# Radiation problem for an interface-piercing cylinder in a two-layer fluid

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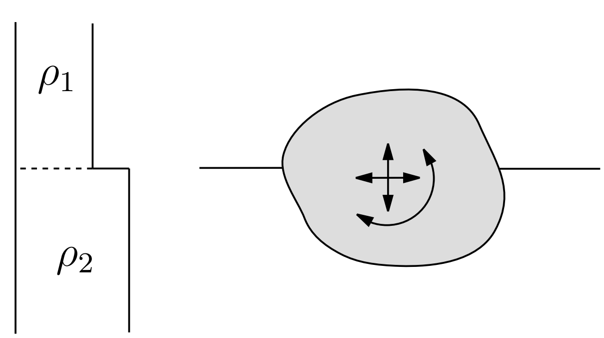
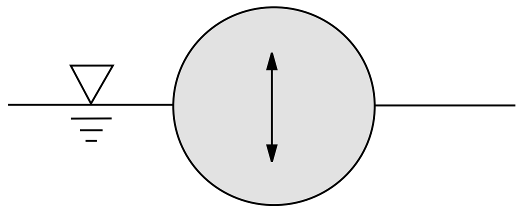
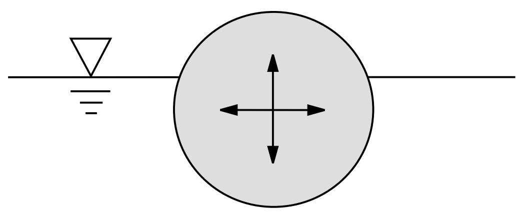


Heaving half-submerged circular cylinder



Ursell (1949)

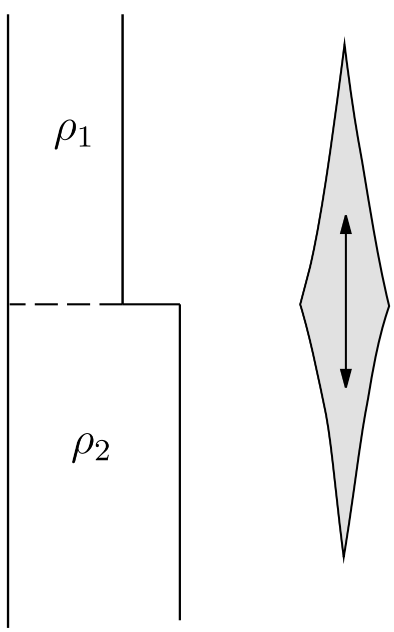
Circular cylinder floating on the free surface



Greenhow & Ahn (1988)

Eatock Taylor & Hu (1991)

Vertical oscillations of interface-intersecting slender body




Akulenko & Nesterov (1987)

Akulenko *et al* (1988)

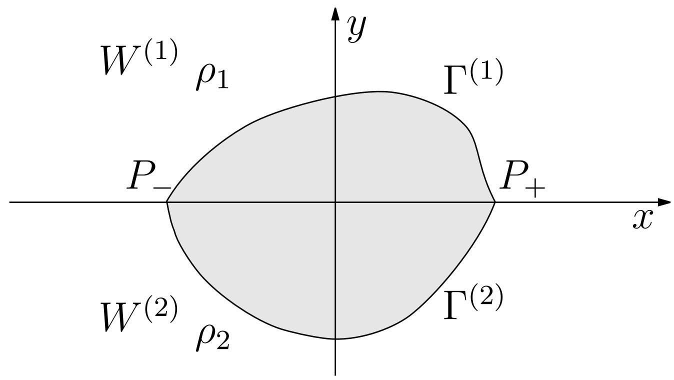
Akulenko, Mikhailov & Nesterov (1990)

Pyl'nev & Razumeenko (1991)

RADIATION PROBLEM

$\rho_2 = \rho_1(1+\varepsilon) \quad (\varepsilon > 0)$

$P_+ = (b, 0), \quad P_- = (-b, 0)$




$$\Phi^{(s)}(x,y,t) = \mathrm{Re}\left[\mathrm{i}\omega \sum_{j=1}^{3} \eta_j \phi_j^{(s)}(x,y) \exp(\mathrm{i}\omega t)\right]$$

In the fluid

$$\Delta \phi_j^{(s)} = 0 \quad (x,y) \in W^{(s)} \quad (s = 1,2)$$

Boundary conditions: on the interface $y = 0$

$$\frac{\partial \phi_j^{(1)}}{\partial y} = \frac{\partial \phi_j^{(2)}}{\partial y}, \quad (1+\varepsilon)\nu\phi_j^{(2)} - \nu\phi_j^{(1)} = \varepsilon \frac{\partial \phi_j^{(1)}}{\partial y}, \quad \nu = \frac{\omega^2}{g}$$

In the far field

$$\nabla \phi_j^{(s)} \to 0 \quad ((-1)^{s+1} y \to +\infty)$$

$$\frac{\partial \phi_j^{(s)}}{\partial x} \sim \pm i k_0 \phi_j^{(s)} \quad (x \to \mp\infty), \quad k_0 = \frac{(2+\varepsilon)\nu}{\varepsilon}$$

On the body surface

$$\frac{\partial \phi_j^{(s)}}{\partial n} = n_j \quad (x,y) \in \Gamma^{(s)}$$

Local finiteness of energy

$$\int_{E \cap W^{(s)}} \left|\nabla \phi_j^{(s)}\right|^2 \mathrm{d}x\, \mathrm{d}y < \infty, \quad P_\pm \in E$$

# BOUNDARY INTEGRAL EQUATIONS

Presentation of potential

$$\phi_j^{(s)}(z) = \sum_{\ell=1}^{2} \int_{\Gamma^{(\ell)}} \sigma_j^{(\ell)} G^{(s,\ell)}(z,\zeta)\,\mathrm{d}\Gamma_\zeta,\ \ z = x + \mathrm{i}y,\ \ \zeta = \xi + \mathrm{i}\eta\ (s = 1,2)$$

Green function $G^{(s,\ell)}(x, y; \xi, \eta)$; where $(x, y) \in L^{(s)}$, $(\xi, \eta) \in L^{(\ell)}$.

By Gorgui & Kassem (1978)

$$G^{(1,1)} = \ln|z-\zeta| + e_1 \ln|z-\bar{\zeta}| - e_2\Big[p\,v \int_0^\infty \frac{\mathrm{e}^{-k(y+\eta)}}{k-k_0} \cos k(x-\xi)\,\mathrm{d}k - \\ - \mathrm{i}\pi \mathrm{e}^{-k_0(y+\eta)} \cos k_0(x-\xi)\Big]$$

$$G^{(2,1)} = e_2\Big[\ln|z-\zeta| + p\,v \int_0^\infty \frac{\mathrm{e}^{k(y-\eta)}}{k-k_0} \cos k(x-\xi)\,\mathrm{d}k - \mathrm{i}\pi \mathrm{e}^{k_0(y-\eta)} \cos k_0(x-\xi)\Big]$$

$$G^{(1,2)} = e_3\Big[\ln|z-\zeta| + p\,v \int_0^\infty \frac{\mathrm{e}^{k(\eta-y)}}{k-k_0} \cos k(x-\xi)\,\mathrm{d}k - \mathrm{i}\pi \mathrm{e}^{k_0(\eta-y)} \cos k_0(x-\xi)\Big]$$

$$G^{(2,2)} = \ln|z-\zeta| - e_1 \ln|z-\bar{\zeta}| - e_3\Big[p\,v \int_0^\infty \frac{\mathrm{e}^{k(y+\eta)}}{k-k_0} \cos k(x-\xi)\,\mathrm{d}k - \\ - \mathrm{i}\pi \mathrm{e}^{k_0(y+\eta)} \cos k_0(x-\xi)\Big]$$

where

$$e_1 = \frac{\rho_2-\rho_1}{\rho_1+\rho_2}, \quad e_2 = \frac{2\rho_1}{\rho_1+\rho_2}, \quad e_3 = \frac{2\rho_2}{\rho_1+\rho_2}$$

System of integral equation

$$\pi\,\sigma_j^{(s)}(z) - \sum_{\ell=1}^{2} \int_{\Gamma^{(\ell)}} \sigma_j^{(\ell)}(\zeta) \frac{\partial G^{(s,\ell)}}{\partial n_z}(z,\zeta)\,\mathrm{d}\Gamma_\zeta = n_j(z) \quad (s = 1,2)$$

## FAR FIELD ASYMPTOTICS

As $x \to \pm\infty$

$$(\phi_1^{(1)}, \phi_1^{(2)}) \sim \pm(\psi^{(1)}, \psi^{(2)})\chi_1 \exp(\mp \mathrm{i} k_0 x)$$

$$(\phi_2^{(1)}, \phi_2^{(2)}) \sim \pm(\psi^{(1)}, \psi^{(2)})\chi_2 \exp(\mp \mathrm{i} k_0 x)$$

where

$$\psi^{(1)} = -\mathrm{e}^{-k_0 y}, \quad \psi^{(2)} = \mathrm{e}^{k_0 y}$$

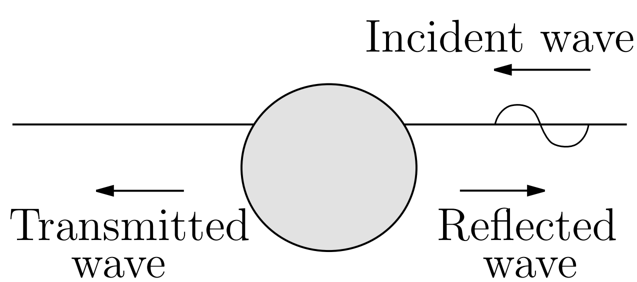
$$\chi_1 = a\pi e_3 k_0 A_1 \mathrm{e}^{k_0 h}, \quad \chi_2 = i\pi e_3 B_0 \mathrm{e}^{k_0 h}$$

Energy relation for the diagonal damping coefficients

$$\lambda_{11} = \omega\rho_1(2+\varepsilon)|\chi_1|^2, \quad \lambda_{22} = \omega\rho_1(2+\varepsilon)|\chi_2|^2$$

Reflection coefficient for
the diffraction problem
M.McIver (1996)

$$R_d = 2^{-1}\left[B_0/\overline{B}_0 - A_1/\overline{A}_1\right]$$



## LIMIT CASE OF HIGH FREQUENCIES

As $\omega \to \infty$ $(\nu \to \infty)$, $(1+\varepsilon)\phi_j^{(2)} - \phi_j^{(1)} = 0$, $y = 0$

No waves and hydrodynamic loads consists of added mass only

Consider $h \leqslant 0$. Multipoles $(m \geqslant 1)$

$$S_m^{(1)} = a^m e_3 \frac{\sin m\theta}{r^m}, \quad S_m^{(2)} = a^m \left[ \frac{\sin m\theta}{r^m} + e_1 (-1)^m \frac{\sin m\tau}{R^m} \right]$$

$$C_0^{(1)} = e_3 \ln \frac{r}{a}, \quad C_0^{(2)} = \ln \frac{r}{a} - e_1 \ln \frac{R}{a}$$

$$C_m^{(1)} = a^m e_3 \frac{\cos m\theta}{r^m}, \quad C_m^{(2)} = a^m \left[ \frac{\cos m\theta}{r^m} - e_1 (-1)^m \frac{\cos m\tau}{R^m} \right]$$

## LIMIT CASE OF LOW FREQUENCIES $\omega \to 0$ $(\nu \to 0)$

Homogeneous fluid

Greenhow & Yanbao (1987)

Greenhow & Ahn (1988)

P. McIver (1994)

$$\mu_{11}(\omega) \to \mu_{11}(0)$$
$$\mu_{22}(\omega) = O(\log \nu)$$
$$\Lambda_{11} \equiv \lambda_{11}(\omega)/\omega \to 0$$
$$\Lambda_{22} \equiv \lambda_{22}(\omega)/\omega \to \Lambda_{22}(0)$$

Two-layer fluid

Matched asymptotics. Outer solutions for horizontal and vertical oscillations:

$$\hat{\phi}_1^{(s)}(x,y) = \sum_{j=1}^{2} \beta^{(j)} k_0^{-1} \frac{\partial G^{(s,j)}}{\partial x}(x,y,0,0)$$

$$\hat{\phi}_2^{(s)}(x,y) = \sum_{j=1}^{2} \alpha^{(j)}\, G^{(s,j)}(x,y,0,0)$$

## LIMIT CASE OF LOW FREQUENCIES (2)

As $\omega \to 0$ $(\nu \to 0)$, $\partial\phi_j^{(s)}/\partial y = 0$, $y = 0$

Inner problems:

consider $\vartheta_j^{(s)}(x,y)$ $(j = 1,2)$, harmonic in $W^{(s)}$ and such that

$$\frac{\partial\vartheta_j^{(s)}}{\partial n} = n_j \quad (x,y) \in \Gamma^{(s)}, \qquad \frac{\partial\vartheta_j^{(s)}}{\partial y} = 0 \quad (y = 0, |x| > b)$$

$$\vartheta_1^{(s)} \sim -\chi^{(s)}\frac{x}{|z|^2}, \quad \vartheta_2^{(s)} = O(\log|z|), \quad \text{as} \ \ |z| \to \infty$$

Then, by matching inner and outer solutions,

$$\mu_{11}(0) = \sum_{s=1}^{2} \rho_s \left[\pi\chi^{(s)} - V^{(s)}\right]$$

$$\Lambda_{11} = (\rho_1 + \rho_2)\frac{\pi^2 k_0^2}{4}\left[e_2\chi^{(1)} - e_3\chi^{(2)}\right]^2$$

$$\mu_{22} = -\frac{4b^2}{\pi}(\rho_1 + \rho_2)\big[\ln(k_0 b) + C\big] + \sum_{s=1}^{2}\rho_s \int_{\Gamma^{(s)}} \vartheta_2^{(s)} n_2 \, \mathrm{d}\Gamma$$

$$\Lambda_{22}(0) = 4b^2(\rho_1 + \rho_2)$$

$V^{(s)}$ is the area of body part in $L^{(s)}$, $C \approx 0,5772$ is Euler's constant,

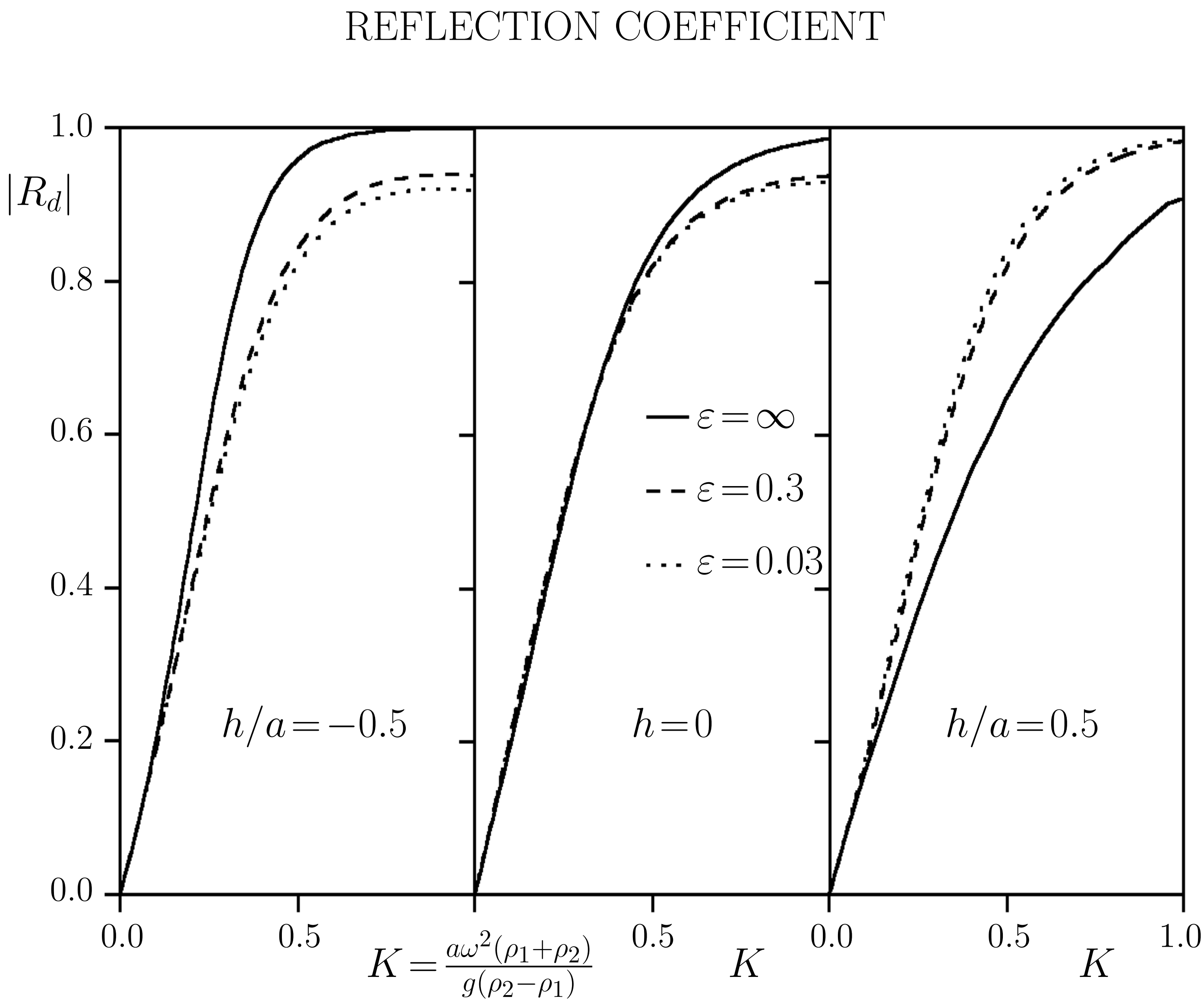
$b$ is the half of water-line, $k_0 = \nu(\rho_2 + \rho_1)/(\rho_2 - \rho_2)$

For circular cylinder

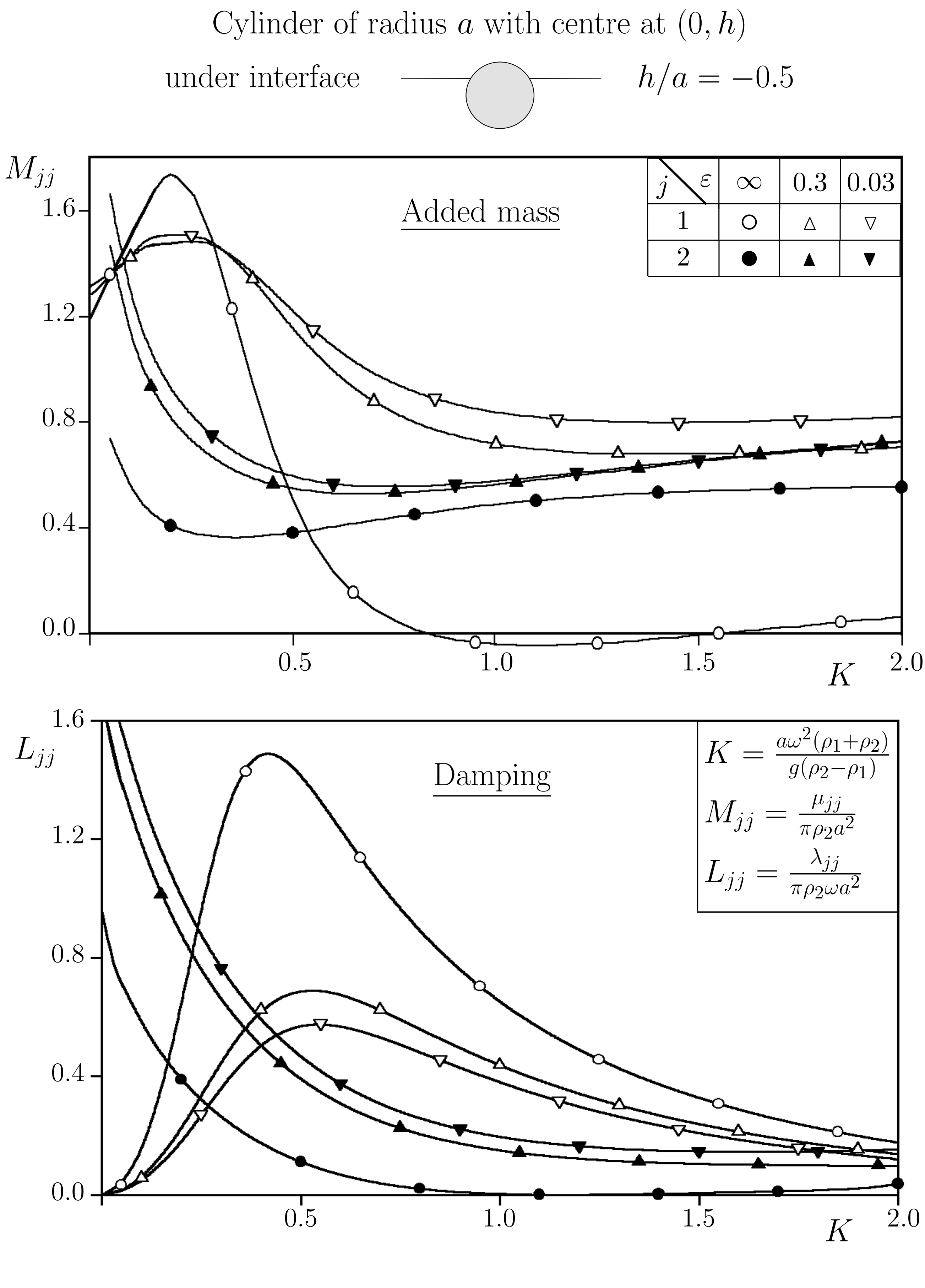
$$\chi^{(s)} = \frac{a^2\gamma(4\pi - \gamma)}{6(2\pi - \gamma)^2}(1 - \cos\gamma), \ V^{(s)} = \frac{a^2}{2}(\gamma - \sin\gamma), \ \gamma = 2\arccos\frac{(-1)^s h}{a}$$

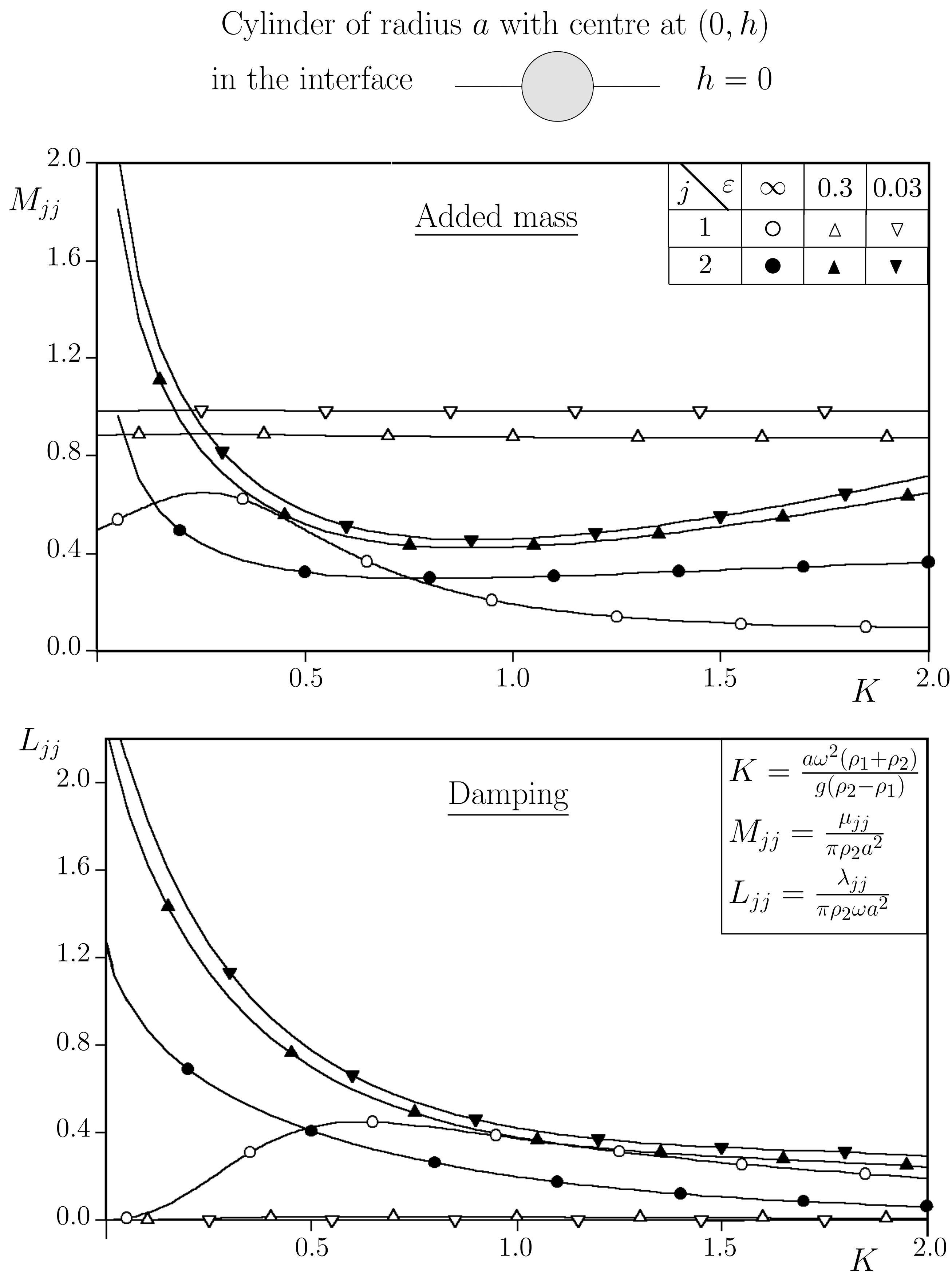
For $h = 0$

$$\int_{\Gamma^{(1)}} \vartheta_2^{(1)} n_2 \, \mathrm{d}\Gamma = \int_{\Gamma^{(2)}} \vartheta_2^{(2)} n_2 \, \mathrm{d}\Gamma = \frac{4a^2}{\pi}\left(\frac{3}{2} - 2\ln 2\right)$$

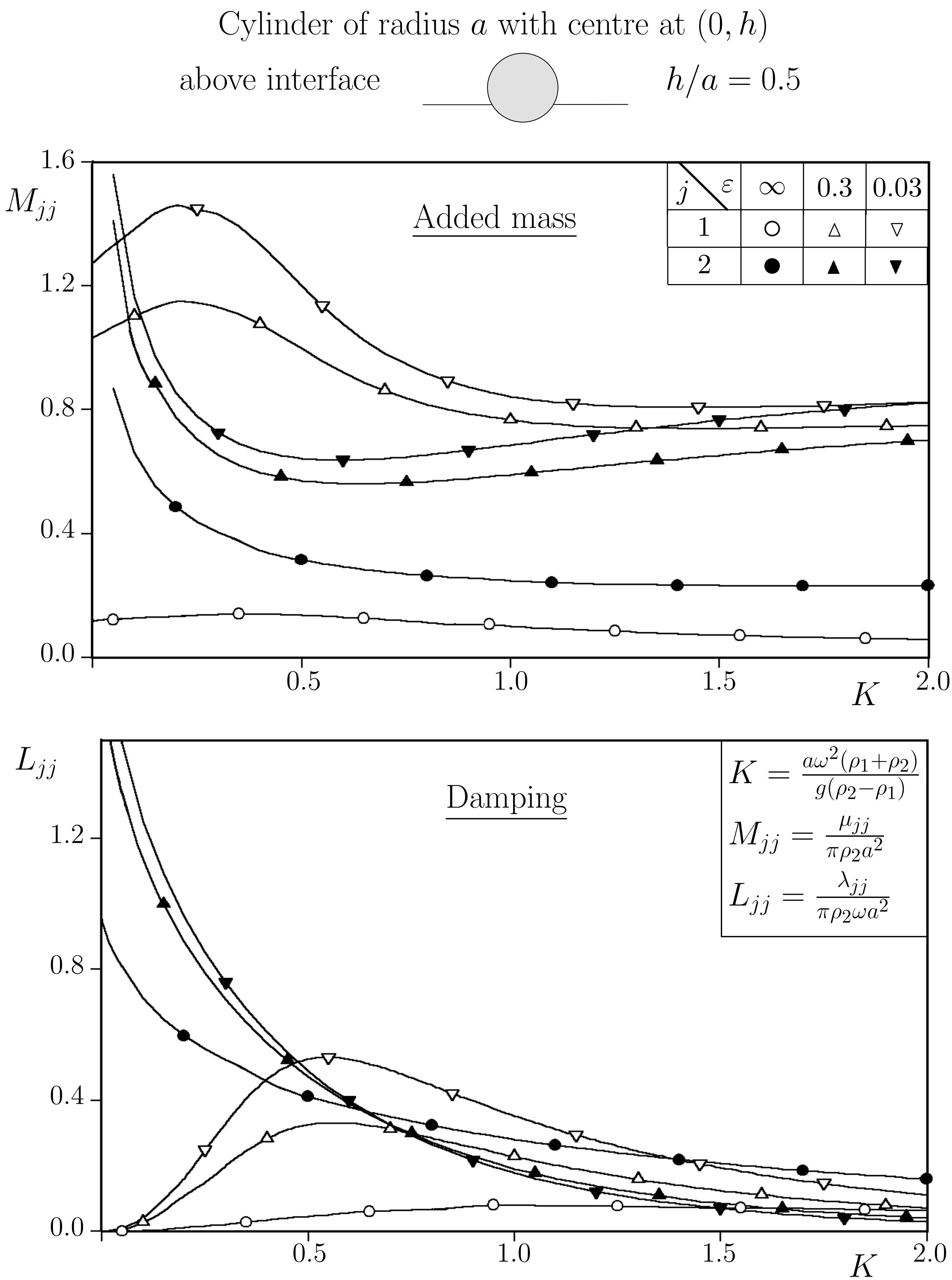


REFLECTION COEFFICIENT
$|R_d|$
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.5
0.5
0.0
0.5
1.0
$K=\frac{a\omega^2(\rho_1+\rho_2)}{g(\rho_2-\rho_1)}$
$K$
$K$
$h/a=-0.5$
$h=0$
$h/a=0.5$
$\varepsilon=\infty$
$\varepsilon=0.3$
$\varepsilon=0.03$


Cylinder of radius $a$ with centre at $(0, h)$
under interface
$h/a = -0.5$
$M_{jj}$
Added mass
$j$ \ $\varepsilon$ | $\infty$ | 0.3 | 0.03
1 | ○ | △ | ▽
2 | ● | ▲ | ▼
1.6
1.2
0.8
0.4
0.0
0.5
1.0
1.5
2.0
$K$
$L_{jj}$
Damping
$K = \frac{a\omega^2(\rho_1+\rho_2)}{g(\rho_2-\rho_1)}$
$M_{jj} = \frac{\mu_{jj}}{\pi\rho_2 a^2}$
$L_{jj} = \frac{\lambda_{jj}}{\pi\rho_2\omega a^2}$
1.6
1.2
0.8
0.4
0.0
0.5
1.0
1.5
2.0
$K$



Cylinder of radius $a$ with centre at $(0, h)$
in the interface
$h = 0$
Added mass
$M_{jj}$
$K$
$j$ \ $\varepsilon$ | $\infty$ | 0.3 | 0.03
1 | ○ | △ | ▽
2 | ● | ▲ | ▼
Damping
$L_{jj}$
$K = \frac{a\omega^2(\rho_1+\rho_2)}{g(\rho_2-\rho_1)}$
$M_{jj} = \frac{\mu_{jj}}{\pi\rho_2 a^2}$
$L_{jj} = \frac{\lambda_{jj}}{\pi\rho_2\omega a^2}$



Cylinder of radius $a$ with centre at $(0, h)$
above interface
$h/a = 0.5$
Added mass
$M_{jj}$
$K$
$j$ \ $\varepsilon$ | $\infty$ | 0.3 | 0.03
1
2
Damping
$L_{jj}$
$K = \frac{a\omega^2(\rho_1+\rho_2)}{g(\rho_2-\rho_1)}$
$M_{jj} = \frac{\mu_{jj}}{\pi\rho_2 a^2}$
$L_{jj} = \frac{\lambda_{jj}}{\pi\rho_2\omega a^2}$

## SUMMARY:

- Linear 2-D water-wave problem, describing small oscillations of a horizontal cylinder in the interface of two unbounded superposed fluids of different density, is considered.
- For a body of arbitrary shape a system of boundary integral equations is derived, a solution is sought in the form of a single layer potential.
- It is shown that the potential density can have a singularity at the points, where wetted contour coincides with the interface.
- Asymptotic formulae for the added mass and damping coefficients at low frequency are derived.
- For the circular cylinder the problem is solved by the multipole expansion method.
- Influence of stratification and position of the cylinder at the interface is shown to be essential for radiation loads and reflection coefficient in the diffraction problem.
- Unlike the case of a body, which is totally immersed in one of the layers, radiation loads by interface-piercing body depend on the type of oscillations, and the reflection coefficient is not identically zero.

## FUTURE WORK:

- effects of the second order; • 3D problem; • the case of smooth picnocline, e.g. of that described by the function $\rho(y) = \rho_0[1 - \frac{\epsilon}{2}\tanh\frac{y}{d}]$.