On the two-dimensional sloshing problem

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We study an eigenvalue problem with a spectral parameter in a boundary condition. This problem for the two-dimensional Laplace equation is relevant to sloshing frequencies that describe free oscillations of an inviscid, incompressible, heavy fluid in a canal having uniform cross-section and bounded from above by a horizontal free surface. It is demonstrated that there exist domains such that at least one of the eigenfunctions has a nodal line or lines with both ends on the free surface (earlier, Kuttler (1984) tried to prove that there are no such nodal lines for all domains, but his proof is erroneous). It is also shown that the fundamental eigenvalue is simple and for the corresponding eigenfunction the behaviour of the nodal line is characterized. For this purpose, a new variational principle is proposed for an equivalent statement of the sloshing problem in terms of the conjugate stream function.

Keywords: Laplace equation; sloshing problem; eigenfunction; nodal domain; nodal line; simple eigenvalue

1. Introduction

The present paper deals with a boundary value problem for the Laplace equation when there is a spectral parameter in a boundary condition. This problem, usually referred to as the sloshing problem, describes natural frequencies and the corresponding modes of the free wave motion. Mainly we are concerned with waves in an infinitely long canal having a uniform cross-section, but waves in a bounded container will also be discussed.

An inviscid, incompressible, heavy fluid (water) occupies a canal bounded from above by a free surface of finite width. The surface tension is neglected and we assume the water motion to be irrotational and of small-amplitude. The latter assumption allows us to linearize boundary conditions on the free surface which leads to the following statement of the problem in the case of the two-dimensional motion in planes normal to the generators of the canal bottom. Let rectangular Cartesian coordinates (x, y) be taken in the plane of the motion with the origin and the x-axis in the mean free surface, whereas the y-axis is directed upwards. With a time-harmonic factor removed, the velocity potential $u(x, y)$ for the flow must satisfy the boundary value problem:

$$u_{xx} + u_{yy} = 0 \quad \text{in } W, \tag{1.1}$$

$$u_y = \nu u \quad \text{on } F, \tag{1.2}$$

$$\partial u/\partial n = 0 \quad \text{on } B. \tag{1.3}$$

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Here the cross-section W of the canal is a bounded simply connected domain whose piecewise smooth boundary ∂W has no cusps. One of the open arcs forming ∂W is an interval F of the x-axis (the free surface of water), and the bottom $B = \partial W \setminus \overline{F}$ is the union of open arcs, lying in the half-plane $y < 0$, complemented by corner points (if there are any) connecting these arcs. We suppose that the orthogonality condition

$$\int_F u \, \mathrm{d}x = 0 \tag{1.4}$$

holds, thus excluding the zero eigenvalue of (1.1)–(1.3), in which case the spectral parameter ν is equal to ω^2/g, where ω is the radian frequency of the water oscillations and g is the acceleration due to gravity.

The sloshing problem has been the subject of a great number of studies over more than two centuries (a historical review was given by Fox & Kuttler (1983)). It has been well-known since the 1950s that problem (1.1)–(1.4) has a discrete spectrum; that is, there exists a sequence of eigenvalues

$$0 < \nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_n \leqslant \ldots, \tag{1.5}$$

each having a finite multiplicity equal to the number of repetitions in (1.5), and such that $\nu_n \to \infty$ as $n \to \infty$. (Some authors, in particular Kuttler (1984), count the sloshing eigenvalues in a different way starting with the zero eigenvalue as the first one.) The corresponding eigenfunctions $\{u_n\}_1^\infty \subset H^1(W)$ form a complete system in an appropriate Hilbert space. These results can be found in many sources the most recent of which is the book by Kopachevsky & Krein (2001).

The behaviour of nodal lines is a classical topic of the spectral theory for boundary value problems that goes back to works of Courant & Hilbert (1953). However, there is only one note by Kuttler (1984) concerning nodal lines of problem (1.1)–(1.4). The approach of Kuttler (1984) is based on the following key lemma.

Nodal lines of an eigenfunction of problem (1.1)–(1.4) *have one end on the free surface and the other one on the bottom.*

Examining the proof of this lemma shows that there is a gap in Kuttler's reasoning that depends on a contradiction to which he tries to come in the following way. He constructs a *nonharmonic* function Φ that is a linear combination of certain functions φ_i that are admissible for the Rayleigh quotient of the sloshing problem:

$$\frac{\int_W (u_x^2 + u_y^2) \, \mathrm{d}x\mathrm{d}y}{\int_F u^2 \, \mathrm{d}x}.$$

Noting that Φ satisfies the boundary condition (1.2), he claims without proof that Φ minimises the above quotient, and so is *harmonic* which leads to a contradiction. However, since Φ is a linear combination of φ_i, it has a discontinuous gradient on a set of curves which obstructs the fact that the Rayleigh quotient achieves a minimum for Φ because the argument based on Green's theorem and applicable to a single φ_i does not holds for Φ.

Our attempt to fill in the gap resulted in constructing an example of sloshing eigenfunction that has a nodal line with both ends on the free surface (see Theorem 2.6 in Section 2). The construction involves the same velocity potentials in

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$\mathbb{R}^2_-$ with singularities on $\partial\mathbb{R}^2_-$ which earlier were used for demonstrating the existence of point eigenvalues embedded in the continuous spectrum of the water-wave problem.

Since all results formulated by Kuttler (1984) are proved by using the above fallacious lemma, it is necessary to check whether they are true. It occurs that one of the main results of Kuttler (1984), the simplicity of the fundamental eigenvalue, is valid; this is demonstrated in Theorem 3.1 (Section 3) by means of a new variational principle for an equivalent spectral problem in which stream function appears instead of the velocity potential. The denominator of the corresponding Rayleigh quotient involves a nonlocal operator, whereas the Dirichlet integral stands in the numerator. In Theorem 3.1, we also prove that the fundamental eigenfunction has only one nodal line connecting F and $\overline{B}$. This nodal line cannot re-enter $\overline{F}$ at the endpoints when W lies within the vertical semistrip bounded by $\overline{F}$ from above (this condition is usually referred to as John's condition). However, such a behaviour of this nodal line is an open question for domains of general geometry with connected F. In Section 4 we discuss other open questions such as the simplicity of all eigenvalues and give some numerical results illustrating the plethora of patterns of nodal lines.

2. Nodal lines and domains of the velocity potential

In this section, we construct an example of the sloshing problem, possessing an eigenfunction that has only one nodal line whose both ends are on F (Subsection 2.1), and consider some simple properties of nodal domains.

(*a*) *Example*

Our example involves a particular pair velocity potential/stream function introduced in (Kuznetsov et al., 2002, Subsection 4.1.1). The simplest example of this kind was proposed by McIver (1996), but for our purpose we need another one that has more nodal lines. Here we investigate nodal lines of u and v simultaneously in order to obtain the required example, whereas Kuznetsov et al. (2002) studied properties of the level lines only for v .

For $\nu = 3/2$ we consider the following two functions:

$$u(x,y) = \int_0^\infty \frac{\cos k(x-\pi) + \cos k(x+\pi)}{k-\nu}\, \mathrm{e}^{ky}\, \mathrm{d}k, \tag{2.1}$$

$$v(x,y) = \int_0^\infty \frac{\sin k(x-\pi) + \sin k(x+\pi)}{\nu - k}\, \mathrm{e}^{ky}\, \mathrm{d}k, \tag{2.2}$$

where both numerators vanish at $k = \nu = 3/2$, and so the integrals are the usual infinite integrals. It is easy to verify that u and v are conjugate harmonic functions in $\mathbb{R}^2_-$ such that

$$u(-x,y) = u(x,y) \quad \text{and} \quad v(-x,y) = -v(x,y).$$

Moreover, u and v are infinitely smooth up to $\partial\mathbb{R}^2_- \setminus \{x = \pm\pi,\ y = 0\}$ and well-known facts from the theory of distributions imply that $[u_y - \nu u]_{y=0}$ is equal to a linear combination of Dirac's measures at $x = \pi$ and $x = -\pi$. Therefore,

$$u_y = \nu u \quad \text{on } \partial\mathbb{R}^2_- \setminus \{x = \pm\pi,\ y = 0\}. \tag{2.3}$$

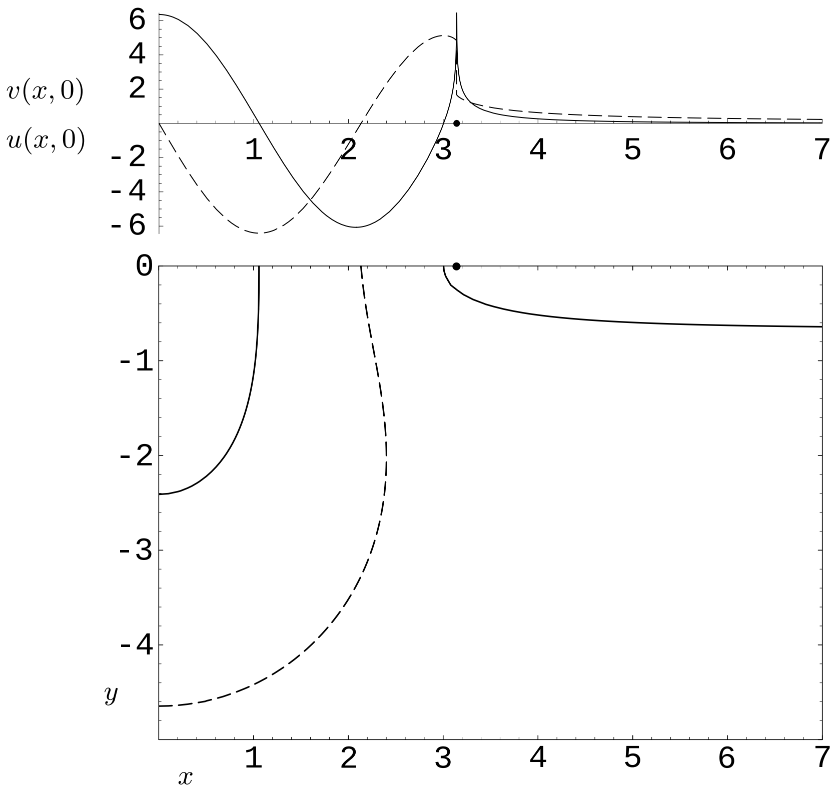

Figure 1. Nodal lines of u (solid lines) and v (dashed line) given by (2.1) and (2.2), respectively, with $\nu = 3/2$.

The calculated nodal lines of u and v are shown in Fig. 1 and we proceed with proving that the location of the lines is as plotted. It is clear that the negative y-axis is a nodal line of v and another nodal line is considered in the following

Proposition 2.1. *Apart from $\{x = 0, y < 0\}$, there is only one nodal line of $v(x, y)$ in $\mathbb{R}^2_-$, which is smooth, symmetric about the y-axis, and has both ends on the x-axis so that the right one, say $(x_0, 0)$, lies between the origin and the point $(\pi, 0)$.*

The latter nodal line serves as the bottom B in our example; the right half of this line is shown by dashed line in Fig. 1, where the bullet marks the position of $(\pi, 0)$ and the solid lines are nodal lines of u. Since (2.3) holds for u and the Cauchy–Riemann equations yield that (1.3) is fulfilled on the so defined bottom B, we see that u satisfies the sloshing problem in the domain W between this B and the x-axis. Moreover, Fig. 1 shows that there is only one nodal line of u in this water domain W and this property will be proved in Theorem 2.6 below.

Our proof of Proposition 2.1 is based on the next lemma illustrated in Fig. 1 (see the top part, where $u(x, 0)$ (solid line) and $v(x, 0)$ (dashed line) are plotted).

Lemma 2.2. *Function $v(x, 0)$ has the following properties on the half-axis $x \geqslant 0$:*

(i) *$v(x, 0)$ is continuous on $[0, \pi]$ and on $[\pi, +\infty)$, but*

$$v(x, 0) \to \mathrm{Si}\,(3\pi) + \frac{\pi}{2} \mp \frac{\pi}{2} \quad \text{as } x \to \pi \pm 0; \tag{2.4}$$

(ii) *there are exactly two zeroes of* $v(x,0)$ *on* $[0,\pi)$, *at* $x=0$ *and at some point* $x_0 \in (2\pi/3, \pi)$;

(iii) $v(x,0) < 0$ *for* $x \in (0, x_0)$ *and there is only one point* $x_m \in (0, x_0)$, *where* $v(x,0)$ *attains minimum*;

(iv) $v(x,0) > 0$ *for* $x > x_0$;

(v) $v(x,0)$ *is a monotonically decreasing convex function for* $x > \pi$ *and it tends to zero as* $x \to +\infty$.

Here and below, Si and Ci are the sine and cosine integrals, respectively, defined by

$$\mathrm{Si}\,(X) = \int_0^X \frac{\sin k}{k}\,\mathrm{d}k \quad \text{and} \quad \mathrm{Ci}\,(X) = -\int_X^\infty \frac{\cos k}{k}\,\mathrm{d}k.$$

Proof. Formula 3.722.5 in Gradshteyn & Ryzhik (1980) allows us to write

$$v(x,0) = \cos\nu x\,[\mathrm{Ci}\,(\nu|x-\pi|) - \mathrm{Ci}\,(\nu(x+\pi))] \\ + \sin\nu x\,[\mathrm{Si}\,(\nu(x-\pi)) - \mathrm{Si}\,(\nu(x+\pi))] - \pi H(\pi - x)\sin\nu x, \quad (2.5)$$

where the equality $\nu = 3/2$ is taken into account and $H(\mu)$ denotes the Heaviside function. The sum of first two terms here is a continuous function because the logarithmic singularity in $\mathrm{Ci}\,(\nu|x-\pi|)$ at $x=\pi$ is suppressed by the first-order zero of $\cos\nu x$ at this point. Hence, we get (2.4), which completes the proof of (i).

Since v is an odd function of x, $v(0,0) = 0$. Combining (2.5) with formula 3.354.1 in Gradshteyn & Ryzhik (1980), we obtain

$$v(x,0) = \int_0^\infty \left[\mathrm{e}^{-|x-\pi|k\nu}\,\mathrm{sgn}\,(x-\pi) + \mathrm{e}^{-(x+\pi)k\nu}\right] \frac{\mathrm{d}k}{1+k^2} - \pi H(\pi-x)\sin\nu x. \quad (2.6)$$

The derivative of the last integral with respect to x is equal to

$$-\nu \int_0^\infty \left[\mathrm{e}^{-(\pi-x)k\nu} + \mathrm{e}^{-(\pi+x)k\nu}\right] \frac{k\mathrm{d}k}{1+k^2} < 0 \quad \text{for } x \in (0,\pi).$$

Therefore, for $x \in (0,\pi)$ the integral is a non-positive concave function, which decreases strictly monotonically and has absolute value smaller than $\pi/2$. These facts prove (ii), (iii), and (iv) for $x \in (x_0, \pi)$. For $x > \pi$, (iv) is an immediate consequence of (2.6). The formula

$$v_x(x,0) = -\nu \int_0^\infty \left[\mathrm{e}^{-(x-\pi)k\nu} + \mathrm{e}^{-(x+\pi)k\nu}\right] \frac{k\mathrm{d}k}{1+k^2} < 0 \quad \text{for } x > \pi$$

proves the last assertion (v) of lemma. □

Proof of Proposition 2.1. The following properties of the nodal lines of harmonic functions are well-known (see, for example, Kuznetsov et al., 2002, Subsection 4.1.1), and so we give only a list of these properties here. There exists a nodal line ℓ_v emanating into $\mathbb{R}^2_-$ from the zero $(x_0, 0)$ of $v(x,0)$; ℓ_v cannot terminate in $\mathbb{R}^2_-$ and re-enter the x-axis at $(x_0, 0)$. Hence, if we exclude any part of the negative y-axis as a continuation of ℓ_v (such a continuation will give a non-smooth nodal line), then there are the following three possibilities for ℓ_v: (a) it goes to infinity; (b) it re-enters the x-axis at the origin; (c) it re-enters the x-axis at $(-x_0, 0)$. Let us show that (a) and (b) are impossible.

For this purpose we consider another representation for $v(x, y)$. From (2.2) we have

$$v_y - \nu v = \int_0^\infty [\sin k(\pi - x) - \sin k(\pi + x)] \, \mathrm{e}^{ky} \, \mathrm{d}k$$
$$= \frac{\pi - x}{y^2 + (\pi - x)^2} - \frac{\pi + x}{y^2 + (\pi + x)^2} = \frac{2x(\pi^2 - x^2 - y^2)}{[y^2 + (\pi - x)^2]\,[y^2 + (\pi + x)^2]}.$$

The solution of this differential equation is

$$v(x, y) = \mathrm{e}^{\nu y} \left[v(x, 0) + 2x \int_y^0 \frac{k^2 - (\pi^2 - x^2)}{[k^2 + (\pi - x)^2]\,[k^2 + (\pi + x)^2]} \, \mathrm{e}^{-k\nu} \, \mathrm{d}k \right]. \qquad (2.7)$$

Taking into account (2.6), this formula shows that $x^{-1}v(x, y)$ has the same nodal lines as $v(x, y)$ with exception of the negative y-axis. Since the limit of $x^{-1}v(x, 0)$ as $x \to 0$ is negative (see the proof of Lemma 2.2) and the integral in (2.7) vanishes for $y = 0$, (b) is impossible.

It immediately follows from (2.7) and Lemma 2.2, (v), that

$$v(x, y) > 0 \quad \text{when } x > \pi \text{ and } y \leqslant 0. \qquad (2.8)$$

For all $x \geqslant 0$ the integral in (2.7) tends to $+\infty$ like $\mathrm{e}^{-\nu y}/(\nu y^2)$ as $y \to -\infty$, and so there exists $y_0 < 0$ independent of x such that

$$x^{-1}v(x, y) > 0 \quad \text{when } x \geqslant 0 \text{ and } y < y_0.$$

This fact together with (2.8) shows that (a) is impossible.

Finally, the integral in (2.7) with $x = 0$ has the following behaviour as the function of $y \leqslant 0$. It is equal to zero at $y = 0$, has only one negative minimum at $y = -\pi$, and tends to $+\infty$ as $y \to -\infty$, which was demonstrated above. Therefore, the nodal line ℓ_v crosses the negative y-axis, thus realising (c). The proof is complete. □

For analysing the nodal lines of $u(x, y)$ given by (2.1), we begin with investigating the behaviour of $u(x, 0)$ for $x \geqslant 0$ (see the top part of Fig. 1, where $u(x, 0)$ is plotted as the solid line). Combining formulae 3.722.7 and 3.354.2 in Gradshteyn & Ryzhik (1980), we obtain

$$u(x, 0) = 2\pi H(\pi - x) \cos \nu x + \int_0^\infty \left[\mathrm{e}^{-|x-\pi|k\nu} + \mathrm{e}^{-(x+\pi)k\nu} \right] \frac{k\mathrm{d}k}{1 + k^2}, \qquad (2.9)$$

It is clear that the last integral has a logarithmic singularity at $x = \pi$, and so we have (see the top of Fig. 1):

$$u(x, 0) \to +\infty \quad \text{as } x \to \pi \pm 0.$$

Differentiating (2.9), we get

$$u_x(x, 0) = -\nu \int_0^\infty \left[\mathrm{e}^{-(x-\pi)k\nu} + \mathrm{e}^{-(x+\pi)k\nu} \right] \frac{k^2 \, \mathrm{d}k}{1 + k^2} \quad \text{for } x > \pi,$$

which implies together with (2.9) that for $x > \pi$ the function $u(x, 0)$ is positive, monotonically decreasing, and convex. Moreover, it tends to zero as $x \to +\infty$ (cf. (v) in Lemma 2.2).

The behaviour of $u(x, 0)$ is more complicated when $x \in [0, \pi)$, because the integral in (2.9) is a positive convex function of x, which increases monotonically from a certain positive value to $+\infty$. Therefore, for x belonging to some neighbourhoods of $x = 0$ and $x = \pi$ the inequality $u(x, 0) > 0$ holds.

Lemma 2.3. *There are exactly two zeroes of $u(x, 0)$ on $(0, \pi)$ and the function changes sign at these zeroes. The first zero is at $x = x_m$ (see* (iii) *in Lemma 2.2) and the second one at $x = x_M$, $x_m < x_M < \pi$.*

Proof. It is clear that $u(x, 0)$ cannot have more than two zeroes because the second zero of cosine in (2.9) is at $x = \pi$ and the integral is a positive, monotonically increasing, convex function since its derivative with respect to x is equal to

$$\nu \int_0^\infty \left[\mathrm{e}^{-(\pi - x)k\nu} - \mathrm{e}^{-(\pi + x)k\nu} \right] \frac{k^2 \, \mathrm{d}k}{1 + k^2} > 0 \quad \text{for } x \in [0, \pi).$$

Moreover, if $u(x, 0)$ has zeroes, then there are exactly two of them between $x = \pi/3$ and $x = \pi$, thus it is sufficient to show that the function has one zero between $x = 0$ and $x = \pi$. Since it is difficult to evaluate the rate of increasing of the integral in (2.9), we will derive the existence of a zero from the fact that $v(x, 0)$ attains its minimum at $x = x_m$ (see Lemma 2.2, (iii)), and so $v_x(x_m, 0) = u_y(x_m, 0) = 0$. Now the boundary condition (1.2) gives that $u(x_m, 0) = 0$. Formula (2.9) together with the fact that the integral in (2.9) increases monotonically on $[0, \pi)$ implies that the second zero x_M of $u(x, 0)$ is between x_m and a and that the function changes sign at its zeroes, which completes the proof. □

Corollary 2.4. *Two nodal lines of $u(x, y)$ emanate from $(x_m, 0)$ and $(x_M, 0)$.*

Proof. Since $u(x, 0)$ changes sign at its zeroes, zero is not an isolated value of $u(x, y)$, and so two nodal lines do exist in $\{x > 0,\ y < 0\}$. □

Of course, all nodal lines of u in $\mathbb{R}^2_-$ are symmetric about the y-axis.

Lemma 2.5. *There are two nodal lines of u in $\{x > 0,\ y < 0\}$; one of them emanates from $(x_m, 0)$ and crosses the negative y-axis and the other one emanates from $(x_M, 0)$ and goes to infinity.*

Proof. As in the proof of Proposition 2.1 we have from (2.1):

$$u_y - \nu u = \int_0^\infty [\cos k(x - \pi) + \cos k(x + \pi)] \, \mathrm{e}^{ky} \, \mathrm{d}k = \frac{-2y(y^2 + x^2 + \pi^2)}{[y^2 + (x - \pi)^2]\,[y^2 + (x + \pi)^2]}.$$

The solution of this differential equation is

$$u(x, y) = \mathrm{e}^{\nu y} \left[u(x, 0) + 2 \int_y^0 \frac{k(k^2 + x^2 + \pi^2)}{[k^2 + (\pi - x)^2]\,[k^2 + (\pi + x)^2]} \, \mathrm{e}^{-k\nu} \, \mathrm{d}k \right]. \qquad (2.10)$$

The last integral is a monotonically decreasing function of y; it is equal to zero for $y = 0$ and tends to $-\infty$ as $y \to -\infty$. Therefore, there is a single nodal line of u below every interval of the positive x-axis, where $u(x, 0) > 0$; that is, between the origin and $(x_m, 0)$ and to the right of $(x_M, 0)$. □

Theorem 2.6. *Inside the domain bounded from below by the nodal line ℓ_v, having endpoints at* $(\pm x_0, 0)$*, the sloshing eigenfunction* u *given by* (2.1) *has a single nodal line with endpoints* $(\pm x_m, 0)$*.*

The line ℓ_v (dashed line in Fig. 1, bottom) serves as the bottom B for the water domain whose right half is shown. The nodal line of u (solid line in Fig. 1, bottom) has both ends on the free surface F.

Proof. Note that the nodal line of u with endpoints $(\pm x_m, 0)$ cannot have a pair of common points with the nodal line ℓ_v. Indeed, if there are such points symmetric about the y-axis both functions u and v must vanish identically in a domain between nodal lines because on each nodal line one of the functions satisfies the homogeneous Dirichlet condition, whereas the homogeneous Neumann condition holds for the other function on the same line. Since vanishing is impossible, there are no pairs of common points and it remains to prove that the nodal line of u cannot have a common point with ℓ_v on the negative y-axis.

At the point, say $(0, y_0)$, where ℓ_v intersects the negative y-axis,

$$\nabla v(0, y_0) = \nabla u(0, y_0) = 0,$$

where ∇ is Hamilton's operator for the gradient, because the negative y-axis is itself a nodal line of v. However, it follows from (2.10) that

$$u_y(0, y_n) = -2 \frac{y_n(y_n^2 + \pi^2)}{[y_n^2 + \pi^2]\,[y_n^2 + \pi^2]} < 0, \quad \text{where } y_n \text{ is such that } u(0, y_n) = 0.$$

Hence $y_n \neq y_0$, which completes the proof. □

The example of the sloshing problem studied in this subsection disproves Lemma of Kuttler (1984), but, of course, there are water domains for which nodal lines connect F with B as supposed by Kuttler (a rectangle with the free surface as the top side is the simplest example).

(*b*) *Properties of nodal domains*

Let $N(u) = \{(x, y) \in \overline{W} : u(x, y) = 0\}$ be the set of nodal lines of a sloshing eigenfunction u. A connected component of $W \setminus N$ will be called a nodal domain. On account of (1.1) and (1.3), one concludes that each nodal domain has a piecewise smooth boundary without cusps. The following simple assertion of Kuttler (1984) is proved here for the sake of completeness.

Proposition 2.7. *If* R *is a nodal domain of* u*, then* $\overline{R} \cap F$ *contains an interval of the* x*-axis.*

Proof. Let $\overline{R} \cap F$ be empty or consist of a finite number of points. Applying Green's identity to u in R, we get from the boundary conditions that $\int_R |\nabla u|^2 \, \mathrm{d}x\mathrm{d}y = 0$. Hence u vanishes in R, and so u is identically equal to zero in W by the analyticity of harmonic functions. □

Proposition 2.8. *The number of nodal domains corresponding to* u_n *is less or equal to* $n + 1$*.*

Kuttler's reasoning (Kuttler, 1984), which is a version of the Courant's original proof (Courant & Hilbert, 1953), turns out to be the proof when the unnecessary reference to the fallacious lemma is omitted.

An immediate consequence of Propositions 2.7 and 2.8 is the following

Corollary 2.9. *The sloshing eigenfunction u_n cannot change sign more than $2n$ times on F.*

It should be noted that the number of nodal domains corresponding to u_n is less then $n+1$ in some cases. For instance, the eigenfunction constructed as the example in Subsection 2.1 has two nodal domains. However, the corresponding eigenvalue $\nu = 3/2$ is not the fundamental one. (This follows from Theorem 3.1 (ii), which says that the fundamental eigenfunction has only one nodal line connecting F and $\overline{B}$.) Therefore, the number of nodal domains in the example which is equal to two is less than the maximal number permitted by Proposition 2.8 which is at least three. On the other hand, the eigenfunctions in a rectangle have the maximal number of nodal domains.

3. The fundamental eigenvalue is simple

The aim of this section is to prove the following

Theorem 3.1. (i) *The fundamental eigenvalue of problem* (1.1)–(1.4) *is simple.*
(ii) *The corresponding eigenfunction has only one nodal line connecting F and $\overline{B}$.*

This theorem is proved in Subsection 3.2.

(*a*) *Variational principle for the stream function*

Our proof of Theorem 3.1 is based on a variational principle for a boundary value problem that is equivalent to (1.1)–(1.4) and involves a conjugate to u harmonic function v (stream function). The latter satisfies

$$v_{xx} + v_{yy} = 0 \quad \text{in } W, \tag{3.1}$$

$$-v_{xx} = \nu v_y \quad \text{on } F, \tag{3.2}$$

$$v = 0 \quad \text{on } B, \tag{3.3}$$

where condition (3.2) is derived from (1.2) by differentiation and application of the Cauchy–Riemann equations; condition (3.3) is obtained from (1.3) by an appropriate choice of the additive constant in v.

It is clear that the multiplicity of ν as an eigenvalue of (1.1)–(1.4) is the same as its multiplicity as an eigenvalue of (3.1)–(3.3).

Without loss of generality we assume that $F = \{-1 < x < 1, y = 0\}$. The next step for formulating a variational principle for problem (3.1)–(3.3) consists in rewriting (3.2) in the form:

$$v = \nu\, \mathcal{K}\, v_y \quad \text{on } F, \tag{3.4}$$

where

$$(\mathcal{K} f)(x) = \int_{-1}^{1} K(x,\xi)\, f(\xi)\, \mathrm{d}\xi, \quad K(x,\xi) = K(\xi, x), \quad \text{and}$$
$$K(x,\xi) = (1-x)(\xi+1)/2 \text{ for } \xi < x.$$

It is clear that $\mathcal{K}$ is a symmetric, positive operator in $L_2(F)$. Finally, by $\mathcal{D}_N$ we denote the so-called Dirichlet–Neumann operator that maps ϕ given on F into $\mathcal{D}_N\,\phi = \Phi_y\big|_F$, where Φ must be found from the following Dirichlet problem:

$$\nabla^2\Phi = 0 \quad \text{in } W, \quad \Phi = \phi \quad \text{on } F, \quad \Phi = 0 \quad \text{on } B.$$

It is known (see, for example, Aubin, 1972, Chapter 7, Section 1) that $\mathcal{D}_N$ is a positive, self-adjoint operator in $L_2(F)$. It follows from (3.1), (3.3), and (3.4) that for finding the fundamental eigenvalue ν_1 one can use the following variational principle:

$$\nu_1 = \min_{w\in H^1_B(W)} \frac{\int_W |\nabla w|^2\,\mathrm{d}x\mathrm{d}y}{\int_F \mathcal{D}_N\,w\;(\mathcal{K}\,\mathcal{D}_N)\;w\,\mathrm{d}x}, \tag{3.5}$$

where $H^1_B(W)$ is the subspace of $H^1(W)$ that consists of functions with vanishing traces on B. Since the operator defined by the quadratic form in the denominator is compact in $H^1_B(W)$, there exists a nontrivial function w^* for which the quotient (3.5) attains the minimum. Moreover, it is easy to verify that $\nabla^2 w^* = 0$ in W. Therefore, $\mathcal{D}_N\,w^* = w^*_y$, and so w^* is an eigenfunction of (3.1)–(3.3).

Let ν_i and ν_k be two different eigenvalues of problem (3.1), (3.3), and (3.4), and let v_i and v_k be the corresponding eigenfunctions. Combining the second Green's formula with the boundary condition (3.4), one obtains the following orthogonality condition for the eigenfunctions:

$$\int_F v_i\,v_{ky}\,\mathrm{d}x = \int_F v_i\,\mathcal{D}_N\,v_k\,\mathrm{d}x = 0.$$

This condition allows us to extend (3.5) for finding the whole sequence of eigenvalues. Let $v_1,\dots,v_{n-1}$ be linearly independent eigenfunctions corresponding to the first $n-1$ eigenvalues. Then for finding the eigenvalue ν_n we have the following variational principle:

$$\nu_n = \min \frac{\int_W |\nabla w|^2\,\mathrm{d}x\mathrm{d}y}{\int_F \mathcal{D}_N\,w\;(\mathcal{K}\,\mathcal{D}_N)\;w\,\mathrm{d}x},$$

where the minimum is taken over all nonzero $w \in H^1_B(W)$ such that

$$\int_F w\,\mathcal{D}_N\,v_k\,\mathrm{d}x = 0, \quad k = 1,\dots,n-1.$$

(*b*) *Proof of Theorem 3.1*

The first statement in Theorem 3.1 is an immediate consequence of the following

Proposition 3.2. *The fundamental eigenvalue of problem* (3.1)–(3.3) *is simple and the corresponding eigenfunction may be chosen to be positive in* $W \cup F$.

Proof. Let us suppose that there exists an eigenfunction v that corresponds to ν_1 and changes sign in W. In view of (3.1) and (3.3), v also changes sign on F. By v_+ and v_- we denote the positive and negative part of v, respectively. Let us suppose that

$$\int_W |\nabla v_\pm|^2\,\mathrm{d}x\mathrm{d}y > \nu_1 \int_F \mathcal{D}_N\,v_\pm\;(\mathcal{K}\,\mathcal{D}_N)\;v_\pm\,\mathrm{d}x. \tag{3.6}$$

The definition of ν_1 and v gives

$$\int_W |\nabla v|^2 \, \mathrm{d}x\mathrm{d}y = \nu_1 \int_F \mathcal{D}_N \, v \, (\mathcal{K} \, \mathcal{D}_N) \, v \, \mathrm{d}x. \tag{3.7}$$

Summing up two equalities (3.6) and subtracting (3.7), we get

$$0 > \int_F \mathcal{D}_N \, v_- \, (\mathcal{K} \, \mathcal{D}_N) \, v_+ \, \mathrm{d}x. \tag{3.8}$$

Here we also used the fact that $\mathcal{K}$ is a self-adjoint operator. Now the equalities

$$v_+ = v + v_- \quad \text{and} \quad \mathcal{D}_N \, v = v_y$$

imply that

$$\int_F \mathcal{D}_N \, v_- \, (\mathcal{K} \, \mathcal{D}_N) \, v_+ \, \mathrm{d}x = \int_F \mathcal{D}_N \, v_- \, \mathcal{K} \, v_y \, \mathrm{d}x + \int_F \mathcal{D}_N \, v_- \, (\mathcal{K} \, \mathcal{D}_N) \, v_- \, \mathrm{d}x. \tag{3.9}$$

The boundary condition (3.4) allows us to write the first term on the right as

$$\nu_1^{-1} \int_F v \, \mathcal{D}_N \, v_- \, \mathrm{d}x = \nu_1^{-1} \int_F v_- \, \mathcal{D}_N \, v \, \mathrm{d}x,$$

the latter equalty uses $\mathcal{D}_N^* = \mathcal{D}_N$. Applying the relation $\mathcal{D}_N \, v = v_y$ and the boundary condition (3.2), we arrive at

$$\int_F \mathcal{D}_N \, v_- \, \mathcal{K} \, v_y \, \mathrm{d}x = -\nu_1^{-2} \int_F v_- \, v_{xx} \, \mathrm{d}x = \nu_1^{-2} \int_F (v_-)_x^2 \, \mathrm{d}x.$$

This together with (3.9) shows that the right-hand side in (3.8) cannot be negative. The obtained contradiction yields that (3.6) cannot hold for v_+ and v_- simultaneously. Besides, according to the definition of ν_1, the inequality opposite to (3.6) also cannot be true. Hence at least one of the functions v_+ and v_-, say v_+, delivers the minimum to the quotient (3.5). Then v_+ is harmonic in W, which implies that either v_+ or v_- is equal to zero identically. Thus v does not change sign in W (it is positive without loss of generality), which guarantees that ν_1 is simple.

The function v is positive on F because v is a positive harmonic function in W that satisfies the boundary condition (3.2). The proof is complete. □

Proof of Theorem 3.1. (ii). We consider the trace $v(x, 0)$ of the fundamental eigenfunction of problem (3.1)–(3.3) (this function is positive on F and vanishes at the endpoints of this interval). Let us show that $v(x, 0)$ cannot have more than one critical point on F. If we suppose the contrary, then there must be either three points, say x_j, $j = 1, 2, 3$, such that $v_x(x_j, 0) = 0$, or two critical points one of which is multiple, that is v_{xx} vanishes at this point. In the first case, $u(x_j, 0) = 0$ by the Cauchy–Riemann equations and the boundary condition (1.2), and so the nodal lines of u divide W into at least three nodal domains which contradicts Proposition 2.8. In the second case, at least two nodal lines of u emanate from the point on F, where $v(x, 0)$ has a multiple critical point. Again the nodal lines of u divide W into at least three nodal domains which contradicts Proposition 2.8.

The existence of only one critical point of $v(x, 0)$ on F implies that the same point is the unique zero of $u(x, 0)$ on F. The nodal line emanating from this point has the second end on $\overline{B}$. This completes the proof of Theorem 3.1. □

Let us consider water domains satisfying an extra condition that W is contained within the semistrip bounded by $\overline{F}$ and two vertical rays going downwards from the endpoints of $\overline{F}$. This condition was first introduced in the work by John (1950) (now it is usually referred to as John's condition), where the so-called water-wave problem was considered (see also Kuznetsov et al., 2002, Chapters 3 and 4). It occurs, that if W satisfies John's condition, then the second statement of Theorem 3.1 may be improved.

Proposition 3.3. *Let v be the fundamental eigenfunction of problem* (3.1)–(3.3), *then $v \in C^1(\overline{F})$. Moreover, if W satisfies John's condition, then $\mp v_x(\pm 1, 0) > 0$. We recall that without loss of generality F is assumed to coincide with $\{-1 < x < 1,\, y = 0\}$.*

Proof. To be specific we consider the corner point $(1, 0)$. The angle enclosed between the corresponding unilateral tangents and directed into W we denote by α_+ and r is the distance to the point $(1, 0)$.

In order to apply the standard results on the asymptotics near a boundary corner point for a solution to the Laplacian Dirichlet problem (see, for example, Nazarov & Plamenevsky, 1994, Chapter 2), we integrate condition (3.2) twice with respect to x. This results in the presence of an additional linear term in the local asymptotic expansion as $r \to 0$. Besides, the next term is either $O\left(r^2\right)$ or $O\left(r^{\pi/\alpha_+}\right)$. Since $0 < \alpha_\pm < \pi$ (α_- is the second angle adjacent to F), this implies that $v \in C^1(\overline{F})$.

For proving the second statement it is more convenient to consider the endpoint $(1, 0)$. From (3.4) we get that

$$v_x(1,0) = \nu \int_{-1}^{1} K_x(1,\xi)\, v_y(\xi,0)\,\mathrm{d}\xi = -(\nu/2) \int_{-1}^{1} (1+\xi)\, v_y(\xi,0)\,\mathrm{d}\xi.$$

Let w be a solution to the following Dirichlet problem

$$\nabla^2 w = 0 \quad \text{in } W, \quad w = 1 + x \quad \text{on } F, \quad w = 0 \quad \text{on } B. \tag{3.10}$$

Then the second Green's formula allows us to write

$$v_x(1,0) = -(\nu/2) \int_{-1}^{1} w_y\, v\,\mathrm{d}\xi.$$

Let us show that

$$w_y(x,0) \geqslant 0 \quad \text{for } x \in (-1,1). \tag{3.11}$$

This proves the second statement of our proposition because the equality in (3.11) cannot hold identically.

For proving (3.11), we seek w in the form $1 + x + w_1$ and get from (3.10) that w_1 must satisfy

$$\nabla^2 w_1 = 0 \quad \text{in } W, \quad w_1 = 0 \quad \text{on } F, \quad w_1 = -(1+x) \quad \text{on } B.$$

According to John's condition $w_1 \leqslant 0$ on B, then the maximum principle guarantees that $w_1 \leqslant 0$ in W. Hence

$$w_y(x,0) = w_{1y}(x,0) \geqslant 0 \quad \text{on } F,$$

and so $v_x(1,0) < 0$. In the same way one obtains the inequality $v_x(-1,0) > 0$, which completes the proof. □

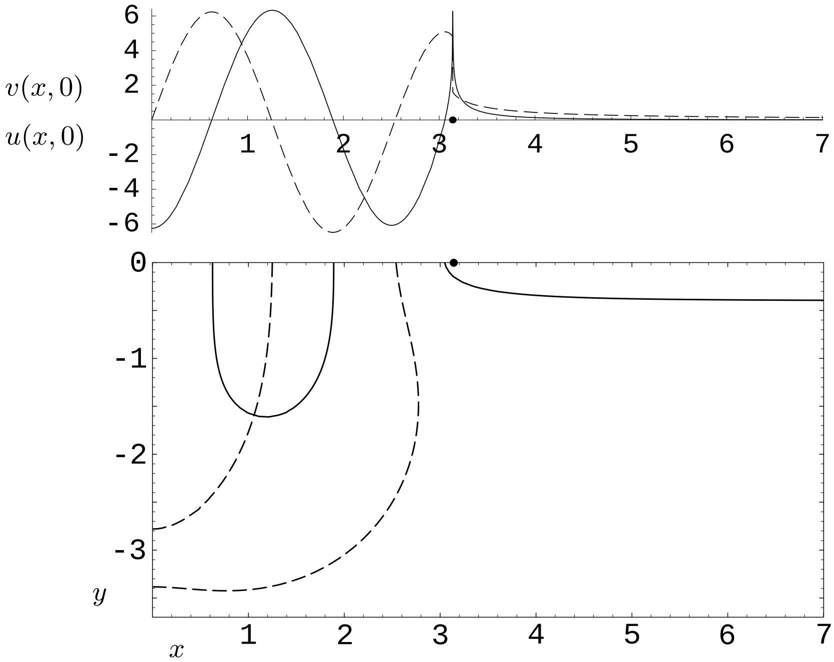

Figure 2. Nodal lines of u (solid lines) and v (dashed lines) given by (2.1) and (2.2), respectively, with $\nu = 5/2$.

Corollary 3.4. *Let W satisfy John's condition, then the endpoints of F do not belong to the nodal line of the fundamental eigenfunction u.*

Proof. If we suppose that $u(1,0) = 0$, then we get from the local asymptotics that

$$u = O\left(r^{\pi/\alpha_+}\right) \quad \text{and} \quad u_y = O\left(r^{\pi/\alpha_+ - 1}\right) \quad \text{as } r \to 0,$$

where r is the distance to the point $(1,0)$ (see the proof of Proposition 3.3 for the corresponding reference). On the other hand, $u_y(1,0) = -v_x(1,0)$, which does not vanish according to Proposition 3.3. This contradicts to the second asymptotic formula above. The proof is complete. □

4. Discussion

(*a*) *The case of connected free surface*

In Section 2 we constructed an example of the water domain such that there exists a sloshing eigenfunction u having a nodal line whose both ends are on the free surface (see Fig. 1). The velocity potential (2.1) and the stream function (2.2) used for this purpose allow us to obtain more examples with the same property. One of them is shown in Fig. 2, where the right half of the water domain symmetric about the y-axis is bounded by the exterior dashed line which is again a nodal line of the stream function. In this example the value $\nu = 5/2$ is used and for the justification one has to follow the same method as in Subsection 2.1. Of course, there are two nodal lines of u having both ends on F in the whole W, but the main novelty of this example is the presence of the nodal line of the stream function v with both ends on F (the interior dashed line).

Let us turn to some open questions concerning the eigensolutions to problem (1.1)–(1.4). The first of them is related to the number of sign changes on F of the eigenfunction u_n. Our Corollary 2.9 gives only a rough upper bound $2n$ for this number and the question is whether one can replace $2n$ by n as stated in (Kuttler, 1984), where the proof is based on the fallacious lemma. Of course, it follows from the explicit expression that the nth sloshing eigenfunction has n changes of sign when W is a rectangle and F is its top side.

Another open question is whether *all* eigenvalues of problem (1.1)–(1.4) are simple. There are a number of particular geometries for which all eigenvalues are proved to be simple. Of course, this is obvious for rectangular domains whose top side is the free surface (by separation of variables one obtains the explicit expressions for both eigenvalues and eigenfunctions in this case). A less trivial result is given implicitly in § 258 of the book by Lamb (1932), where Kirchhoff's solution is presented for the case when B is formed by two segments at $\pi/4$ to the vertical (we recall that it is assumed that $F = \{-1 < x < 1,\, y = 0\}$). For this triangle the eigenvalues are

$$\nu_n = \mu_n \left(\tanh \mu_n\right)^{(-1)^n}, \quad \text{where } \mu_n,\ n = 1, 2, \ldots,$$

are positive roots of $\cos 2\mu \cosh 2\mu = 1$ (actually, our formulae are a unified form of those in Lamb (1932), where the symmetric and antisymmetric sloshing modes are considered separately). Since the roots of the last transcendental equation are simple, the sloshing eigenvalues are also simple. Recently, Kuznetsov & Motygin (2003) established that all eigenvalues are simple for $W = \mathbb{R}^2_-$ when F consists either of one gap or of two equal gaps in the rigid dock covering W. Finally, for the domains which intersect the x-axis at right angles all eigenvalues with sufficiently large numbers are simple. This follows from the asymptotic formula

$$\nu = \frac{\pi n}{2} - \frac{\kappa_+ + \kappa_-}{4\pi} + o\left(n^{-1}\right) \quad \text{as } n \to \infty,$$

that was proved by Davis (1969) and a simplified proof was given by Ursell (1974). Here κ_+ (κ_-) is the curvature of B at the right (left) intersection with the x-axis.

The following heuristic reasoning demonstrates a possibility of the existence of a domain for which two nodal lines emanate from one point on B. Let us transform continuously a rectangle having the top side as the free surface into the domain obtained by reflection in the y-axis of the domain shown in Fig. 1. The second sloshing eigenfunction in the rectangle has two vertical nodal lines, but there is only one nodal line having both ends on the free surface in the domain obtained by the described reflection. Therefore, a domain for which two nodal lines emanate from one point on B should arise at a certain intermediate stage of such continuous transformation.

Besides, it is unknown whether two nodal lines can emanate from one point on F or from a corner point between F and B, but this is admissible when F consists of several intervals (see next subsection).

(b) *Other geometries*

Numerical computations demonstrate that a nodal line emanating from the corner point, where $\overline{F}$ and $\overline{B}$ meet, does exist in the water domain whose free

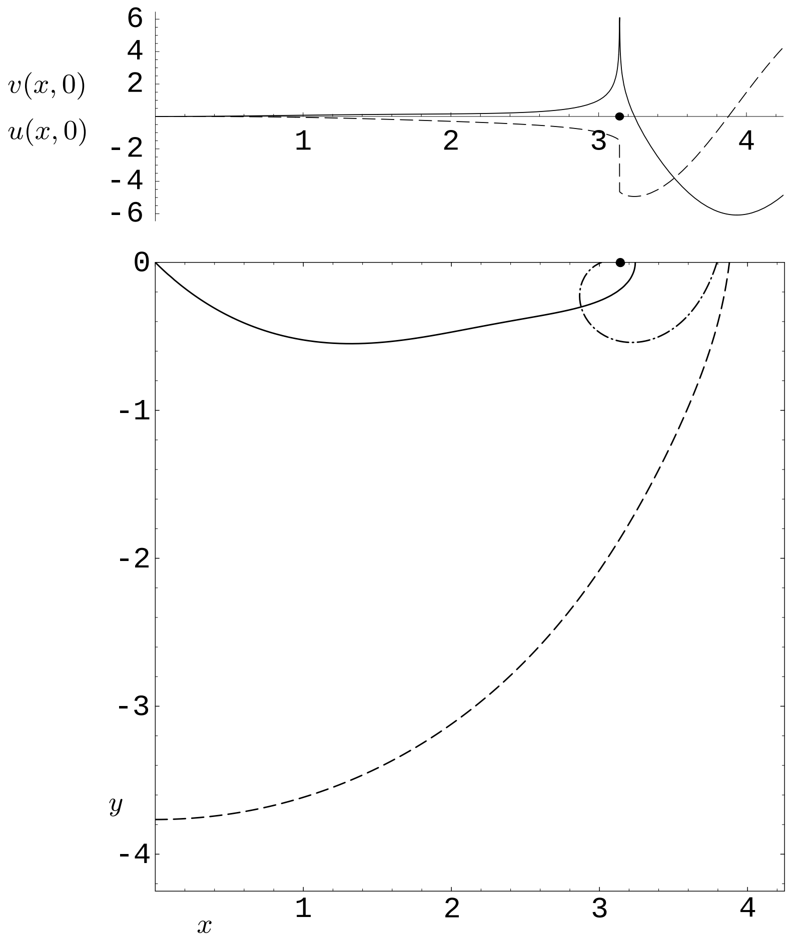

Figure 3. Nodal lines of u (solid line) and v (dashed line) given by (2.1) and (2.2), respectively, with $\nu = 2.00228\ldots$; the dot-dash line is the level line $v(x,y) = -1$.

surface consists of two intervals (see Fig. 3). This domain is constructed with the help of level lines of the stream function v given by (2.2) with $\nu = 2.00228\ldots$. (In this case u and v describe propagation of waves in $\mathbb{R}^2_-$, and the corresponding integrals are understood as the Cauchy principal values, but this is unimportant because we are concerned only with a bounded subdomain of $\mathbb{R}^2_-$.) The domain shown in Fig. 3 is bounded by:

1. Two nodal lines of the stream function, one of which is the negative y-axis and the other one is plotted by the dashed line.

2. The level line $v = -1$ shown by the dot-dash line that has both ends on the x-axis and thus cuts out the source point at $(\pi, 0)$ from the water domain.

Hence the free surface, bounding this water domain from above, consists of two intervals. The nodal line of the velocity potential u is shown by the solid line. The existence of ν, for which the behaviour of the nodal line is as is shown in Fig. 3, can be proved in the same way as in Subsection 2.1 starting with the investigation of the graphs of $u(x,0)$ and $v(x,0)$ (see Fig. 3, top). It is clear that by reflecting the water domain shown in Fig. 3 one obtains the case when two nodal lines of u emanate from one point on F, which now consists of three intervals now.

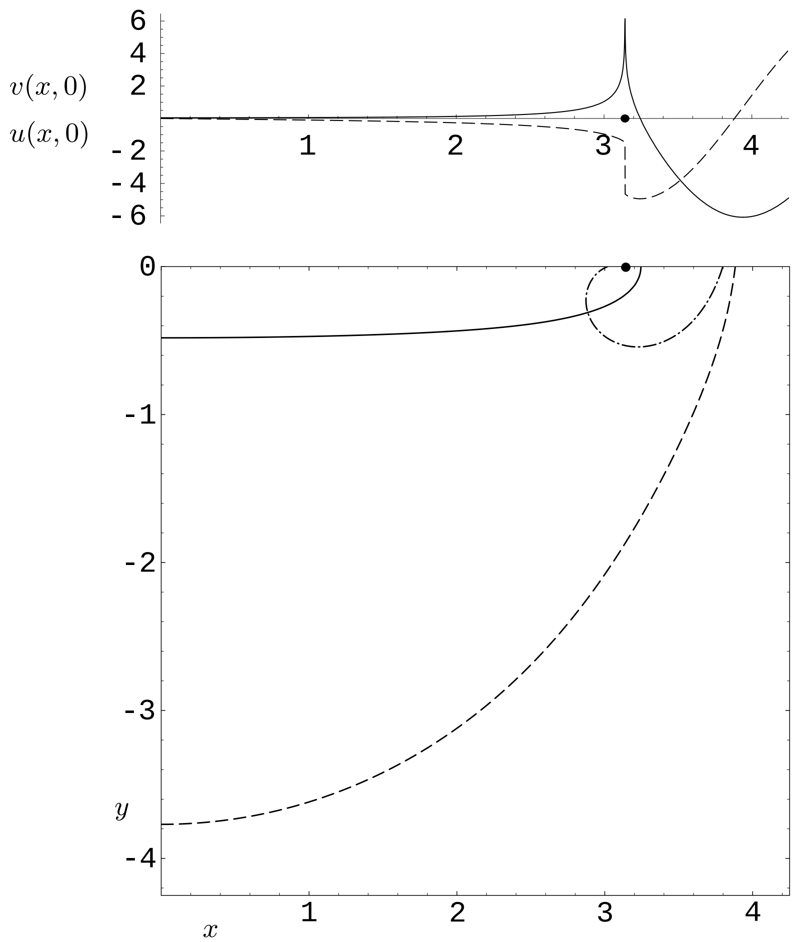

Figure 4. Nodal lines of u (solid line) and v (dashed line) given by (2.1) and (2.2), respectively, with $\nu = 2$; the dot-dash line is the level line $v(x, y) = -1$.

It is interesting to note that in the two-dimensional sloshing problem with disconnected F a nodal line of u can connect two rigid boundaries, where condition (1.3) holds (this is impossible when F is connected). In Fig. 4 obtained for $\nu = 2$ in (2.1) and (2.2) (again the integrals defining u and v must be understood as the Cauchy principal values), the water domain is similar to that in Fig. 3, but the nodal line of u connects two rigid boundaries. It is clear that this is impossible when F is connected and W is simply connected.

In conclusion, we make some notes about the three-dimensional sloshing problem. First, the assertions similar to Propositions 2.7 and 2.8 are also true in this case. However, even the fundamental eigenvalue (not to mention any others) is multiple for some container geometries. One can verify this by separation of variables for vertical cylinders having the horizontal bottom and either circular or square cross-section. Second, it should be emphasized that the plethora of possibilities for nodal surfaces in the three-dimensional sloshing problem is much greater than in the two-dimensional case. This can be easily seen in the case of a container that has vertical sidewalls and a horizontal bottom because this geometry allows us to reduce the sloshing problem to the free membrane problem by separating the vertical variable.

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