Proc. of 6th Int. Conf. on Math. and Num. Aspects of Wave Propagation Waves'2003, Univ. of Jyväskylä, Finland, 2003, p. 463–469

Eigenvalues of the Steklov problem in an infinite cylinder

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Abstract. The Steklov problem is considered in cylindrical domains; the coefficient in the boundary condition has a compact support and is an even function of a coordinate varying along the generators. We study the dependence of eigenvalues on the spacing between two symmetric parts of the coefficient's support. It is proved that the antisymmetric (symmetric) eigenvalues are monotonically decreasing (increasing) functions of the spacing and formulae for their derivatives are obtained. Application to the sloshing problem in a channel covered by a dock with two equal rectangular gaps is given.

1 Sloshing in a channel covered by a dock with two equal rectangular gaps

First we consider in more detail a particular case of the Steklov problem that has a clear hydrodynamic interpretation; for the general statement results are outlined in Sect. 2.

Let us begin with the problem of sloshing frequencies in a channel having infinite depth, parallel vertical walls, and occupied by an inviscid, incompressible, heavy fluid covered by a rigid dock so that the free surface consists of two parallel gaps of a unit length at distance $2b$ (see fig. 1). Neglecting the surface tension, we consider free, small-amplitude, time-harmonic oscillations of the fluid; its motion is assumed to be irrotational.

The sloshing problem with a single gap over a half-plane has received much consideration (see [1,2] and references cited therein) because eigenvalues of this

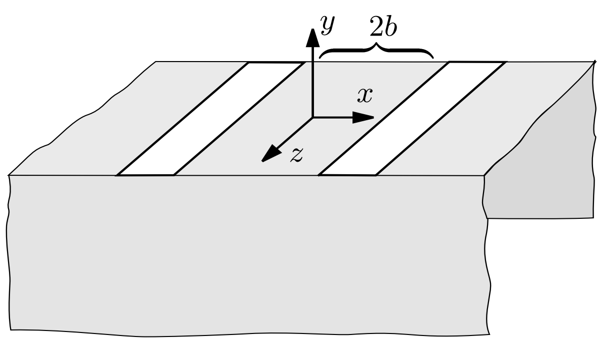

Fig. 1.

problem furnish universal upper bounds for sloshing frequencies in the two-dimensional domains having the same free surface. The case of a two-gap dock over a half-plane was considered by the authors in [3].

The aim of the present note is to give more general application of the approaches suggested in [3], which allows us to establish that the eigenvalues are monotonic functions of spacing between gaps and to derive formulae for their derivatives with respect to spacing in terms of energy integrals of the corresponding eigenfunctions.

Applying the ansatz $U(x,y,z,t) = u(x,y)\left\{\begin{matrix}\cos mz\\ \sin mz\end{matrix}\right\}\cos\omega t$ (where t is time and ω is the radian frequency of oscillations) and taking into account symmetry of the domain we arrive at the problems for symmetric $u^{(+)}(x,y)$ and antisymmetric $u^{(-)}(x,y)$ modes

$$\nabla^2 u^{(\pm)} = m^2 u^{(\pm)} \quad \text{in} \quad Q, \tag{1}$$

$$\partial_x u^{(+)} = 0, \quad u^{(-)} = 0 \quad \text{when} \quad x = 0,\ y < 0, \tag{2}$$

$$\partial_y u^{(\pm)} = 0 \quad \text{when} \quad y = 0,\ 0 < x < b,\ x > b+1, \tag{3}$$

$$\partial_y u^{(\pm)} = \nu^{(\pm)} u^{(\pm)} \quad \text{when} \quad y = 0,\ b < x < b+1, \tag{4}$$

$$\int_Q \left|\nabla u^{(\pm)}\right|^2 \mathrm{d}x\,\mathrm{d}y < \infty, \tag{5}$$

where $F = \{b < x < b+1,\ y = 0\}$, and it is convenient to restrict our considerations to the quadrant $Q = \{x \geq 0,\ y \leq 0\}$.

Following [3], where the case $m = 0$ was considered, we can prove that the antisymmetric problem (1)–(5) is equivalent to spectral problem for the following integral equations on $(b, b+1)$: for $m = 0$,

$$u^{(-)}(x) = \frac{\nu^{(-)}}{\pi} \int_b^{b+1} u^{(-)}(\xi) \log \frac{x+\xi}{|x-\xi|}\, \mathrm{d}\xi, \tag{6}$$

and for $m \neq 0$,

$$u^{(-)}(x) = \frac{\nu^{(-)}}{\pi} \int_b^{b+1} \left[K_0\big(m|x-\xi|\big) - K_0\big(m(x+\xi)\big)\right]\, u^{(-)}(\xi)\, \mathrm{d}\xi, \tag{7}$$

where $u^{(-)}(x) = u^{(-)}(x,0)$ and K_0 is modified Bessel function.

The symmetric problem (1)–(5) can be shown to be equivalent to spectral problem for the integral equation

$$u^{(+)}(x) = \nu^{(+)} \int_b^{b+1} G(x,0;\xi)\, u^{(+)}(\xi)\, \mathrm{d}\xi, \quad x \in (b, b+1). \tag{8}$$

The kernel is as follows

$$G(x,\xi) = G_0(x,\xi) - f(x) - f(\xi) + C_0,$$

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where

$$G_0(x,\xi) = \begin{cases} -\pi^{-1}\big(\log(x+\xi) + \log|x-\xi|\big) & \text{for } m = 0, \\ \pi^{-1}\left[K_0\big(m(x+\xi)\big) + K_0\big(m|x-\xi|\big)\right] & \text{for } m \neq 0, \end{cases}$$

and the rest terms are defined as follows:

$$f(x) = \int_b^{b+1} G_0(x,\xi)\,\mathrm{d}\xi, \quad C_0 = \int_b^{b+1} f(\xi)\,\mathrm{d}\xi.$$

They are introduced because solution to the integral equation must satisfy the condition:

$$\int_b^{b+1} u^{(+)}(x,0)\,\mathrm{d}x = 0,$$

which follows from (1)–(5) (see [1,3] for details). Many properties of $\nu^{(\pm)}$ and $u^{(\pm)}$ can be derived from (6)–(8), in particular, it follows that the eigenvalues and eigenfunctions are continuous functions of the parameter $b > 0$.

Our purpose is to prove the following formulae for derivatives

$$\frac{\mathrm{d}\nu_n^{(-)}}{\mathrm{d}b} = -\frac{\int_{-\infty}^0 \big|\partial_x u_n^{(-)}(0,y)\big|^2\,\mathrm{d}y}{\int_b^{b+1} \big|u_n^{(-)}(x,0)\big|^2\,\mathrm{d}x}, \tag{9}$$

$$\frac{\mathrm{d}\nu_n^{(+)}}{\mathrm{d}b} = \frac{\int_{-\infty}^0 \Big[\big|\partial_y u_n^{(+)}(0,y)\big|^2 + m^2\big|u_n^{(+)}(0,y)\big|^2\Big]\,\mathrm{d}y}{\int_b^{b+1} \big|u_n^{(+)}(x,0)\big|^2\,\mathrm{d}x}. \tag{10}$$

These formulae immediately prove monotonicity of eigenvalues as functions of distance between the gaps and describe the monotonic behaviour quantitatively.

We shall derive the formula for symmetric modes, the computations for antisymmetric case are analogous. Let $u_n^{(+)}(x,y;b)$ be a symmetric eigenmode corresponding to the sloshing eigenvalue $\nu_n^{(+)}(b)$. We can see from (8) that $\nu_n^{(+)}(b)$ is a differentiable function of $b > 0$. Let Δ be a sufficiently small number (such that $b + \Delta > 0$). After extending $u_n^{(+)}(x,y;b+\Delta)$ to the whole half-plane $y < 0$ by application of the Schwarz Reflection Principle to the harmonic potential U (a solution to (1) is not harmonic when $m \neq 0$), we consider $u_n^{(+)}(x+\Delta,y;b+\Delta)$ defined in the closed quadrant $\{x \geq 0,\ y \leq 0\}$ even when $\Delta < 0$. The latter function satisfies the similar boundary conditions as $u_n^{(+)}(x,y;b)$ on

$$\{0 < x < b,\ y = 0\}, \quad \{b < x < b+1,\ y = 0\} \quad \text{and} \quad \{b+1 < x < +\infty,\ y = 0\}$$

respectively. Further we apply the second Green's formula to $u_n^{(+)}(x,y;b)$ and $u_n^{(+)}(x+\Delta,y;b+\Delta)$ in $\{x > 0,\ y < 0\}$. This gives

$$\int_b^{b+1} \big[u_n^{(+)}(x,0;b)\,\partial_y u_n^{(+)}(x+\Delta,0;b+\Delta) - u_n^{(+)}(x+\Delta,0;b+\Delta)\,\partial_y u_n^{(+)}(x,0;b)\big]\,\mathrm{d}x$$
$$= \int_{-\infty}^0 \big[u_n^{(+)}(0,y;b)\,\partial_x u_n^{(+)}(\Delta,y;b+\Delta) - \partial_x u_n^{(+)}(\Delta,y;b+\Delta)\,u_n^{(+)}(0,y;b)\big]\,\mathrm{d}y$$

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because (5) guarantees that the integral over a large quarter-circle tends to zero as its radius goes to infinity; the homogeneous Neumann condition on the dock is also applied here. Using (2) and the Lagrange theorem in the second integral, and the free surface conditions in the first one, we get

$$\Big[\nu_n^{(+)}(b+\Delta) - \nu_n^{(+)}(b)\Big] \int_b^{b+1} u_n^{(+)}(x,0;b)\; u_n^{(+)}(x+\Delta,0;b+\Delta)\,\mathrm{d}x$$
$$= \Delta \int_{-\infty}^{0} u_n^{(+)}(0,y;b)\; \partial_x^2 u_n^{(+)}(\theta(y)\Delta, y; b+\Delta)\,\mathrm{d}y,$$

where $0 < \theta(y) < 1$ for $y \in (-\infty, 0)$. Letting $\Delta \to 0$ in this equation divided by Δ produces

$$\frac{\mathrm{d}\nu_n^{(+)}}{\mathrm{d}b} \int_b^{b+1} \big|u_n^{(+)}(x,0;b)\big|^2\,\mathrm{d}x = \int_{-\infty}^{0} u_n^{(+)}(0,y;b)\; \partial_x^2 u_n^{(+)}(0,y;b)\,\mathrm{d}y.$$

In order to obtain (10), it remains to transform the last integral using the equation (1) and then applying integration by parts. The out of integral terms vanish because $\partial_y u_n^{(+)}(0,y;b)$ satisfies the no flow condition on the dock and decays at infinity.

2 The Steklov problem in a cylinder: dependence of eigenvalues on a parameter varying along generators

The above scheme for proving formulae (9), (10) is applicable to more general situations. As an example we consider the following version of the problem proposed by Steklov [4] in 1902 and now referred to as the Steklov problem. Consider an infinitely long cylinder $\Omega = D \times \mathbb{R}^1$, where $D \subset \mathbb{R}^2$ is a domain with piecewise smooth boundary. By p_+ we denote a bounded positive function on $\Gamma = \partial D \times \mathbb{R}^1$ having a compact support $S_+ \subset \partial D \times \mathbb{R}^1_+$ (see fig. 2). For $b > -\operatorname{dist}\big(S_+, \{x = 0\}\big)$ and $(x, y, z) \in \Gamma$ we introduce p_b as follows:

$$p_b(x,y,z) = p_+(|x| - b, y, z).$$

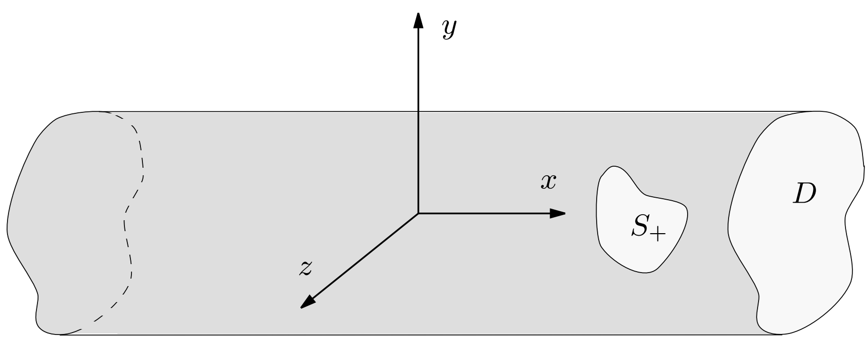

Fig. 2.

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For this geometry we can also distinguish symmetric and antisymmetric in x solutions $u^{(+)}(x,y,z)$ and $u^{(-)}(x,y,z)$, respectively, satisfying the following problem

$$\nabla^2 u^{(\pm)} = 0 \quad \text{in} \quad \Omega,$$
$$\partial_x u^{(+)} = 0, \quad u^{(-)} = 0 \quad \text{when} \quad x = 0,\ (y,z) \in D,$$
$$\partial_n u^{(\pm)} = \nu^{(\pm)}\, p_b\, u^{(\pm)} \quad \text{on} \quad \Gamma,$$
$$\int_\Omega \left|\nabla u^{(\pm)}\right|^2 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z < \infty,$$

where ∂_n stands for the derivative in the direction of the exterior unit normal to Γ.

In this case it is not possible to obtain such simple integral equations as in the case of two rectangular gaps. Providing that yet continuous dependence of $u^{(\pm)}$, $\nu^{(\pm)}$ on b is established, application of the scheme given in the previous section immediately leads to the formulae

$$\frac{\mathrm{d}\nu_n^{(-)}}{\mathrm{d}b} = -2\frac{\int_D \left|\partial_x u_n^{(-)}(0,y,z)\right|^2 \mathrm{d}y\,\mathrm{d}z}{\int_\Gamma p_b \left|u_n^{(-)}\right|^2 \mathrm{d}s},$$
$$\frac{\mathrm{d}\nu_n^{(+)}}{\mathrm{d}b} = 2\frac{\int_D \left|\nabla_{y,z}\, u_n^{(+)}(0,y,z)\right|^2 \mathrm{d}y\,\mathrm{d}z}{\int_\Gamma p_b \left|u_n^{(+)}\right|^2 \mathrm{d}s}.$$

Note that the above formulae give the derivatives with respect to b of the Rayleigh quotient

$$\nu_n^{(\pm)} = \frac{\int_\Omega \left|\nabla\, u_n^{(\pm)}(x,y,z)\right|^2 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\int_\Gamma p_b \left|u_n^{(\pm)}\right|^2 \mathrm{d}s}$$

Acknowledgement. The authors acknowledge a support from the Russian Foundation of Basic Research through the grant 01-01-00973.

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