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{(11)} & \mathcal{M}_{(12)} \\ \mathcal{M}_{(21)} & \mathcal{M}_{(22)} \end{pmatrix}.$$

Note that with another argument the formulae of type (4.12) were obtained in (Zorin 1987).

4. Laminated plates. Let us focus on the case of laminated plates, when the matrix A is piecewise constant in z , i.e. $A^n = A^n(y)$, $n = 1, \dots, N$. By calculation of integrals in (4.12) we have

$$\mathcal{M} = \sum_{n=1}^N \mathcal{M}^n,$$

where

$$\mathcal{M}^n = \begin{pmatrix} \mathbf{A}^n (H_n - H_{n-1}) & -\alpha^{-1} \mathbf{A}^n (H_n^2 - H_{n-1}^2) \\ -\alpha^{-1} \mathbf{A}^n (H_n^2 - H_{n-1}^2) & \frac{1}{3} \alpha^{-2} \mathbf{A}^n (H_n^3 - H_{n-1}^3) \end{pmatrix} \quad (4.13)$$

and $\mathbf{A}^n = \mathbf{A}(\zeta)$ with $\zeta \in (H_n, H_{n-1})$.

Of interest is to find a representation of \mathcal{M} in terms of the matrices \mathcal{M}_n^0

$$\mathcal{M}_n^0 = \begin{pmatrix} \mathbf{A}^n & \mathbb{O} \\ \mathbb{O} & \frac{1}{6} \mathbf{A}^n \end{pmatrix}, \quad (4.14)$$

where \mathcal{M}_n^0 corresponds to the plate $\omega \times (-h/2, h/2)$ made of material with the Hooke's matrix A^n . We define the thickness and the middle plane of the n -th layer:

$$h_n = H_n - H_{n-1}, \quad H_n^* = \frac{H_n + H_{n-1}}{2}.$$

Then, the matrix \mathcal{M}^n takes the form

$$\mathcal{M}^n = h_n \operatorname{diag} \{ \mathbf{A}, \mathbf{A} \} K_n,$$

where $\operatorname{diag} \{ \mathbf{A}, \mathbf{A} \}$ is a 6×6-matrix composed from the matrix (4.11),

$$K_n = \begin{pmatrix} \mathbb{I} & -\alpha^{-1} H_n^* \mathbb{I} \\ -\alpha^{-1} H_n^* \mathbb{I} & \alpha^{-2} (H_n^{*2} + h_n^2/12) \mathbb{I} \end{pmatrix} = L_n^t L_n, \quad L_n = \begin{pmatrix} \mathbb{I} & -\alpha^{-1} H_n^* \mathbb{I} \\ \mathbb{O} & -h_n 6^{-\frac{1}{2}} \mathbb{I} \end{pmatrix}.$$

Then,

$$\mathcal{M}^n = h_n L_n^t \operatorname{diag} \{ \mathbf{A}, \mathbf{A} \} L_n.$$

Since

$$\begin{aligned} \operatorname{diag} \{ \mathbf{A}, \mathbf{A} \} &= \operatorname{diag} \left\{ \mathbb{I}, 6^{\frac{1}{2}} \mathbb{I} \right\} \mathcal{M}_0^n \operatorname{diag} \left\{ \mathbb{I}, 6^{\frac{1}{2}} \mathbb{I} \right\}, \\ \operatorname{diag} \left\{ \mathbb{I}, 6^{\frac{1}{2}} \mathbb{I} \right\} &= \operatorname{diag} \left\{ 1, 1, 1, 6^{\frac{1}{2}}, 6^{\frac{1}{2}}, 6^{\frac{1}{2}} \right\} \end{aligned}$$

we thus arrive at

$$\mathcal{M} = \sum_{n=1}^N h_n J_n^t \mathcal{M}_0^n J_n, \quad J_n = \begin{pmatrix} \mathbb{I} & -\alpha^{-1} H_n^* \mathbb{I} \\ \mathbb{O} & -h_n \mathbb{I} \end{pmatrix}. \quad (4.15)$$

We recall that the n -th summand is the matrix (4.13) constructed for the n -th isolated layer of the laminated plate and, thus, the bordering by $h_n J_n^t$ and J_n in (4.15) translates the matrix (4.14) of the resultant system (2.34) for the plate $\omega \times (-h/2, h/2)$ to the matrix of the system for the plate $\omega \times (hH_{n-1}, hH_n)$. The formula (4.15) means that the matrix \mathcal{M} for laminated plates represents a sum of analogous matrices for its isolated layers.

We emphasize that the above formulae manifest changes in the matrix \mathcal{M} due to a shift of the reference plane.

5. Properties of the mapping $A \mapsto \mathcal{M}$. The formulae (4.11) and (4.12) defines mapping $\mathfrak{M} : A \mapsto \mathcal{M}$ where the matrix \mathcal{M} contains 6 independent components only, whilst A has 21 different components. Thus, there exists a local 15-parameter transformation of the matrix A which leaves the corresponding matrix \mathcal{M} unchanged. Using (4.12) we describe the transformation with the help of two 3×3-matrices B and C where B is arbitrary non-singular and C is arbitrary symmetric, such that $A_{(22)} + C$ is non-singular. Then, the 9-parameter transformation $A \rightarrow A'$ can be introduced as follows

$$A'_{(11)} = A_{(11)}, \quad A'_{(22)} = B^t A_{(22)} B, \quad A'_{(12)} = A_{(12)} B, \quad A'_{(21)} = B^t A_{(21)}.$$

Other six parameters are taken into account by the transformation

$$\begin{aligned} A'_{(22)} &= A_{(22)} + C, \quad A'_{(12)} = A_{(12)}, \quad A'_{(21)} = A_{(21)}, \\ A'_{(11)} &= A_{(11)} + A_{(12)} \left\{ (A_{(22)} + C)^{-1} - A_{(22)}^{-1} \right\} A_{(21)}. \end{aligned}$$

We emphasize that according to (4.11) and (4.12) the upper right 3×3-block of the matrix \mathcal{M} turns out to be symmetric. Thus, even the plates of the most general structure we have considered here do not exhaust all symmetric, positive definite matrices \mathcal{M} in (2.34). Nevertheless, the hypothesis can be formulated that arbitrary symmetric matrix \mathcal{M} may appear in (2.34) as a result of homogenization of a thin cylindrical plate with highly oscillating elastic properties (see, e.g. Caillerie 1984, Nazarov 1995).

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