On frequency bounds for modes trapped near a channel-spanning cylinder

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A channel of infinite length and depth and of constant width contains inviscid heavy fluid having free surface. The fluid is bounded internally by a submerged cylinder which spans the channel and has its generators normal to the sidewalls. The existence of trapped modes, i.e. states with finite energy corresponding to localized fluid oscillations, is well established in the linearised theory of water waves and the modes have been proven to occur at some frequencies for any geometry of the submerged cylinder. The purpose of this work is to find lower bounds for these trapped-mode frequencies. An integral identity suggested by Grimshaw (1974) is applied to a possible trapped-mode potential and a comparison, or trial, function. This identity yields uniqueness of the problem if the trial function has special properties. A number of trial functions are suggested possessing these properties for some sets of parameters of the problem. The potentials are constructed with the help of singular solutions, namely modified Bessel functions and the Green's function of the problem. A comparison is given between the bounds obtained here and known bounds and examples of trapped modes.

Keywords: uniqueness; trapped mode; eigenvalue bound; linear surface wave theory; integral inequality; maximum principles; Green's function

1. Introduction

In 1846 Stokes produced a simple solution to the linearised water wave equations which is now called an edge wave and represents a wave travelling in the long-shore direction over a uniformly sloping beach and decays in the direction of increasing depth. It was not until over a century later that a further localized solution was discovered by Ursell (1950), who proved the existence of modes which are symmetric about a vertical plane and are trapped near a totally submerged circular cylinder in a channel of infinite depth. The proof, using multipole expansions and infinite determinants, required the radius of the cylinder to be sufficiently small. This restriction was removed by Jones (1953) who proved that trapped modes exist for a wider class of submerged cylinders which are symmetric about the vertical axis. Ursell (1987) has since given a simplified existence proof for trapped waves as well as a number of comparison theorems. Detailed computations of the symmetric trapped modes for circular cylinders of unrestricted radius were made by McIver &

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Evans (1985). They computed trapped-wave frequencies for the submerged circular cylinder and found that more trapped waves appear as the cylinder approaches the free surface.

McIver & Evans (1985) also derived lower bounds for frequencies of symmetric modes trapped near a cylinder. These bounds are based on a comparison theorem by Grimshaw (1974), which estimates trapped-mode frequencies for one domain by the frequencies for another (associated) domain whose lower boundary is everywhere higher. Two types of associated domains were used in (McIver & Evans 1985): plane beach and rectangular shelf. Another method to obtain lower bounds of the frequency was suggested by Simon (1992) as an extension of the uniqueness theorem by Simon & Ursell (1984). This scheme based on integral inequalities is more flexible, in particular, the bound by McIver & Evans based on comparison with the edge waves was rederived in (Simon 1992) to estimate the frequencies of all trapped modes, not only the symmetric ones. However, only the bound mentioned above was given in explicit form by Simon (1992), though it is very plausible that better bounds can be found by the use of his method.

In this work in order to obtain bounds of the trapped-mode frequencies we shall use an integral identity derived following Grimshaw (1974) and McIver & Linton (1995). The method relies on finding a strictly positive trial function which satisfies a certain field inequality within the fluid, a condition at infinity and boundary inequalities on the free surface and cylinder's contour. The integral identity relates the trial potential to the possible trapped-mode potential in such a way that it may be deduced that the latter must be identically equal to zero throughout the fluid. A number of the trial functions are suggested, which satisfy the conditions for some set of the parameters of the problem where, hence, the uniqueness is established. The motivation of the approach comes from ideas on maximum principles and can be found in Ch. 2 of (Protter & Weinberger 1984).

The plan of the paper is as follows. The problem is formulated in § 2 and the integral identity, which relates the solution to the problem to a trial function, is derived in § 3. The simplest singular trial functions based on the modified Bessel functions are presented in § 4 and bounds are obtained with the help of these functions. Green's function of the problem is introduced in § 5 and some auxiliary assertions on its properties are proven. The bounds using the Green function as the trial one are given in § 6. A comparison of the bounds with known bounds and examples of trapped modes is done in § 7, based on numerical results and asymptotics.

2. Statement of the problem

A horizontal cylinder of arbitrary cross-section B and with parallel generators is submerged in a deep-water channel of constant width. Cartesian coordinates (x, y, z) are chosen with origin in the mean free surface, the z-axis is directed parallel to the length of the cylinder and the y-axis is directed vertically upwards. We denote by W the cross-section of the domain occupied by fluid, F is the mean free surface $y = 0$, $S = \partial B$ and $\vec{n}$ is the normal coordinate on S directed into W. This notation is illustrated in fig. 1.

Under the usual assumptions of linearised surface-wave theory, the fluid motion can be described by a velocity potential $\Phi(x, y, z, t)$ where t is time. Solutions are

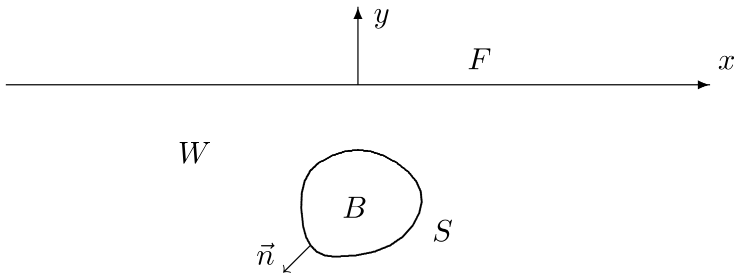

Figure 1. A sketch of the geometrical notation

sought that describe oscillatory motions of radian frequency ω and wavenumber k along the cylinder. The potential can be written as

$$\Phi(x, y, z, t) = \phi(x, y) \sin kz \cos \omega t.$$

Here $\phi(x, y)$ satisfies

$$\nabla^2 \phi - k^2 \phi = 0 \quad \text{in} \quad W, \tag{2.1}$$

$$\frac{\partial \phi}{\partial y} - \nu \phi = 0 \quad \text{on} \quad F, \tag{2.2}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad S, \tag{2.3}$$

where $\nu = \omega^2/g$ and g is the acceleration due to gravity.

We also have to complete the problem by a condition at infinity. We shall assume that for some $M > 0$

$$|\nabla \phi| = O(R^M), \quad \text{as} \quad R \to \infty \tag{2.4}$$

uniformly in ϑ, where $x + \mathrm{i}y = R\mathrm{e}^{\mathrm{i}\vartheta}$. This condition is used in the expansion theorem (Ursell 1968, p. 815, see also Notes on the theorem on p. 822), which guarantees that the potential ϕ decays exponentially at infinity when $k > \nu$ unlike the case $k < \nu$ when there are wave terms in asymptotics of solution at infinity.

Further we shall only consider the case $k > \nu$, when the fluid motion cannot propagate energy to infinity and

$$\phi(x, y) = O(|x + \mathrm{i}y|^{-n}) \quad \text{for any } n > 0. \tag{2.5}$$

The problem (2.1)–(2.4) was proven (Ursell 1950, Jones 1953, Ursell 1987) to have a finite number of eigenvalues $\nu \in (0, k)$ for any geometry of the body B. The aim of the present paper is to find lower bounds for the eigenvalues.

3. An integral identity

In this section we derive a modification of the integral identity suggested by Grimshaw (1974), who applied it to obtain bounds for the dispersion relation, which connects frequency $\sqrt{\nu g}$ and wavenumber k of the so-called edge waves in ocean of finite depth. The scheme of the present section follows McIver & Linton (1995) who used this integral identity in order to prove non-existence of trapped modes in acoustic waveguides having bounded cross-section and containing bodies.

Without loss of generality we can assume that $\phi(x,y)$ is a real-valued function in the fluid domain W. Let another real-valued function w be defined in W and strictly positive so that

$$w(x,y) > 0 \quad \text{in} \quad \overline{W}. \tag{3.1}$$

In view of the latter inequality the function

$$v = \phi/w \tag{3.2}$$

is defined in W. Also we consider the functional L, where

$$L(v,w,\Pi) = \int_{\Pi} \nabla\cdot\left[w^2 v\nabla v\right] \mathrm{d}x\,\mathrm{d}y, \quad \Pi\subset W \tag{3.3}$$

Expanding the integrand shows that

$$L(v,w,\Pi) = \int_{\Pi}\left[w^2 v\left(\nabla^2 v + \frac{2}{w}\nabla v\cdot\nabla w\right) + w^2\left(\nabla v\right)^2\right]\mathrm{d}x\,\mathrm{d}y.$$

From (3.2) it can be shown that

$$\nabla^2 v = w^{-1}\left(-2\nabla v\cdot\nabla w + \nabla^2\phi - v\nabla^2 w\right)$$

and, thus, we obtain

$$L(v,w,\Pi) = \int_{\Pi}\left[\phi\nabla^2\phi - \frac{\phi^2}{w}\nabla^2 w + w^2\left(\nabla v\right)^2\right]\mathrm{d}x\,\mathrm{d}y.$$

Since ϕ satisfies the modified Helmholtz equation (2.1), we have

$$L(v,w,\Pi) = \int_{\Pi}\left[w^2\left(\nabla v\right)^2 - \frac{\phi^2}{w}\left(\nabla^2 w - k^2 w\right)\right]\mathrm{d}x\,\mathrm{d}y. \tag{3.4}$$

From (3.3), using the divergence theorem, we find

$$L(v,w,\Pi) = -\int_{\partial\Pi} w^2 v\frac{\partial v}{\partial n}\,\mathrm{d}s,$$

where $\partial/\partial n$ indicates the normal derivative ($\vec{n}$ is directed into the region Π). Substituting for v from (3.2) in the last formula and using (3.4) we arrive at

$$\int_{\Pi_R}\left[w^2\left(\nabla v\right)^2 - \frac{\phi^2}{w}\left(\nabla^2 w - k^2 w\right)\right]\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Pi_R}\left\{\frac{\phi^2}{w}\frac{\partial w}{\partial n} - \phi\frac{\partial\phi}{\partial n}\right\}\mathrm{d}s.$$

where we fix $\Pi = \Pi_R = W\cap\{|x+\mathrm{i}y|\leqslant R\}$. The limit process $R\to\infty$ can be performed in the last equality if absolute value of the expression in curly brackets on $\partial\Pi_R$ has estimate $O(R^{-1-\delta})$, $\delta>0$, as $R\to\infty$. In view of (2.5) this condition is obviously satisfied if for some fixed $n>0$

$$\frac{|\nabla w(x,y)|}{w(x,y)} = O(R^n), \quad \text{as} \quad |x+\mathrm{i}y| = R\to\infty. \tag{3.5}$$

Assuming that w satisfies (3.5) and taking the limit $R \to \infty$ leads to

$$\int_W \left[w^2 (\nabla v)^2 - \frac{\phi^2}{w} \left(\nabla^2 w - k^2 w \right) \right] \mathrm{d}x\, \mathrm{d}y = \int_F \frac{\phi^2}{w} \left\{ \nu w - \frac{\partial w}{\partial y} \right\} \mathrm{d}x + \int_S \frac{\phi^2}{w} \frac{\partial w}{\partial n}\, \mathrm{d}s, \tag{3.6}$$

where we also make of use (2.2), (2.3) and the equality $\partial/\partial n = -\partial/\partial y$ on the free surface.

Suppose that we can find a function w, which along with (3.1) and (3.5), also satisfies

$$\nabla^2 w - k^2 w \leqslant 0 \quad \text{in} \quad W, \tag{3.7}$$

$$\frac{\partial w}{\partial y} - \nu w \geqslant 0 \quad \text{on} \quad F, \tag{3.8}$$

$$\frac{\partial w}{\partial n} \leqslant 0, \quad \text{on} \quad S. \tag{3.9}$$

Then, the left-hand side of (3.6) is non-negative and the right-hand side is non-positive and we can conclude that both these expressions must vanish identically. Thus, we find that $(\nabla v)^2 \equiv 0$ throughout W, i.e. v is constant and

$$\phi = A w \tag{3.10}$$

with some constant A. In the case when relationships (3.7), (3.8) and (3.9) are equalities, the potential w is a positive trapped mode and (3.10) means that this solution of the spectral problem is unique up to a constant multiplier.

Let, on the contrary, at least one of the expressions $\nabla^2 w - k^2 w$, $\partial w/\partial y - \nu w$ or $\partial w/\partial n$ be non-zero at a point ζ belonging to $\Upsilon = W$, F or S respectively. Then, obviously this expression is also non-zero in some vicinity $\Upsilon_0 \subset \Upsilon$ of the point ζ and at the same time the integral in (3.6), which contains this expression multiplied by ϕ^2/w, is equal to zero. Hence, in view of (3.1), $\phi = 0$ in Υ_0 which implies that $A = 0$ in (3.10) and, hence, $\phi \equiv 0$ in the domain occupied by fluid.

Remark 3.1. *If a trial function w satisfies (3.1), (3.7)–(3.9) for some fixed geometry and values ν_0, k_0 of parameters ν, k, then the function also satisfies these conditions for all ν and k such that $\nu \leqslant \nu_0$ and $k \geqslant k_0$, because in view of (3.1),*

$$\nabla^2 w - k^2 w = \nabla^2 w - k_0^2 w + \left(k_0^2 - k^2\right) w < 0 \quad \text{if} \quad k > k_0,$$
$$\frac{\partial w}{\partial y} - \nu w = \frac{\partial w}{\partial y} - \nu_0 w + \left(\nu_0 - \nu\right) w > 0 \quad \text{if} \quad \nu < \nu_0.$$

Thus, if for some geometry and for some values $k = k_0$, $\nu = \nu_0$ the problem (2.1)–(2.4) is proven with the help of (3.6) to have only the trivial solution, this is also true for all values $k \in (k_0, +\infty)$, $\nu \in (0, \nu_0)$.

We make of use the property described in the latter remark and prove the following assertion.

Proposition 3.1. *Let the parameter k be fixed and ϕ_n, $n = 1, \ldots, N$ be solutions to the spectral problem (2.1)–(2.4) corresponding to the eigenvalues*

$$0 < \nu_1 = \nu_2 = \ldots = \nu_{j-1} < \nu_j \leqslant \ldots \leqslant \nu_N,$$

where $N \geqslant 1$ by (Ursell 1950, Jones 1953, Ursell 1987). We also assume that ϕ_n satisfy the condition (3.5). Then only the modes corresponding to $\nu_1, \dots, \nu_{j-1}$ can be strictly positive (negative) in the whole fluid domain W. If one of the modes ϕ_n, $n = 1, \dots, j-1$ is strictly positive (negative), then $j = 2$.

Proof. Assume, on the contrary, that for some $\ell \geqslant j$, ϕ_ℓ is a strictly positive mode. Then, by remark 3.1 the potential satisfies the condition (3.8) for $\nu = \nu_m$, $m < \ell$. Since ϕ_ℓ also satisfies (3.7), (3.9) and the condition (3.5) by assumption, it can be applied as a trial function w in (3.6) with $\phi = \phi_m$ to prove that $\phi_m \equiv 0$, $m < n$, which contradicts the assumption. In the discussion of the formula (3.10) we have found that a strictly positive trapped mode is unique, which completes the proof. □

Analogously, for a fixed value of ν we can consider a set of trapped modes ϕ_n corresponding to the eigenvalues k_n. Then the scheme of the latter proof leads to the conclusion that only the trapped mode with maximum value of k_n can be strictly positive (negative) in the whole domain W and the mode is unique up to an arbitrary factor.

4. Construction of the trial function w

In this section we construct examples of the trial function w using the modified Bessel function K_n,

$$\mathrm{K}_n(z) = \int_0^\infty \mathrm{e}^{-z\cosh\mu} \cosh n\mu \, \mathrm{d}\mu \tag{4.1}$$

(see e.g. 8.432.1 in Gradsteyn & Ryzhik 1994). We introduce the polar system of coordinates with origin in the point $(0, -d)$ such that

$$x = r\sin\theta, \quad y = -d + r\cos\theta. \tag{4.2}$$

The potential $\mathrm{K}_0(kr)$ in the above notation seems to be a good candidate for the trial function w, because $\mathrm{K}_0(kr)$ is strictly positive and satisfies the equation (2.1). Besides, by 9.6.8 in (Abramowitz & Stegun 1965)

$$\mathrm{K}_0(kr) \sim -\log kr \quad \text{as} \quad r \to 0$$

and so $\mathrm{K}_0(kr)$ also satisfies (3.9) at least for sufficiently small circular cylinders, having their centre at $(0, -d)$.

In order to fulfil the condition (3.8), we consider the following potential

$$w_0(x, y) = \mathrm{K}_0(kr) + C_0\,\mathrm{e}^{ky}, \tag{4.3}$$

where $C_0 = C_0(k, \nu, d)$ is a positive constant. We have to check that the potential w_0 satisfies (3.5). In view of 8.486.18 in (Gradsteyn & Ryzhik 1994) we have

$$|\nabla w_0| \leqslant |\nabla\,\mathrm{K}_0(kr)| + |\nabla C_0\,\mathrm{e}^{ky}| = k\,\mathrm{K}_1(kr) + kC_0\,\mathrm{e}^{ky},$$

and, obviously,

$$\frac{|\nabla w_0(x,y)|}{w_0(x,y)} \leqslant \frac{k\,\mathrm{K}_1(kr)}{\mathrm{K}_0(kr) + C_0\,\mathrm{e}^{ky}} + \frac{kC_0\,\mathrm{e}^{ky}}{\mathrm{K}_0(kr) + C_0\,\mathrm{e}^{ky}} \leqslant k\left\{1 + \frac{\mathrm{K}_1(kr)}{\mathrm{K}_0(kr)}\right\}.$$

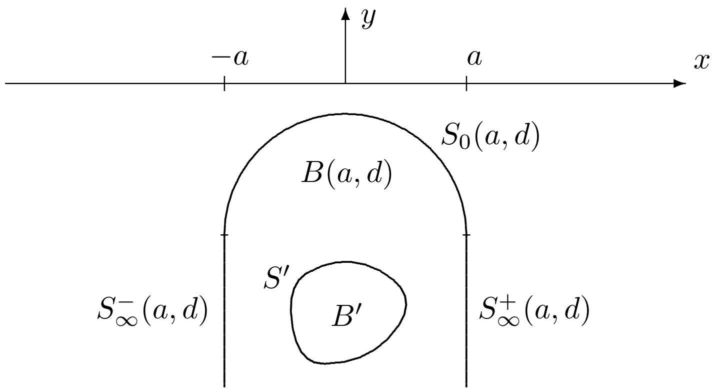

Figure 2. Auxiliary geometrical notations

From the last inequality and the asymptotics of Bessel functions given by 9.7.2 in (Abramowitz & Stegun 1965) it follows that

$$|\nabla w_0(x,y)|/w_0(x,y) = O(1) \quad \text{as} \quad |x+\mathrm{i}y| \to \infty.$$

The constant C_0 in (4.3) is defined by the condition (3.8). We have for $y=0$

$$\frac{\partial w_0}{\partial y} - \nu w_0 = -k\,\mathrm{K}_1\big(k\sqrt{x^2+d^2}\big)\cos\theta - \nu\,\mathrm{K}_0\big(k\sqrt{x^2+d^2}\big) + (k-\nu)C_0.$$

Since the functions K_0, K_1 are monotonically decreasing, the condition (3.8) is satisfied if

$$C_0(k,\nu,d) = \frac{k\,\mathrm{K}_1(kd) + \nu\,\mathrm{K}_0(kd)}{k-\nu}.$$

It is to note that this expression is positive only when $k > \nu$.

Finally, uniqueness of the problem (2.1)–(2.4) for a body B, which contains the point $(0,-d)$ inside, will follow from the identity (3.6) with the trial potential w_0 if this potential satisfies (3.9) on the contour $S = \partial B$. In the following remark we shall show that it is possible to consider the condition for special geometries of bodies instead of checking the condition for every particular contour in question. We shall need the monotonicity principle of Theorem 5.1 in (Ursell 1987), which is formulated in our notations as follows:

Suppose that $S^{(1)}$ lies inside $S^{(2)}$, and that the problem (2.1)–(2.4) for $S^{(1)}$ has p eigenvalues such that $0 < \nu_1^{(1)} \leqslant \nu_2^{(1)} \leqslant \ldots \leqslant \nu_p^{(1)} < k$. Then the eigenvalue problem for $S^{(2)}$ has at least p eigenvalues, and $\nu_s^{(1)} > \nu_s^{(2)}$, $s = 1,2,\ldots,p$.

Remark 4.1. *We introduce the family of contours $S(a,d) = \partial B(a,d)$, defined as $S(a,d) = S_0(a,d) \cup S^+_\infty(a,d) \cup S^-_\infty(a,d)$, where $S_0(a,d) = \big\{(r,\theta) : r = a, |\theta| \leqslant \pi/2\big\}$ and $S^\pm_\infty(a,d) = \big\{(x,y) : x = \pm a, y \leqslant -d\big\}$ (see fig. 2). Consider a body $B' \subset B(a,d)$ and a trial potential w satisfying for fixed values ν_0, k_0 of parameters ν, k the conditions (3.1), (3.5), (3.7), (3.8). Let also $\partial w/\partial n \leqslant 0$ on $S(a,d)$ and on the lower side of $B_\sigma(a,d) = B(a,d) \cap \{y \geqslant -\sigma\}$ for a sequence $\{\sigma_n\}$, such that $\sigma_n \to \infty$ as $n \to \infty$ and $B' \subset B_{\sigma_n}(a,d)$. We use the monotonicity principle for the domains B' and $B_{\sigma_n}(a,d)$ tending n to infinity. Since the non-existence of trapped modes for $B_{\sigma_n}(a,d)$ follows from (3.6), we can make of use remark 3.1 and, thus, arrive at the conclusion that under the above assumptions on w, trapped modes do not exist for $k \geqslant k_0$, $\nu \leqslant \nu_0$ for any body of finite size lying inside the contour $S(a,d)$.*

Consider the normal derivative of w_0 on $S(a,d)$. As the function $\mathrm{K}_0(kr)$ decays in r we find

$$\pm\frac{\partial w_0}{\partial x} < 0, \quad x = \pm a, \quad y \leqslant -d.$$

On $S_0(a,d)$ we have

$$\frac{\partial w_0}{\partial r}\Big|_{r=a} = -k\,\mathrm{K}_1(ka) + C_0 k\,\mathrm{e}^{ky}\frac{\partial y}{\partial r} \leqslant -k\,\mathrm{K}_1(ka) + C_0 k\,\mathrm{e}^{ky}.$$

Thus, if $\mathcal{F}_0(k,\nu,a,d) \geqslant 0$, where

$$\mathcal{F}_0(k,\nu,a,d) = \mathrm{K}_1(ka) - C_0(k,\nu,d)\,\mathrm{e}^{k(a-d)},$$

then the potential w_0 satisfies the condition $\partial w_0/\partial n \leqslant 0$ on $S(a,d)$. Thus, appealling to remark 4.1 and applying (3.6) with w_0 we prove that the problem (2.1)–(2.4) has only the trivial solution for frequencies and contours belonging to the set $\Omega_0 = \Omega(\mathcal{F}_0)$, where

$$\Omega(\mathcal{F}) = \{(k,\nu,B) : B \subset B(a,d),\ (k,\nu,a,d) \in \omega(\mathcal{F})\}, \tag{4.4}$$
$$\omega(\mathcal{F}) = \{(k,\nu,a,d) :\ \mathcal{F}(k,\nu,a,d) \geqslant 0\}. \tag{4.5}$$

Further we shall construct more complicated potentials w and corresponding functions $\mathcal{F}$ which will be proven to give wider than Ω_0 sets of uniqueness.

We define

$$w_1(x,y) = \mathrm{K}_0(kr) + \mathrm{K}_0(kr') + C_1(k,\nu,d)\,\mathrm{e}^{ky},$$

where r is given by (4.2) and

$$x = r'\sin\theta', \quad y = d - r'\cos\theta'. \tag{4.6}$$

By arguments analogous to those used for w_0 we find

$$|\nabla w_1| \leqslant k\,\mathrm{K}_1(kr) + k\,\mathrm{K}_1(kr') + kC_1\,\mathrm{e}^{ky},$$

and using 9.7.2 in (Abramowitz & Stegun 1965), we have

$$\frac{|\nabla w_1(x,y)|}{w_1(x,y)} \leqslant k\left\{1 + \frac{\mathrm{K}_1(kr)}{\mathrm{K}_0(kr)} + \frac{\mathrm{K}_1(kr')}{\mathrm{K}_0(kr')}\right\} = O(1) \quad \text{as} \quad |x + \mathrm{i}y| \to \infty.$$

The constant C_1 is defined by the condition (3.8). Since for $y = 0$,

$$\frac{\partial w_1}{\partial y} - \nu w_1 = -2\nu\,\mathrm{K}_0\big(k\sqrt{x^2+d^2}\big) + (k-\nu)C_1(k,\nu,d),$$

we fix

$$C_1 = C_1(k,\nu,d) = \frac{2\nu}{k-\nu}\,\mathrm{K}_0(kd).$$

Consider the normal derivative of the potential w_1 on $S(a,d)$. Taking into account the monotonicity of K_0 we have

$$\pm\frac{\partial w_1}{\partial x} < 0, \quad x = \pm a,\ y \leqslant -d.$$

On the upper semicircle $S_0(a,d)$ we obtain

$$\frac{\partial w_1}{\partial r} = -k\,\mathrm{K}_1(ka) + \frac{k(2d\cos\theta - a)}{\varrho(a,2d,\theta)}\,\mathrm{K}_1\big(k\,\varrho(a,2d,\theta)\big) + C_1 k\,\mathrm{e}^{ky}\frac{\partial y}{\partial r}. \tag{4.7}$$

where

$$\varrho(a,2d,\theta) = \sqrt{a^2+(2d)^2-4ad\cos\theta}. \tag{4.8}$$

It is easy to see that the maximum of the second term in the right-hand side of (4.7) is achieved for $\theta = 0$. Thus, $\partial w_1/\partial n \leqslant 0$ on $S(a,d)$ if

$$\mathcal{F}_1(k,\nu,a,d) \geqslant 0,$$

where

$$\mathcal{F}_1(k,\nu,a,d) = \mathrm{K}_1(ka) - \mathrm{K}_1\big(k(2d-a)\big) - C_1(k,\nu,d)\,\mathrm{e}^{k(a-d)}. \tag{4.9}$$

Using the notation (4.4) we define the set $\Omega_1 = \Omega(\mathcal{F}_1)$. This set of parameters k,ν and geometries B, for which the problem (2.1)–(2.4) has only the trivial solution, is wider than the set Ω_0. The inclusion $\Omega_0 \subset \Omega_1$ is a subsequence of the following assertion.

Proposition 4.1. *The inequality*

$$\mathcal{F}_0(k,\nu,a,d) \leqslant \mathcal{F}_1(k,\nu,a,d)$$

holds for all values of k, ν, a *and* d, *such that* $0<\nu<k$, $0<a<d$.

Proof. We use the representation of the Bessel functions K_n given by 8.432.8 in (Gradsteyn & Ryzhik 1994):

$$\mathrm{K}_n(kz) = \sqrt{\frac{\pi}{2z}}\,\frac{k^n\,\mathrm{e}^{-kz}}{\Gamma(n+1/2)}\int_0^\infty \mathrm{e}^{-kt}t^{n-\frac12}\left(1+\frac{t}{2z}\right)^{n-\frac12}\mathrm{d}t$$

which is valid when $|\arg z| < \pi$, $\mathrm{Re}\,n > -\frac12$, $k>0$. For real n, $n \geqslant \frac12$, we have

$$\mathrm{K}_n\big(k(z_1+z_2)\big) \leqslant \mathrm{e}^{-kz_2}\sqrt{\frac{\pi}{2z_1}}\,\frac{k^n\,\mathrm{e}^{-kz_1}}{\Gamma(n+1/2)}\int_0^\infty \mathrm{e}^{-kt}t^{n-\frac12}\left(1+\frac{t}{2z_1}\right)^{n-\frac12}\mathrm{d}t.$$

Thus, we establish the inequality

$$\mathrm{K}_n\big(k(z_1+z_2)\big) \leqslant \mathrm{e}^{-kz_2}\,\mathrm{K}_n(kz_1), \quad n \geqslant \frac12.$$

Applying the last formula to the difference of functions $\mathcal{F}_0$ and $\mathcal{F}_1$ we get

$$\begin{aligned}
&\mathcal{F}_0(k,\nu,a,d) - \mathcal{F}_1(k,\nu,a,d)\\
&\quad= \mathrm{K}_1\big(k(2d-a)\big) + \frac{2\nu}{k-\nu}\mathrm{e}^{k(a-d)}\,\mathrm{K}_0(kd) - \frac{\mathrm{e}^{k(a-d)}}{k-\nu}\Big\{k\,\mathrm{K}_1(kd) + \nu\,\mathrm{K}_0(kd)\Big\}\\
&\quad= \mathrm{K}_1\big(k(2d-a)\big) - \mathrm{e}^{k(a-d)}\,\mathrm{K}_1(kd) - \frac{\nu}{k-\nu}\,\mathrm{e}^{k(a-d)}\Big\{\mathrm{K}_1(kd) - \mathrm{K}_0(kd)\Big\} \leqslant 0,
\end{aligned}$$

where we also make of use the inequality $\mathrm{K}_n(z) > \mathrm{K}_m(z)$, for real z and $n>m$, which is obvious in view of (4.1). The proof is complete. □

Consider the boundary of the set $\omega_1 = \omega(\mathcal{F}_1)$ (see (4.5)) given by the equation

$$\mathcal{F}_1(k,\nu,a,d) = 0. \tag{4.10}$$

The latter equation defines a single-valued function $a^* = a^*(k,\nu,d)$ and the set ω_1 is located under the surface so that

$$\omega_1 = \{(k,\nu,a,d) : 0 < \nu < k,\ d > 0,\ 0 < a \leqslant a^*(k,\nu,d)]\}$$

which follows from the assertion.

Proposition 4.2. *For any fixed k, ν and d, $0 < \nu < k$, $d > 0$, there exists only one root a of the equation (4.10).*

Proof. By (4.9) and 9.6.9 in (Abramowitz & Stegun 1965) we have

$$\mathcal{F}_1(k,\nu,a,d) \to +\infty \quad \text{as} \quad a \to +0, \qquad \mathcal{F}_1(k,\nu,d,d) = -\frac{2\nu}{k-\nu}\,\mathrm{K}_0(kd) < 0.$$

Besides, $\partial\mathcal{F}_1/\partial a < 0$ in view of the representation

$$\frac{\partial\mathcal{F}_1}{\partial a} = -\frac{k}{2}\Big\{\mathrm{K}_0(ka) + \mathrm{K}_2(ka) + \mathrm{K}_0\big(k(2d-a)\big) + \mathrm{K}_2\big(k(2d-a)\big) + \frac{4\nu\,\mathrm{e}^{k(a-d)}}{k-\nu}\,\mathrm{K}_0(kd)\Big\},$$

which completes the proof. □

In view of the proposition 4.2 it is reasonable to use the dependence of a on other parameters for presentation of the surface defined by the equation (4.10). Results of numerical computations of the dependence of a/d on k/ν for some fixed values of νd are presented in fig. 3, 4 (dashed lines). The set of uniqueness is located under the curves and marked by the letter 'ω'.

Asymptotic behaviour of the solution of (4.10) can be investigated using well-known asymptotics of the Bessel functions K_n. In particular, using the asymptotic representation of $\mathrm{K}_n(z)$ as $z \to \infty$ given by 8.451.6 in (Gradsteyn & Ryzhik 1994) one can find that $a/d \sim 1 - \nu d/(dk)^2$ as $k \to \infty$.

5. Green's function of the problem (2.1)–(2.4)

In order to construct another example of the potential w for the identity (3.6) we shall use the Green function of the problem (2.1)–(2.4). The potential $G(x,y,-d)$ of a source located at the point $(0,-d)$ must satisfy:

$$\nabla^2_{x,y}G - k^2G = -2\pi\delta(x,y+d), \quad y < 0,$$
$$\frac{\partial G}{\partial y} - \nu G = 0, \quad y = 0,$$
$$G \to 0 \quad \text{as} \quad y \to -\infty,$$

where δ is the delta function. Using the representation of the Green function given in (Ursell 1950) and the formula

$$\mathrm{K}_0(k\sqrt{x^2+y^2}) = \int_0^\infty \frac{\mathrm{e}^{y\sqrt{t^2+k^2}}\cos xt\,\mathrm{d}t}{\sqrt{t^2+k^2}} \tag{5.1}$$

(see 3.961.2 in Gradsteyn & Ryzhik 1994), we write

$$G(x,y,-d) = \mathrm{K}_0(kr) + \mathrm{K}_0(kr') + 2\nu \int_0^\infty \frac{\mathrm{e}^{(y-d)\sqrt{t^2+k^2}} \cos xt \,\mathrm{d}t}{\sqrt{t^2+k^2}\big(\sqrt{t^2+k^2}-\nu\big)} \tag{5.2}$$

$$= \mathrm{K}_0(kr) - \mathrm{K}_0(kr') + 2\,I(x, y-d), \tag{5.3}$$

where r and r' are defined in (4.2) and (4.6) and

$$I(x,y) = \int_0^\infty \frac{\mathrm{e}^{y\sqrt{t^2+k^2}} \cos xt \,\mathrm{d}t}{\sqrt{t^2+k^2}-\nu} \tag{5.4}$$

The Green function satisfies the condition (3.7), (3.8) and in order to use G as the trial function in the identity (3.6) we have to check positiveness of the function and to consider behaviour of $|\nabla G|/G$ as $|x+\mathrm{i}y| \to \infty$ and of the normal derivative on $S(a,d)$ (see (3.1), (3.5) and (3.9)). We start with proof of positiveness of the Green function. We need the following auxiliary assertion.

Lemma 5.1. *For all values of x, y, k and ν such that $y \leqslant c < 0$ and $0 < \nu < k$, there holds the inequality $I(x,y) > 0$.*

Proof. We start with noting that the function I defined by (5.4) is a holomorphic function of complex ν in the disc $|\nu| < k$ and has a singularity on the boundary of the circle at $\nu = k$. We can write the expansion of I at the point $\nu = 0$ when the other parameters are fixed as

$$I_{(x,y,k)}(\nu) = \sum_{n=0}^{\infty} c_n \nu^n, \tag{5.5}$$

where

$$c_n(x,y,k) = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\nu^n} I_{(x,y,k)}(\nu)\,\Big|_{\nu=0} = \int_0^\infty \frac{\mathrm{e}^{y\sqrt{t^2+k^2}} \cos xt \,\mathrm{d}t}{\big(t^2+k^2\big)^{\frac{n+1}{2}}}. \tag{5.6}$$

Using (5.1), we write

$$c_0(x,y,k) = \mathrm{K}_0(k\sqrt{x^2+y^2}).$$

This term is strictly positive (see (4.1)). Furthermore, it is easily seen that

$$c_1(x,y,k) = \int_{-\infty}^{y} \int_0^\infty \frac{\mathrm{e}^{\xi\sqrt{t^2+k^2}} \cos xt}{\big(t^2+k^2\big)^{\frac{1}{2}}} \,\mathrm{d}t \,\mathrm{d}\xi = \int_{-\infty}^{y} \mathrm{K}_0(k\sqrt{x^2+\xi^2}) \,\mathrm{d}\xi > 0. \tag{5.7}$$

In the same way we have

$$c_n(x,y,k) = \int_{-\infty}^{y} c_{n-1}(x,\xi,k) \,\mathrm{d}\xi. \tag{5.8}$$

Thus, by (5.7) and (5.8), $c_n > 0$ for $n = 0, 1, 2, \ldots$ and since the radius of convergence of (5.5) is equal to k, for all real ν, such that $0 < \nu < k$, we have $I_{(x,y,k)}(\nu) > 0$. The proof is complete. □

Theorem 5.2. *For any k, ν and d, such that $0 < \nu < k$, $d > 0$, the Green function $G(x, y, -d)$ of the problem (2.1)–(2.4) is strictly positive for all x, y, where $(x, y) \neq (0, -d)$, $y \leqslant 0$.*

Proof. In view of lemma 5.1 we should only note that by monotonicity of K_0 the sum of first two terms in (5.3) is strictly positive for $y < 0$ and equal to zero for $y = 0$. □

Theorem 5.3. *For any k, ν and d, such that $0 < \nu < k$, $d > 0$, the x-derivative of Green's function $G_x(x, y, -d)$ is negative (positive) for all x, y, where $x > 0$ ($x < 0$), $y \leqslant 0$ and $(x, y) \neq (0, -d)$.*

Proof. Using (5.3) we write

$$G_x(x, y, -d) = -k\,\mathrm{K}_1(kr)\frac{\partial r}{\partial x} + k\,\mathrm{K}_1(kr')\frac{\partial r'}{\partial x} + 2\,I_x(x, y - d). \tag{5.9}$$

$$I_x(x, y - d) = \int_0^\infty \frac{t\,\mathrm{e}^{(y-d)\sqrt{t^2+k^2}}\sin xt}{\sqrt{t^2+k^2} - \nu}\mathrm{d}t.$$

Let us consider the case $x > 0$. Negativeness of the sum of first two terms in (5.9) is obvious in view of monotonicity of K_1. Further, following the scheme applied in lemma 5.1 we expand I_x in a series in ν, $|\nu| < k$

$$I_x(x, y, k, \nu) = -\sum_{n=0}^{\infty} f_n \nu^n, \qquad f_n = \int_0^\infty \frac{t\,\mathrm{e}^{y\sqrt{t^2+k^2}}\sin xt}{(t^2+k^2)^{\frac{n+1}{2}}}\mathrm{d}t. \tag{5.10}$$

Since by formula 3.961.1 in (Gradsteyn & Ryzhik 1994),

$$f_0 = \frac{kx}{\sqrt{x^2+y^2}}\,\mathrm{K}_1(k\sqrt{x^2+y^2}) > 0, \quad x > 0,$$

the proof of lemma 5.1 can be repeated literally to result in the inequality

$$I_x(x, y - d) < 0, \quad x > 0,$$

which completes the proof. □

The Green function also satisfies the condition (3.5) which is established in the following assertion.

Lemma 5.4. *The estimate holds*

$$\frac{|\nabla_{x,y} G(x, y, -d)|}{G(x, y, -d)} = O(1) \quad \text{as} \quad |x + \mathrm{i}y| \to \infty.$$

Proof. We start with noting that the third term in the right-hand side of (5.2) is positive, since it has the following expansion in ν, $|\nu| < k$ (see the proof of lemma 5.1):

$$\int_0^\infty \frac{\mathrm{e}^{(y-d)\sqrt{t^2+k^2}}\cos xt\,\mathrm{d}t}{\sqrt{t^2+k^2}\big(\sqrt{t^2+k^2} - \nu\big)} = \sum_{n=0}^{\infty} c_{n+1}(x, y - d, k)\nu^n,$$

where positive coefficients c_n are defined by (5.6). Further, by (5.2) we have

$$G_y(x, y, -d) = -k\,\mathrm{K}_1(kr)\frac{\partial r}{\partial y} - k\,\mathrm{K}_1(kr')\frac{\partial r'}{\partial y} + 2\nu\, I(x, y-d).$$

Then, using (5.2) and (5.3) and taking into account positiveness of the third term in the right-hand side of (5.2) and of the sum of the first two terms in the right-hand side of (5.3), we obtain

$$\frac{|G_y(x, y, -d)|}{G(x, y, -d)} \leqslant 2\nu + k\left\{\frac{\mathrm{K}_1(kr)}{\mathrm{K}_0(kr)} + \frac{\mathrm{K}_1(kr')}{\mathrm{K}_0(kr')}\right\} = O(1) \quad \text{as} \quad |x + \mathrm{i}y| \to \infty. \tag{5.11}$$

Estimation of the term $|G_x|/G$ is not so straightforward. We write the term I_x in (5.9) in the form

$$I_x(x, y-d) = \frac{\mathrm{i}k^2}{2}\int_{-\infty}^{\infty} \frac{\sinh\mu\cosh\mu}{k\cosh\mu - \nu}\exp\{-kr'\cosh(\mu - \mathrm{i}\theta')\}\,\mathrm{d}\mu, \tag{5.12}$$

where r' and θ' are defined in (4.6). The change of variable in the latter formula follows the consideration in (Ursell 1968, § 2). In the same way we have

$$I(x, y-d) = \frac{k}{2}\int_{-\infty}^{\infty} \frac{\cosh\mu}{k\cosh\mu - \nu}\exp\{-kr'\cosh(\mu - \mathrm{i}\theta')\}\,\mathrm{d}\mu. \tag{5.13}$$

The expressions under integral sign in (5.12) and (5.13) have simple poles in the complex μ-plane, $\mu = \pm\mathrm{i}(\tau + 2\pi n)$, where $n = 0, 1, 2, \ldots$ and $\tau = \arccos(\nu/k)$.

We consider the case $x \geqslant 0$ and define the function

$$\begin{aligned} I_*(x, y-d) &= \frac{2}{k^2}\left[I_x(x, y-d) + \sqrt{k^2 - \nu^2}\, I(x, y-d)\right] \\ &= \int_{-\infty}^{\infty} f(\mu, k, \nu)\exp\{-kr'\cosh(\mu - \mathrm{i}\theta')\}\,\mathrm{d}\mu, \end{aligned} \tag{5.14}$$

where the function

$$f(\mu, k, \nu) = \frac{\mathrm{i}\big(\sinh\mu - \sinh(\mathrm{i}\tau)\big)\cosh\mu}{k\cosh\mu - \nu}$$

is holomorphic in μ. Moving contour of integration in (5.14) we find

$$\begin{aligned} I_*(x, y-d) &= \int_{-\infty+\mathrm{i}\theta'}^{\infty+\mathrm{i}\theta'} f(\mu, k, \nu)\exp\{-kr'\cosh(\mu - \mathrm{i}\theta')\}\,\mathrm{d}\mu \\ &= \int_{-\infty}^{\infty} f^*(\mu, \theta', k, \nu)\exp\{-kr'\cosh(\mu)\}\,\mathrm{d}\mu, \end{aligned}$$

where $f^*(\mu, \theta', k, \nu) = \mathrm{Re}\big\{f(\mu + \mathrm{i}\theta', k, \nu)\big\}$ and

$$f^*(\mu, \theta', k, \nu) = \frac{\alpha_0 + \alpha_1\cosh(\mu) + \alpha_2\cosh(2\mu) + \alpha_3\cosh(3\mu)}{2\nu^2 + k^2\cos(2\,\theta') - 4k\nu\cos(\theta')\cosh(\mu) + k^2\cosh(2\mu)}, \tag{5.15}$$

$$\alpha_0 = k\gamma\cos(2\,\theta'), \quad \alpha_1 = -2\nu\gamma\cos(\theta') - k\sin(3\,\theta')/2,$$

$$\alpha_2 = k\gamma + \nu\sin(2\,\theta'), \quad \alpha_3 = -k\sin(\theta')/2, \quad \gamma = \sqrt{1 - \nu^2/k^2}.$$

The numerator in the right-hand side of the formula (5.15) is obviously greater than $k\,(k-4\nu\cos(\theta'))\cosh(2\mu)$, and it is easily seen that if θ'_* is large enough, for $\theta'\in[\theta'_*,\pi/2]$ the numerator can be estimated from below by a strictly positive constant. Then, by (5.15) we have for some constant $C=C(k,\nu,\theta'_*)$

$$|f^*(\mu,\theta,k,\nu)|\leqslant C\,\cosh(\mu),\quad \text{for}\quad \theta'\in[\theta'_*,\pi/2],\ \mu\in(-\infty,+\infty),$$

and by using (4.1) we arrive at

$$|\,I_*\,|\leqslant C\ \mathrm{K}_1(kr'),\quad \theta'\in[\theta'_*,\pi/2]. \tag{5.16}$$

We consider the subdomain of W, $W_b=\{(x,y)\in W: x>0, |x+\mathrm{i}y|>b\}$. Let $\partial W_b=F_b\cup\gamma_b\cup V_b$, where F_b is the part of free surface $x>b$, γ_b is the quarter-circle $|x+\mathrm{i}y|=b$, $x\geqslant 0$, $y\leqslant 0$ and V_b is the part of the y-axis, $y<-b$. Let b be large enough so that $\theta'>\theta'_*$ for all points of F_b (θ' is given in (4.6)). We define

$$g(x,y;c)=c\,G(x,y,-d)+G_x(x,y,-d),\quad (x,y)\in W_b.$$

Since the function $G(x,y,-d)$ is strictly positive for $(x,y)\in\partial W_b$ and the function $G_x(x,y,-d)$ is equal to zero on V_b, a majorant $c\,G$ with some constant c can be found for $-G_x=|G_x|$ on $\gamma_b\cup V_b$. Further, since (5.16) holds on F_b, by (5.3), (5.9) and (5.14) the expression $|G_x(x,y,-d)+\sqrt{k^2-\nu^2}G(x,y,-d)|$ can be majorized on F_b by $c_1\,\mathrm{K}_1(kr)+c_2\,\mathrm{K}_1(kr')$ with some constants c_1 and c_2. Thus, from (5.2) and asymptotics of $\mathrm{K}_0,\mathrm{K}_1$ at infinity it follows that a value of the parameter $c>0$ can be found such that $g(x,y,c)>0$ when $(x,y)\in\partial W_b$. At the same time, from Theorem 6 in (Protter & Weinberger 1984, ch. 2), it follows that the function $g(x,y)$ can not attain non-positive minimum at an interior point of W_b. It means that $g(x,y)>0$ in W_b and in view of Theorems 5.2 and 5.3 we have $|G_x(x,y,-d)|/G(x,y,-d)<c$. Combining this inequality with (5.11) completes the proof. □

6. Eigenvalue bounds using the Green function

We define the potential

$$w_G(x,y)=G(x,y,-d).$$

From the above it follows that this potential satisfies (3.5), (3.7) and (3.8), and we have to consider the normal derivative of the potential on the contour $S(a,d)$ (see fig. 2). First we note that from Theorem 5.3 it follows that $\partial w_G/\partial x<0$ for $x=a$, $y\leqslant -d$. Thus, we only have to check the condition (3.9) for w_G on the upper semicircle $S_0(a,d)$. Using (5.2) we write (cf. (4.7))

$$\frac{\partial w_G}{\partial r}\Big|_{r=a}=-k\,\mathrm{K}_1(ka)+\frac{k(2d\cos\theta-a)}{\varrho(a,2d,\theta)}\,\mathrm{K}_1\big(k\,\varrho(a,d,\theta)\big)-2\nu\,H(x,y-d)\sin\theta$$
$$+\,2\nu\,I(x,y-d)\cos\theta,$$

where $\varrho(a,d,\theta)$ is defined in (4.8) and

$$H(x,y)=\int_0^\infty\frac{t\,\mathrm{e}^{y\sqrt{t^2+k^2}}\sin xt}{\sqrt{t^2+k^2}\big(\sqrt{t^2+k^2}-\nu\big)}\,\mathrm{d}t.$$

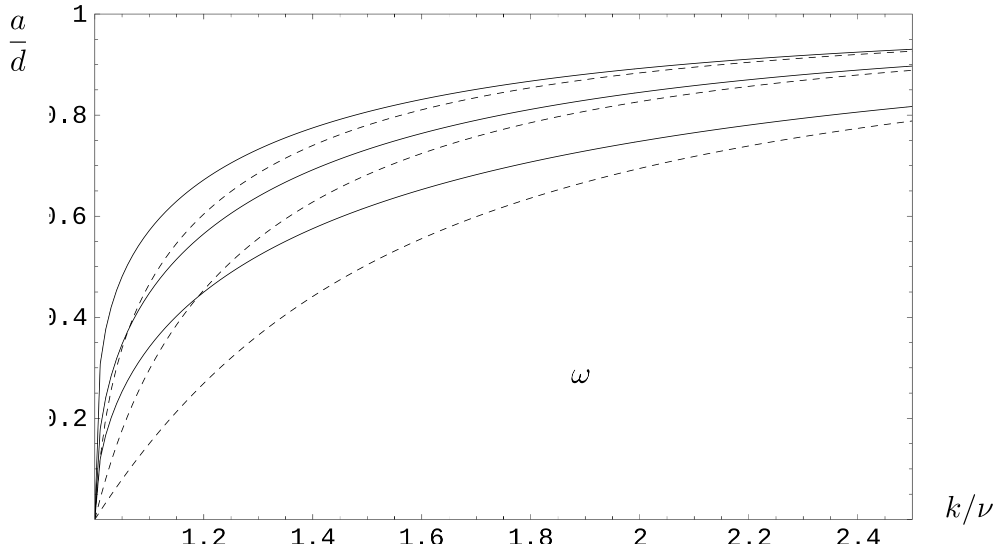

Figure 3. The dependence of a/d on k/ν given by (6.3) (—) and by (4.10) (- - -) for fixed values of $\nu d = 0.4$, 1.2 and 2 (counting from below).

It is easily seen that $H(x, y) > 0$ for $x > 0$, $y < 0$, because this function can be expanded in ν, $|\nu| < k$ (see lemma 5.1) $H_{(x,y,k)}(\nu) = \sum_{n=0}^{\infty} f_{n+1}\nu^n$, where the coefficients f_n are given by (5.10) and $f_n > 0$.

We define

$$I_0(k, \nu, y) = \int_0^{\infty} \frac{\mathrm{e}^{y\sqrt{t^2+k^2}}\,\mathrm{d}t}{\sqrt{t^2+k^2}-\nu}. \tag{6.1}$$

Since for any k, ν, such that $0 < \nu < k$, $I(x, y) \leqslant I_0(k, \nu, y)$ and since $I_0(k, \nu, y)$ is an increasing function of y, we finally arrive at the inequality

$$\frac{\partial w_G}{\partial r}\bigg|_{r=a} \leqslant -k\,\mathcal{F}_G(k, \nu, a, d),$$

where

$$\mathcal{F}_G(k, \nu, a, d) = \mathrm{K}_1(ka) - \mathrm{K}_1\big(k(2d-a)\big) - 2k^{-1}\nu\, I_0(k, \nu, a - 2d). \tag{6.2}$$

Hence, using the notation (4.4) we can define the set $\Omega_G = \Omega(\mathcal{F}_G)$ of parameters k, ν and geometries B, for which the problem (2.1)–(2.4) has only the trivial solution. It can be proved that the set Ω_G contains the set Ω_1. This inclusion is a subsequence of the following assertion.

Proposition 6.1. *The inequality*

$$\mathcal{F}_1(k, \nu, a, d) \leqslant \mathcal{F}_G(k, \nu, a, d)$$

holds for all values of k*,* ν*,* a *and* d *such that* $0 < \nu < k$*,* $0 < a < d$*.*

Proof. Using (4.9) and (6.2) we write

$$\Delta = \frac{1}{2\nu}\Big[\mathcal{F}_G(k, \nu, a, d) - \mathcal{F}_1(k, \nu, a, d)\Big] = \frac{\mathrm{e}^{k(a-d)}}{k-\nu}\,\mathrm{K}_0(kd) - \frac{1}{k}\, I_0(k, \nu, a - 2d).$$

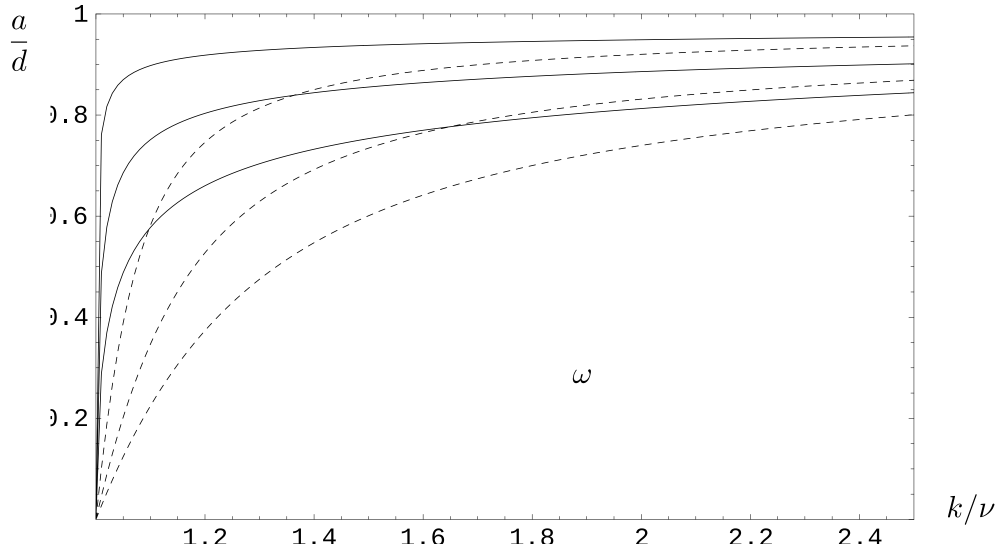

Figure 4. The dependence of a/d on k/ν given by (6.3) (—) and by (4.10) (- - -) for fixed values of $\nu d = 0.01$, 0.03 and 0.07 (counting from above).

By 8.432.9 in (Gradsteyn & Ryzhik 1994) we rewrite Δ as follows

$$\Delta = \frac{e^{k(a-d)}}{k-\nu}\int_0^\infty \frac{e^{-d\sqrt{t^2+k^2}}}{\sqrt{t^2+k^2}}\,dt - \frac{1}{k}\int_0^\infty \frac{e^{(a-2d)\sqrt{t^2+k^2}}}{\sqrt{t^2+k^2}-\nu}\,dt.$$

Then, we have

$$\Delta \geqslant e^{k(a-d)}\int_0^\infty \left\{ \frac{1}{(k-\nu)\sqrt{t^2+k^2}} - \frac{1}{k\big(\sqrt{t^2+k^2}-\nu\big)} \right\} e^{-d\sqrt{t^2+k^2}}\,dt.$$

The term in curly brackets is equal to

$$\frac{\nu\big(\sqrt{t^2+k^2}-k\big)}{k(k-\nu)\sqrt{t^2+k^2}\big(\sqrt{t^2+k^2}-\nu\big)}$$

Under the assumption imposed the latter expression is obviously positive, which completes the proof. □

Results of numerical computations for the equation

$$\mathcal{F}_G(k,\nu,a,d) = 0 \tag{6.3}$$

are presented in fig. 3, 4 and 5 (solid lines), where the domain of uniqueness is marked by the letter 'ω'.

7. Correlation of the eigenvalue bounds with known bounds and examples of trapped modes

In this section we shall compare the bounds for the uniqueness set given by (6.3) with known bounds derived in (McIver & Evans 1985, Simon 1992) and examples

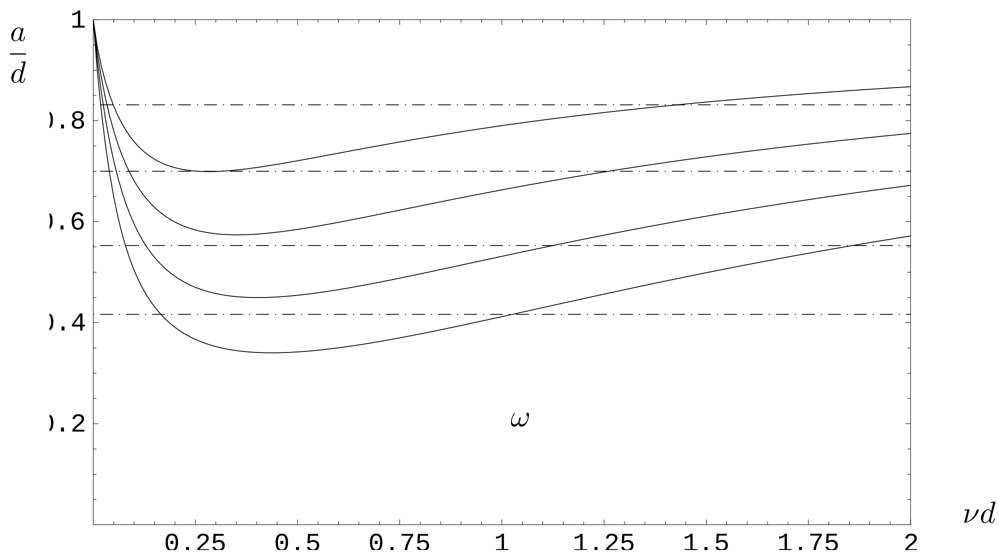

Figure 5. The dependence of a/d on νd given by (6.3) (—) and by (7.1) (–·–) for fixed values of $k/\nu = 1.1$, 1.2, 1.4 and 1.8 (counting from below).

of trapped modes for a submerged cylinder constructed in the papers (Ursell 1950, McIver 1991, Porter & Evans 1998).

First we address to the results by Simon (1992). Consider an edge, formed by two lines: L and its reflection in the y-axis L', which emanate from the origin and go to infinite depth not intersecting a contour S. Let $\beta = \beta(y)$ be the angle that L and L' make with the downward vertical. In (Simon 1992) the eigenvalue bounds for the geometry S were given in the form

$$\frac{k}{\nu} \geqslant \frac{m^2+1}{2m} + k \int_0^\infty \mathrm{e}^{2kmy} \tan^2 \beta(y)\, \mathrm{d}y,$$

where m is an arbitrary positive value. It seems that quite good eigenvalue bounds can be obtained using the latter formulation and, since the edge can include the shape $S(a,d)$ and not vice versa, the bounds by Simon (1992) are valid for a more general class of obstacles than the bounds obtained in the present paper. However, the question of optimal choice of the contour L and the value of m was not discussed at length by Simon (1992) and explicit formula for bounds was given only for the case when L is a straight line. For the geometry $B(a,d)$ introduced in fig. 2 the uniqueness takes place when

$$a/d \leqslant \sqrt{1-\nu^2/k^2}. \tag{7.1}$$

The bound (7.1) was earlier derived by McIver & Evans (1985) for symmetric modes trapped near a circular cylinder. Comparison of the bounds given by (6.3) and the bounds (7.1) is shown in fig. 5. It is to note that the bounds (7.1) turn out to be better in some range of parameters of the problem.

Further we compare the bounds delivered by (6.3) with known examples of trapped modes. As was shown numerically by McIver & Evans (1985), more than one trapped mode for a fixed cylinder can appear and the number of modes increases as the depth of submergence is decreased. For the trapped mode, which is symmetric

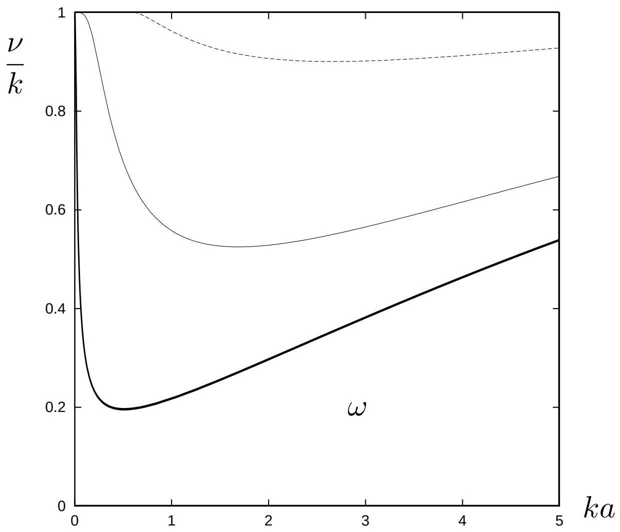

Figure 6. Curves showing the dispersion relation for symmetric (—) and antisymmetric (---) modes constructed by Porter & Evans (1998) and the solution of the equation (6.3) (lower curve) when $d/a = 1.1$.

in x and has smallest value of the ratio k/ν, asymptotics of the dispersion relation near the cut-off (as $k/\nu \to 1{+}0$) was obtained in (Ursell 1950) for a circular cylinder and in (McIver 1991) for a cylinder having symmetric in x but otherwise arbitrary geometry. According to (Ursell 1950), the asymptotics of the mode is as follows

$$\nu a \sim (3\pi)^{-\frac{1}{2}}\, \mathrm{e}^{\nu d} \left(k^2/\nu^2 - 1\right)^{\frac{1}{4}} \quad \text{as} \quad k/\nu \to 1+0, \tag{7.2}$$

where the same notation for the radius a and the submergence d of circular cylinder's centre is used. At the same time, asymptotics of the solution to (6.3) near the cut-off can be obtained with the help of the representation (A.3) derived in Appendix A. We write (6.3) in the form

$$\mathrm{K}_1(ka) = \mathrm{K}_1\big(k(2d-a)\big) + 2k^{-1}\nu\, I_0(k,\nu,a-2d). \tag{7.3}$$

In view of (A.3) the right-hand side of the last equality has estimate

$$2\pi\, \mathrm{e}^{\nu(a-2d)} \left(k^2/\nu^2 - 1\right)^{-\frac{1}{2}} + \phi(k,\nu,a,d) \quad \text{as} \quad k/\nu \to 1+0,$$

where $\phi = O(1)$ uniformly in a when $d > 0$. Thus, by using the asymptotics 9.6.9 in (Abramowitz & Stegun 1965) for the left-hand side of (7.3), we arrive at

$$\nu a \sim (2\pi)^{-1}\, \mathrm{e}^{2\nu d} \left(k^2/\nu^2 - 1\right)^{\frac{1}{2}} \quad \text{as} \quad k/\nu \to 1+0.$$

Obviously, the latter asymptotics of the bound given by (6.3) is consistent with the asymptotics (7.2) of the dispersion relation, but it is not sharp. However, it is to be taken into account that the formula (7.2) is established only for the lower in k/ν and symmetric in x trapped mode and the equation (6.3) delivers bounds for all modes.

Shown in fig. 6 are the bounds given by (6.3) and the symmetric and antisymmetric trapped modes for submerged circular cylinder of radius a and with centre at $(0, -d)$ obtained numerically in (Porter & Evans 1998). The uniqueness set is marked by the letter 'ω'.

8. Conclusion

In this work we consider the linearised water-wave problem describing motion of fluid in presence of a cylindrical body spanning a channel of infinity depth and length and of finite width. Existence of surface waves trapped near the cylinder is well-established in (Ursell 1950, Jones 1953, Ursell 1987). In this work a method, which relies on integral identity by Grimshaw (1974), has been suggested which yields simple bounds for the frequency of the trapped-mode solutions. Comparison with known bounds and examples of trapped modes has been given.

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Appendix A

In this section we consider the function $I_0(k,\nu,y)$ defined in (6.1). We write

$$I_0(k,\nu,y)=\frac{1}{2}\int_{-\infty}^{+\infty} s(k,\nu,y,t)\,\mathrm{d}t,\quad s(k,\nu,y,t)=\frac{\mathrm{e}^{y\sqrt{t^2+k^2}}}{\sqrt{t^2+k^2}-\nu}.$$

We treat $s(k,\nu,t)$ as a single-valued holomorphic function of $t=t_1+\mathrm{i}\,t_2$ in $\mathbb{C}\setminus\{(t_1,t_2):t_1=0,|t_2|\geqslant k\}$ and integrate $s(k,\nu,y,t)$ over the contour C_R shown in fig. 7. As $R\to\infty$, we obtain

$$2\,I_0(k,\nu,y)=-2\pi\mathrm{i}\,\mathrm{Res}_{t=-\mathrm{i}\sqrt{k^2-\nu^2}}\,s(k,\nu,y,t)+\int_{\gamma_-} s(k,\nu,y,t)\,\mathrm{d}t. \tag{A.1}$$

The contour γ_- consists of two sides the lower branch-cut, where the expression $\sqrt{t^2+k^2}$ has different signs. We have

$$\mathrm{Res}_{t=-\mathrm{i}\sqrt{k^2-\nu^2}}\,s(k,\nu,y,t)=\frac{\mathrm{i}\nu\,\mathrm{e}^{\nu y}}{\sqrt{k^2-\nu^2}}. \tag{A.2}$$

Changing variable t for $\mathrm{i}\mu$ in $\int_{\gamma_-} s(k,\nu,y,t)\,\mathrm{d}t$, we arrive at

$$\int_{\gamma_-} s(k,\nu,y,t)\,\mathrm{d}t=2\int_k^{\infty}\frac{\sqrt{\mu^2-k^2}\cos\big(y\sqrt{\mu^2-k^2}\big)+\nu\sin\big(y\sqrt{\mu^2-k^2}\big)}{\mu^2-k^2+\nu^2}\mathrm{d}\mu.$$

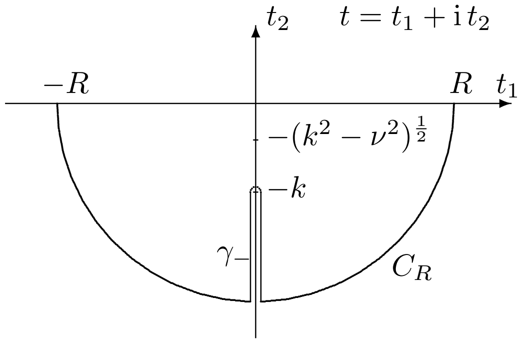

Figure 7. Contour of integration

Combining the latter formula with (A.1), (A.2) and the formula 3.754.2 in (Gradsteyn & Ryzhik 1994), after some simple algebra we find

$$I_0(k,\nu,y) = \frac{\pi\nu\,\mathrm{e}^{\nu y}}{\sqrt{k^2-\nu^2}} + \mathrm{K}_0(-ky) + \nu\int_0^\infty \frac{z\sin(yz)-\nu\cos(yz)}{(z^2+\nu^2)\sqrt{z^2+k^2}}\,\mathrm{d}z. \qquad \text{(A.3)}$$

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