Sloshing problem in a half-plane covered by a dock with two equal gaps

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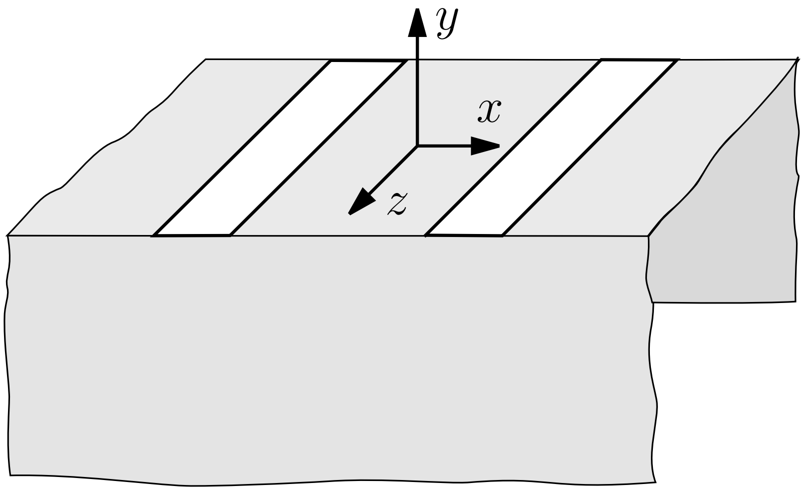
STATEMENT OF THE PROBLEM



Ansatz: $u(x, y, z, t) = u(x, y) \sin mz \cos \omega t$

Problems for symmetric $u^{(+)}$ and antisymmetric $u^{(-)}$ mode

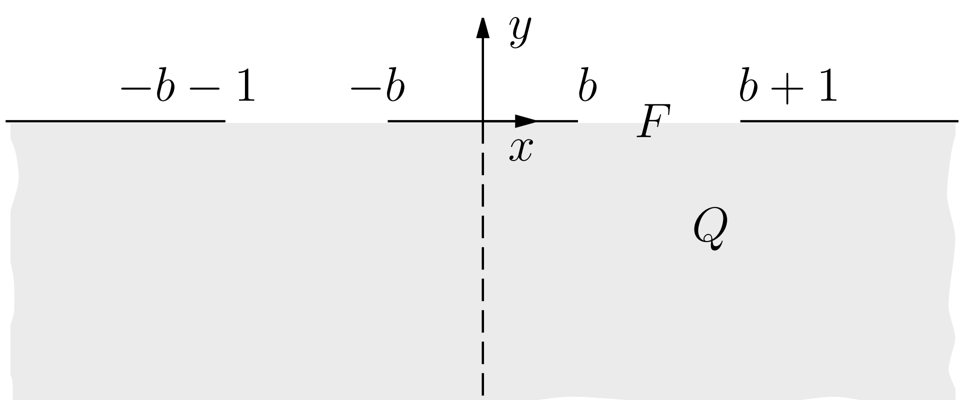
$$\Delta u^{(\pm)} = m^2 u^{(\pm)}, \quad (x, y) \in Q$$

$$\partial_x u^{(+)} = 0, \quad u^{(-)} = 0, \quad x = 0,\ y < 0$$

$$\partial_y u^{(\pm)} = 0, \quad \text{on docks}$$

$$\partial_y u^{(\pm)} - \nu^{(\pm)} u^{(\pm)} = 0, \quad (x, y) \in F$$

$$\iint_Q |\nabla u^{(\pm)}|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_F \left|u^{(\pm)}\right|^2 \mathrm{d}x < \infty$$



REFERENCES

For the case $m = 0$, $b = 0$

A.M.J. Davis (1970)

P. Henrici, B.A. Troesch, L. Wuytack (1970)

J.W. Miles (1972)

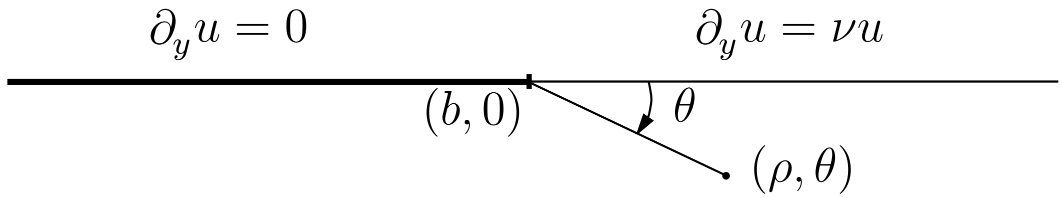
B.A. Troesch, H.R. Troesch (1972)

B.A. Troesch, H.R. Troesch (1974)

D.W. Fox, J.R. Kuttler (1983)

LOCAL ASYMPTOTICS NEAR DOCK TIPS

Polar coordinates (ρ, θ), $(z - b = \rho e^{i\theta}$, $z = x + iy)$:



By Wigley (1964), Nazarov, Plamenevsky (1994)

$u \sim P(\rho, \log \rho)$, where P is a polynomial function

Local asymptotics for the potential

$$u^{(\pm)}(\rho, \theta) = c^{(\pm)}\left\{-\pi/\nu^{(\pm)} + \rho\big(\cos\theta \log\rho + (\pi - \theta)\sin\theta\big)\right\} + d^{(\pm)}\rho\cos\theta + \psi^{(\pm)}(\rho, \theta)$$

and for a complex conjugate stream function

$$\varphi^{(\pm)}(\rho, \theta) = b^{(\pm)} - c^{(\pm)}\rho\big[\sin\theta \log\rho + (\theta - \pi)\cos\theta\big] - d^{(\pm)}\rho\sin\theta + \phi^{(\pm)}(\rho, \theta)$$

with some coefficients $b^{(\pm)}$, $c^{(\pm)}$ and $d^{(\pm)}$

As $\rho \to 0$,

$$\psi^{(\pm)}, \phi^{(\pm)} = O(\rho^{1+\delta}), \quad \left|\nabla\psi^{(\pm)}\right| = \left|\nabla\phi^{(\pm)}\right| = O(\rho^{\delta})$$

where $\delta > 0$

ANTISYMMETRIC MODES. INTEGRAL EQUATION.

Completing Miles (1972) scheme (based on Fourier transform)

<u>For $m = 0$</u>

$$u^{(-)}(x) = \frac{\nu^{(-)}}{\pi} \int_b^{b+1} u^{(-)}(\xi) \log \frac{x+\xi}{|x-\xi|} \, \mathrm{d}\xi, \quad x \in (b, b+1)$$

In the fluid: $$u^{(-)}(x,y) = \frac{1}{2\pi} \int_b^{b+1} u^{(-)}(\xi) \, \log \frac{(x+\xi)^2 + y^2}{(x-\xi)^2 + y^2} \, \mathrm{d}\xi$$

<u>For $m \neq 0$</u>

$$u^{(-)}(x) = -\frac{\nu^{(-)}}{\pi} \int_b^{b+1} \left[K_0\big(m(x+\xi)\big) - K_0\big(m|x-\xi|\big) \right] u^{(-)}(\xi) \, \mathrm{d}\xi, \quad x \in (b, b+1)$$

Restoring the potential in the fluid by

$$u^{(-)}(x,y) = -\frac{1}{2\pi} \int_b^{b+1} u^{(-)}(\xi) \Big[K_0\Big(m\sqrt{(x+\xi)^2 + y^2}\Big) - K_0\Big(m\sqrt{(x-\xi)^2 + y^2}\Big) \Big] \, \mathrm{d}\xi$$

(K_0 is the modified Bessel function)

Consequences:

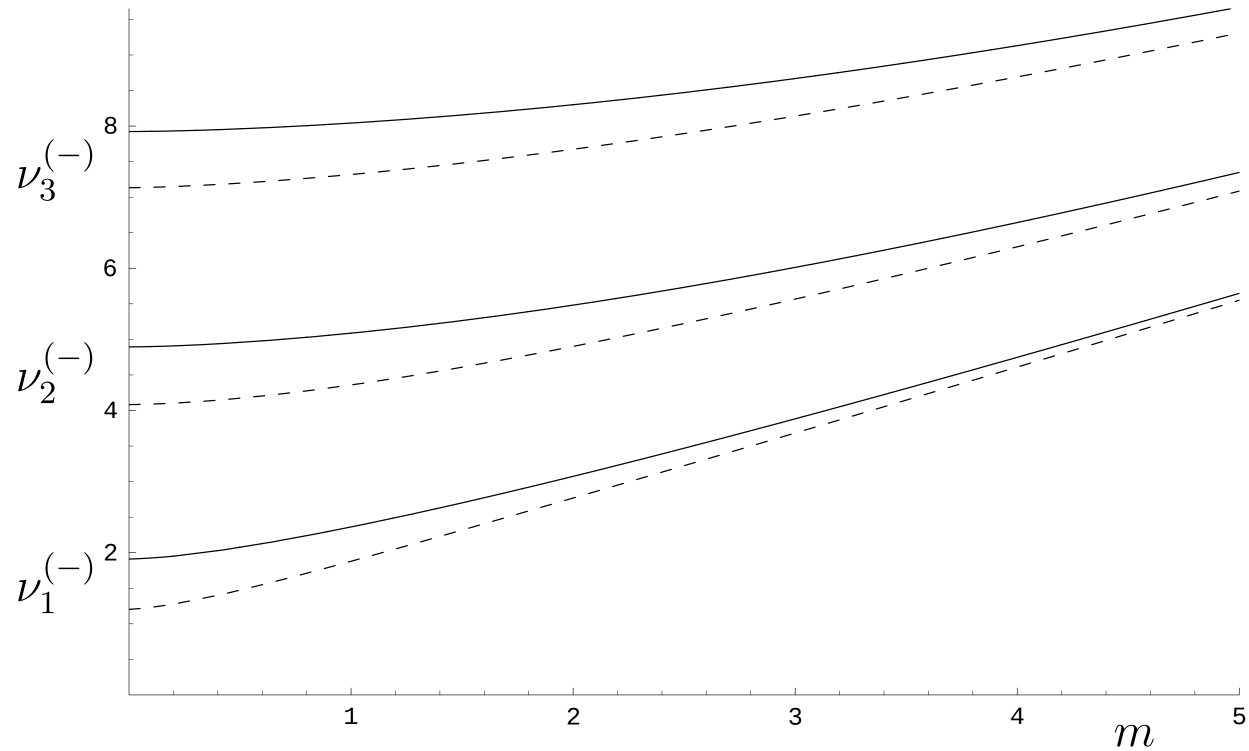
(i) For any $b \geqslant 0$, there exist a sequence of eigenvalues

$$0 < \nu_1^{(-)} < \nu_2^{(-)} \leqslant \ldots \leqslant \nu_n^{(-)} \leqslant \ldots, \quad \nu_n^{(-)}(b) \to \infty \ \text{ as } \ n \to \infty$$

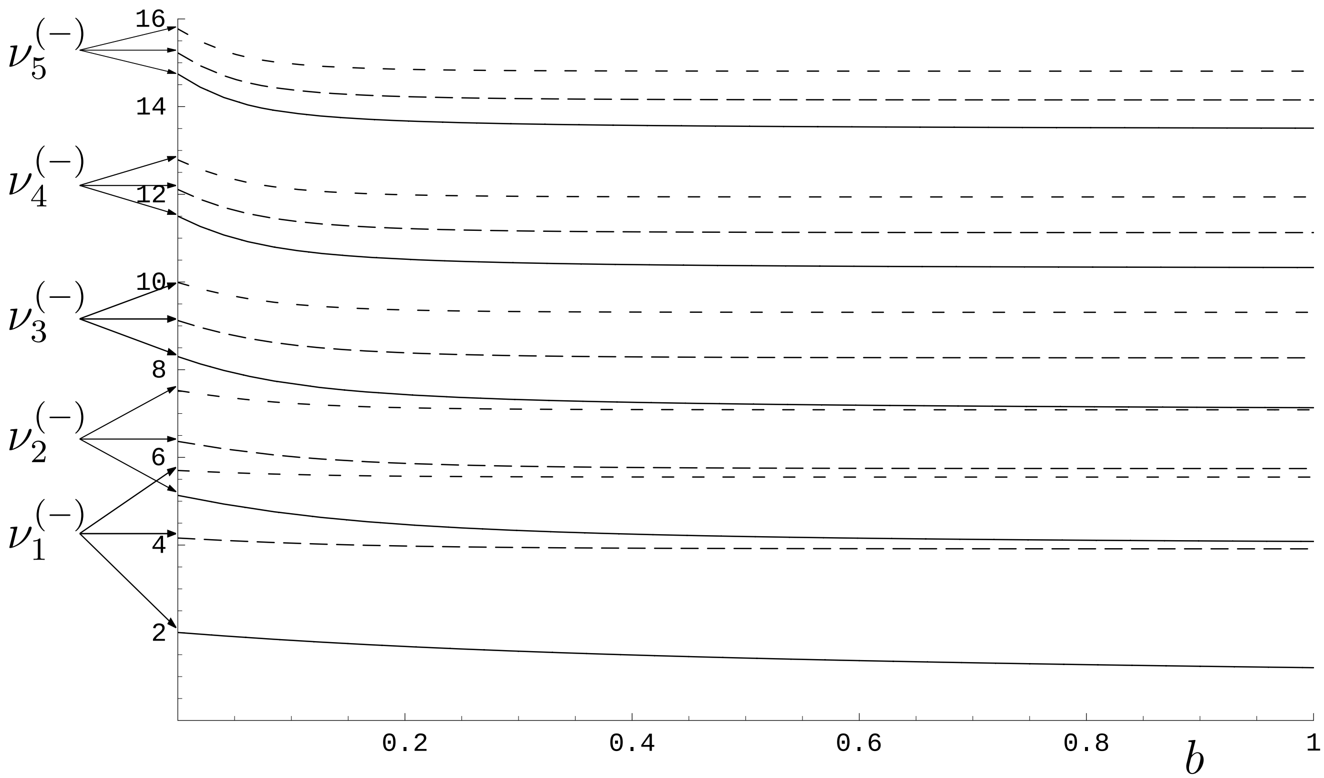
and $u_1^{(-)}(x;b) \geqslant 0$ for $x \in [0,1]$

(ii) The values of $\nu_n^{(-)}(b)$ are monotonically decaying in b

NUMERICAL RESULTS, ANTISYMMETRIC MODES



Solid lines $b = 0.05$, dashed lines $b = 1.0$



Solid lines $m = 0$, dashed lines (— — —) $m = 3.25$, (- - -) $m = 5.0$

SYMMETRIC MODES, INTEGRAL EQUATION, $m = 0$

Fourier transform

$$u^{(+)}(x, y; b) = \frac{2}{\pi} \int_0^\infty f(k)\, \mathrm{e}^{ky} \cos kx \, \mathrm{d}k$$

Applying ∂_y and the inverse transform for $y = 0$

$$k\, f(k) = \int_0^\infty \partial_y u^{(+)}(\xi, 0; b) \cos k\xi \, \mathrm{d}\xi = \nu^{(+)}(b) \int_b^{b+1} u^{(+)}(\xi, 0; b) \cos k\xi \, \mathrm{d}\xi$$

Then,

$$u^{(+)}(x, 0; b) = \nu^{(+)}(b) \frac{2}{\pi} \int_0^\infty k^{-1} \cos kx \, \mathrm{d}k \int_b^{b+1} u^{(+)}(\xi, 0; b) \big[\cos k\xi - 1\big] \, \mathrm{d}\xi$$

(since $\int_b^{b+1} u^{(+)}(x, 0) \, \mathrm{d}x = 0$ by zero flux condition)

Integral equation

$$u^{(+)}(x) = -\nu^{(+)} \pi^{-1} \int_b^{b+1} \big[\log(x + \xi) + \log|x - \xi|\big] \; u^{(+)}(\xi) \, \mathrm{d}\xi, \qquad x \in (b, b+1)$$

The integral equation is not correct !

(it has no solutions orthogonal to constant)

Assuming the contrary, restoring the potential in the fluid by

$$u^{(+)}(x, y) = -\frac{1}{2\pi} \int_b^{b+1} u^{(+)}(\xi) \; \log\big[(x + \xi)^2 + y^2\big)\big((x - \xi)^2 + y^2\big)\big] \, \mathrm{d}\xi$$

$u^{(+)}(x, y) = O(|z|^{-2})$ as $z \to \infty$. Such decaying is not allowed.

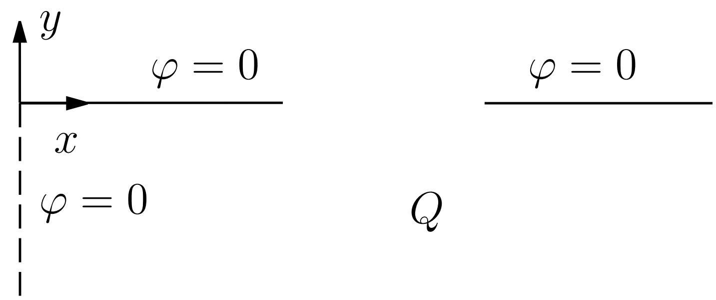
NON-EXISTENCE OF SYMMETRIC MODES DECAYING AT INFINITY $(m = 0)$

$$\text{If } \int_Q \left(\left| \nabla u^{(+)} \right|^2 + \left| u^{(+)} \right|^2 \right) \mathrm{d}x \, \mathrm{d}y < \infty, \text{ then } u^{(+)} \equiv 0 \text{ in Q}$$

The proof follows M. McIver (1999) (the 14th IWWWFB)

Introduce the stream function φ complex conjugate to $u = u^{(+)}$

By the assumption $u, \varphi \to 0$ as $z \to 0$



Green's formula for harmonic functions $u^2 - \varphi^2$ and x

$$\int_{\partial Q} \left\{ \left(u^2 - \varphi^2\right) n_x - x \partial_n \left(u^2 - \varphi^2\right) \right\} \mathrm{d}s = 0$$

+ boundary conditions

$$\int_{-\infty}^{0} u^2 \, \mathrm{d}y + \int_b^{b+1} x \partial_y \left(u^2 - \varphi^2\right) \mathrm{d}x = 0$$

By Cauchy–Riemann conditions $\partial_y \left(u^2 - \varphi^2\right) = -\partial_x \left(2u\varphi\right)$ and (+ boundary conditions) in the gap $2u\varphi = \frac{2}{\nu^{(+)}} \partial_y u \varphi = -\frac{1}{\nu^{(+)}} \partial_x \left(\varphi^2\right)$

$$\Downarrow$$

$$\int_{-\infty}^{0} u^2 \, \mathrm{d}y = -\frac{1}{\nu^{(+)}} \int_b^{b+1} \partial_x \left(\varphi^2\right) \mathrm{d}x = -\frac{1}{\nu^{(+)}} [\varphi(x, 0)]^2 \Big|_{x=b}^{x=b+1} = 0$$

$$u^{(+)}(0, y) = \partial_x u^{(+)}(0, y) = 0 \quad \Rightarrow \quad u^{(+)} \equiv 0 \text{ in } Q$$

SYMMETRIC MODES, CORRECT INTEGRAL EQUATION

Following Davis (1970), introduce

$$W(z;\xi) = -\frac{1}{\pi}\left\{\log\frac{8(z-\xi)}{(1-2z)(1-2\xi)} - \frac{1+2z}{2}\log\left(\frac{1+2z}{1-2z}\right)\right.$$
$$\left. - \frac{1+2\xi}{2}\log\left(\frac{1+2\xi}{1-2\xi}\right) + \frac{1}{2} - \pi\mathrm{i}(z+\xi)\right\}$$

For one gap $(-1,1)$ all modes can be found from

$$u(x,0) = \nu\int_{-1}^{1} u(\xi,0)\,\mathrm{Re}\,W(x,\xi)\,\mathrm{d}\xi$$

The symmetric two-gap problem is equivalent to the spectral problem

$$u^{(+)}(x) = \nu^{(+)}\int_{b}^{b+1} G(x,0;\xi)\; u^{(+)}(\xi)\,\mathrm{d}\xi, \quad x\in(b,b+1)$$

Green's function:

$$G(x,y;\xi) = \tfrac{1}{2}\,\mathrm{Re}\big\{W(z+b+\tfrac{1}{2};\xi+b+\tfrac{1}{2}) + W(z-b-\tfrac{1}{2};\xi-b-\tfrac{1}{2})$$
$$+ W(z+b+\tfrac{1}{2};-\xi+b+\tfrac{1}{2}) + W(z-b-\tfrac{1}{2};-\xi-b-\tfrac{1}{2})\big\} - g_0(b)$$

where

$$g_0(b) = \frac{1}{2\pi}\big[2b^2\log(2b) + 2(1+b^2)\log[2(1+b)] - (1+2b)^2\log(1+2b)\big]$$

and

$$\int_{b}^{b+1} G(x,0;\xi)\,\mathrm{d}x = \int_{b}^{b+1} G(x,0;\xi)\,\mathrm{d}\xi = 0$$

For $y=0$

$$\partial_y\big[G(x,y,\xi) + \frac{1}{\pi}\log|z-\xi|\big] = \begin{cases} 0 & \text{for} \quad x\in(0,b)\cup(b+1,+\infty) \\ -1 & \text{for} \quad x\in(b,b+1)\end{cases}$$

Restoring the potential in the fluid by

$$u^{(+)}(x,y) = \int_b^{b+1} u^{(+)}(\xi)\ G(x,y;\xi)\,\mathrm{d}\xi, \quad (x,y) \in Q$$

Hence, $u^{(+)}(x,y) = c + O(|z|^{-2})$ as $z \to \infty$, $c = \text{const} \neq 0$

<u>Another way to the integral equation</u>

For $m \neq 0$ the kernel by Fourier transform

$$G_0(x,\xi) = \pi^{-1}\left[K_0\big(m(x+\xi)\big) + K_0\big(m|x-\xi|\big)\right]$$

will not deliver solutions satisfying zero flux condition

$$\int_b^{b+1} \partial_y u^{(+)}(x,0)\,\mathrm{d}x = \nu^{(+)} \int_b^{b+1} u^{(+)}(x,0)\,\mathrm{d}x = 0$$

The correct kernel orthogonal to constant is as follows

$$G(x,\xi) = G_0(x,\xi) - f(x) - f(\xi) + C_0$$

where

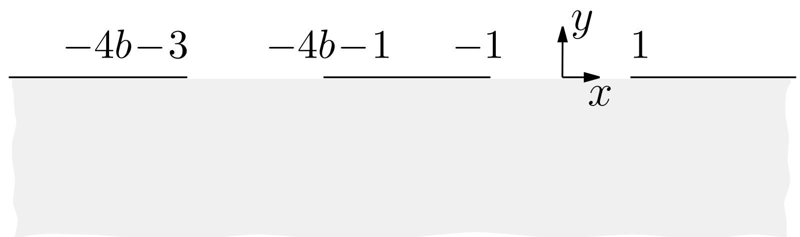
$$f(x) = \int_b^{b+1} G_0(x,\xi)\,\mathrm{d}\xi, \quad C_0 = \int_b^{b+1} f(\xi)\,\mathrm{d}\xi$$

Integral equation

$$u^{(+)}(x) = \nu^{(+)} \int_b^{b+1} G(x,0;\xi)\ u^{(+)}(\xi)\,\mathrm{d}\xi, \quad x \in (b, b+1)$$

NUMERICAL SCHEME, SYMMETRIC MODES ($m = 0$)

Change the coordinate system



to seek solutions to the integral equation

$$u^{(+)}(x) = \nu^{(+)} \int_{-1}^{1} G(x, 0; \xi)\; u^{(+)}(\xi)\, \mathrm{d}\xi, \quad x \in (-1, 1)$$

in the form

$$u^{(+)}(x) = \sum_{k=1}^{\infty} (2k+1)^{\frac{1}{2}} c_k\, P_k(x)$$

where P_k are the Legendre functions of the first kind

Since the system $\{P_k\}$ is orthogonal, the integral equation turns to

$$2c_m + \frac{\nu}{\pi} \sum_{k=1}^{\infty} (2n+1)^{\frac{1}{2}} (2m+1)^{\frac{1}{2}} \big\{ I_{mn} + I^*_{mn} \big\} c_n = 0, n, m = 1, 2, \ldots$$

where

$$I_{mn} = \int_{-1}^{1} \int_{-1}^{1} P_m(x)\, P_n(\xi) \log|x - \xi|\, \mathrm{d}x\, \mathrm{d}\xi,$$

$$I^*_{mn} = \int_{-1}^{1} \int_{-1}^{1} P_m(x)\, P_n(\xi) \log(x + \xi + 4b + 2)\, \mathrm{d}x\, \mathrm{d}\xi$$

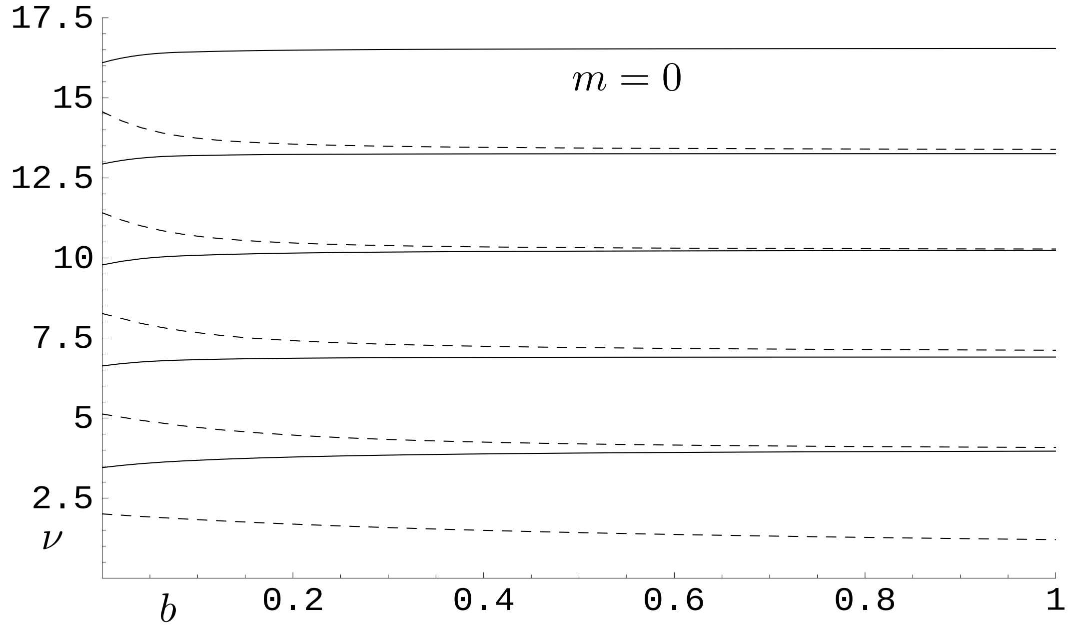
By Davis (1970) $I_{mn} = 0$ when $m - n$ is odd; for even $m - n$

$$I_{mn} = \frac{8}{(m+n)(2+m+n)\big[(m-n)^2 - 1\big]}$$

Besides,

$$I^*_{mn} = \frac{2}{2n+1} \int_{-1}^{1} P_m(\xi) \big\{ Q_{n+1}(-\xi - 4b - 2) - Q_{n-1}(-\xi - 4b - 2) \big\}\, \mathrm{d}\xi$$

where Q_k are the Legendre functions of the second kind



$\nu_1^{(-)}, \ldots, \nu_5^{(-)}$ (dashed lines) and $\nu_1^{(+)}, \ldots, \nu_5^{(+)}$ (solid lines)

Asymptotics for large b

$$\lim_{b\to\infty} \nu_{n+1}^{(-)}(b) = \lim_{b\to\infty} \nu_n^{(+)}(b) = 2\lambda_n$$

where λ_n are eigenvalues of the problem with one gap $(-1, 1)$

$$\lambda_{2n+1} = \nu_{n+1}^{(-)}(0), \quad \lambda_{2n} = \nu_n^{(+)}(0), \quad n = 0, 1, 2, \ldots$$

For $m = 0$, $\nu_0^{(+)} = 0$ corresponds to trivial mode $u_0^{(+)} = \text{const}$;

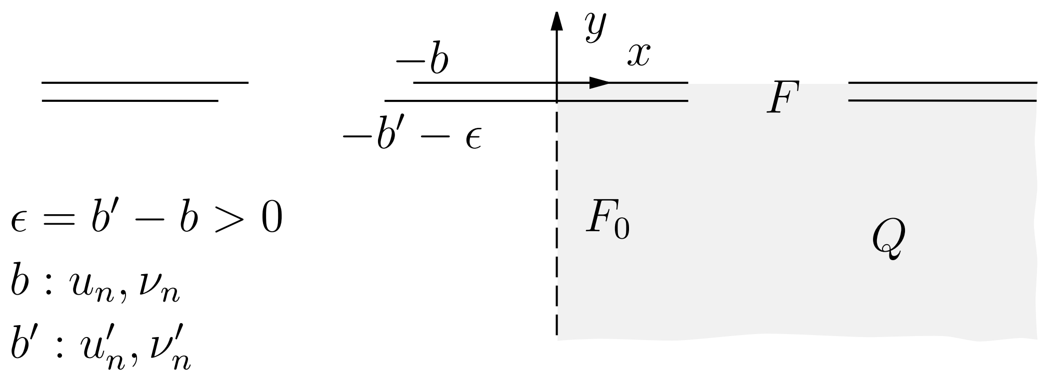
For $m \neq 0$ the mode $u_0^{(+)}$ is not trivial and $\nu_0^{(+)} \neq 0$

For $m = 0$, as $b \to \infty$,

$$\frac{\pi}{\nu_1^{(-)}} = \log(2b) + \frac{3}{2} + O\left(1/\log b\right),$$

$$1 - w_1^{(-)}(x) = \frac{\nu_1^{(-)}}{\pi}\left[\frac{1}{2} + x\log x + (1-x)\log(1-x)\right] + O\left(1/\log^2 b\right) \quad \left(w_1^{(-)} \geqslant 0, \int_0^1 w_1^{(-)}(x)\,\mathrm{d}x = 1\right)$$

DERIVATIVE OF EIGENVALUES AS FUNCTIONS OF SPACING BETWEEN GAPS



By Green's formula over Q,

$$0 = \int_F \Big\{ u_n \partial_y u'_n - u'_n \partial_y u_n \Big\} \, \mathrm{d}x + \int_{F_0} \Big\{ u'_n \partial_x u'_n - u_n \partial_y u'_n \Big\} \, \mathrm{d}y$$

<u>Antisymmetric modes</u>

By the condition on F and on F_0,

$$\big[\nu' - \nu\big] \int_F u_n u'_n \, \mathrm{d}x = \int_{F_0} u'_n \partial_x u_n \, \mathrm{d}y$$

Since $u'_n(x,y) = -\epsilon \, \partial_x u'_n(\theta(y), y)$, $0 \leqslant \theta(y) \leqslant \epsilon$, when $x \in [0, \epsilon]$,

$$\big[\nu' - \nu\big] \int_F u_n(x,0) \, u'_n(x,0) \, \mathrm{d}x = -\epsilon \int_{F_0} \partial_x u'_n(\theta(y), y) \, \partial_x u_n(0,y) \, \mathrm{d}y$$

As $\epsilon \to 0$, this gives

$$\boxed{\frac{\mathrm{d}\nu_n^{(-)}}{\mathrm{d}b} = -\frac{\int_{-\infty}^{0} \big|\partial_x \, u_n^{(-)}(0,y)\big|^2 \, \mathrm{d}y}{\int_b^{b+1} \big|u_n^{(-)}(x,0)\big|^2 \, \mathrm{d}x}}$$

Symmetric modes

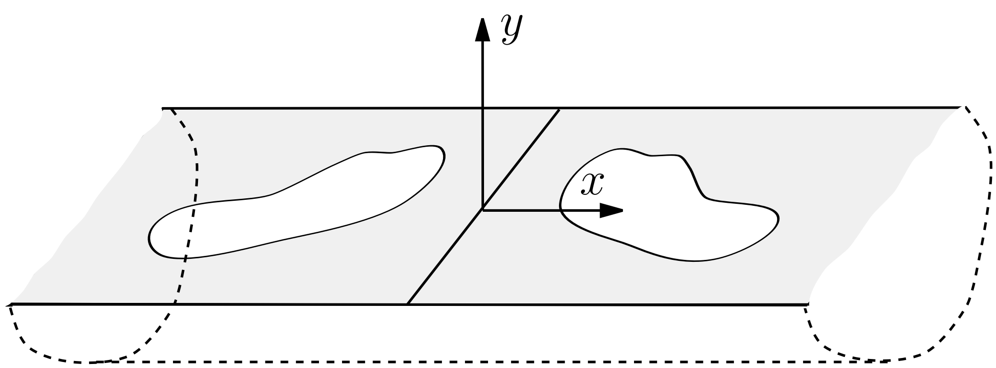
$$\frac{\mathrm{d}\nu_n^{(+)}}{\mathrm{d}b} = \frac{\int_{-\infty}^{0}\left[\left|\partial_y\, u_n^{(+)}(0,y)\right|^2 + m^2\left|u_n^{(+)}(0,y)\right|^2\right]\mathrm{d}y}{\int_b^{b+1}\left|u_n^{(+)}(x,0)\right|^2\mathrm{d}x}$$

Interesting to compare with the Rayleigh quotient

$$\nu_n^{(\pm)} = \frac{\int_{\mathbb{R}^2_-}\left[\left|\nabla\, u_n^{(\pm)}(x,y)\right|^2 + m^2\left|u_n^{(\pm)}(x,y)\right|^2\right]\mathrm{d}x\,\mathrm{d}y}{2\int_b^{b+1}\left|u_n^{(\pm)}(x,0)\right|^2\mathrm{d}x}$$

Example of another configuration (symmetric in x)

for which the method is applicable



SUMMARY:

- Sloshing problem is considered for an inviscid incomressible heavy fluid that occupies a channel with rigid walls and is covered by a rigid dock with two equal, strip-like apertures.
- The problem is reduced to spectral problems for integral equations (different for symmetric and antisymmetric modes).
- It is proved that the antisymmetric (symmetric) sloshing eigenvalues are monotonically decreasing (increasing) functions of spacing between gaps and formulae for their derivatives are obtained.
- Simplicity of eigenvalues is shown and their limit properties when spacing tends to zero and infinity are studied.
- Numerical scheme is developed and computations for eigenvalues are made.
- Asymptotics of solutions to the problem near dock tips are obtained ($m = 0$).
- Non-existence of symmetric modes decaying at infinity is proved ($m = 0$).